Winter School and Workshop
“Riemann-Hilbert correspondences”

Padua, January 29 – February 9, 2018

Recent groundbreaking developments in the theory of irregular holonomic $D$-modules have brought to light new relationships among $D$-module theory and other subjects such as Non Commutative Hodge theory, Mirror Symmetry and Microlocal Analysis. The aim of this event is twofold: to provide an introduction to this new active area for young students and researchers, and to present the state of the art on the subject. The School will be taught during the first week, and center around four mini courses on previous developments. Tutorials will also be offered. The Workshop will run during the second week and include a series of colloquium-style talks by top international experts. Research talks by young participants are also scheduled during the second week. Among the topics discussed: formal and analytic solutions of systems of differential equations; $p$-adic differential equations; analytic and arithmetic $D$-modules; Hodge theory; microlocal analysis.

Web page: http://events.math.unipd.it/rh2018

Local Scientific Committee

Andrea D’Agnolo (Director)
Bruno Chiarellotto
Luisa Fiorot
Pietro Polesello
Luca Prelli

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Program of the School

Mon 29/1
10:00-12:00 Jean-Pierre Schneiders 1
pause for lunch
14:00-16:00 Luca Migliorini 1
coffee break
16:30-18:00 Luisa Fiorot 1

Tue 30/1
10:00-12:00 Luca Migliorini 2
pause for lunch
14:00-16:00 Jean-Pierre Schneiders 2
coffee break
16:30-18:00 Luisa Fiorot 2

Wed 31/1
10:00-12:00 Jean-Pierre Schneiders 3
pause for lunch
14:00-16:00 Luca Migliorini 3
coffee break
16:30-18:00 Luisa Fiorot 3

Thu 1/2
9:00-10:30 Tomoyuki Abe 1
coffee break
11:00-12:30 Andrea D’Agnolo 1
pause for lunch
14:30-16:00 Tomoyuki Abe 2

Fri 2/2
9:00-10:30 Andrea D’Agnolo 2
coffee break
11:00-12:30 Tomoyuki Abe 3
pause for lunch
14:30-16:00 Andrea D’Agnolo 3
Program of the Workshop

**Mon 5/2**
10:00-10:55 Hélène Esnault  
coffee break  
11:30-12:25 Tomoyuki Abe  
pause for lunch  
14:00-14:55 Vivek Shende  
coffee break  
15:30-16:25 Michael Kapranov  
16:30-17:25 Boris Tsygan

**Tue 6/2**
9:00-9:55 Marco Hien  
coffee break  
10:30-11:25 Teresa Monteiro Fernandes  
11:30-12:25 Kiyoshi Takeuchi  
pause for lunch  
14:00-14:30 Tatsuki Kuwagaki  
14:40-15:10 Christophe Dubussy  
coffee break  
15:40-16:10 Takahiro Saito  
16:20-16:50 Avi Steiner

**Wed 7/2**
9:00-9:55 Javier Fresán  
coffee break  
10:30-11:25 Nicolas Templier  
11:30-12:25 Jean-Baptiste Teyssier  
free afternoon

**Thu 8/2**
9:00-9:55 Ahmed Abbes  
10:00-10:55 Michel Gros  
coffee break  
11:30-12:00 Daxin Xu  
pause for lunch  
14:00-14:30 Marco D’Addezio  
14:40-15:10 Simon Pepin Lehalleur  
coffee break  
15:40-16:10 Dimitri Wyss  
16:20-16:50 Julian Holstein

**Fri 9/2**
9:00-9:55 Dmitri Tamarkin  
coffee break  
10:30-11:25 Takuro Mochizuki  
11:30-12:25 Claude Sabbah
School's courses

**Perverse sheaves**

Luca Migliorini  
Università di Bologna, Italy


**Riemann-Hilbert correspondence I**

Jean-Pierre Schneiders  
Université de Liège, Belgium

Holonomic systems and the constructibility of their solution complex. Regular holonomic systems and their main properties. Tempered distributions and the TH functor. Presentation of Kashiwara’s original solution of the Riemann-Hilbert problem.

**Tutorials**

Luisa Fiorot  
Università di Padova, Italy

These sessions are dedicated to address some of the exercises proposed in the courses “Riemann-Hilbert correspondence I” and “Perverse sheaves”. The tutor will give some hints and the participants will be strongly encouraged to share their solutions. The tutor will propose a review on basics on derived categories, $t$-structures, locally constant sheaves... following the needs of the class.
Hilbert’s twenty-first problem (a.k.a. Riemann-Hilbert problem) asks for the existence of linear ordinary differential equations with prescribed regular singularities and monodromy. In higher dimensions, Deligne formulated it as a correspondence between regular meromorphic flat connections and local systems. In the early eighties, Kashiwara generalized it to a correspondence between regular holonomic $D$-modules and perverse sheaves on a complex manifold. This will be discussed in the course “Riemann-Hilbert correspondence I.”

The aim of the present course will then be to discuss the analogous problem for possibly irregular holonomic $D$-modules (a.k.a. the Riemann-Hilbert-Birkhoff problem). Such problem has been standing for a long time. One of the difficulties was to find a substitute target to the category of perverse sheaves. In the eighties, Deligne and Malgrange proposed a correspondence between meromorphic connections and Stokes filtered local systems on a complex curve. Recently, Kashiwara and the speaker solved the problem for general holonomic $D$-modules in any dimension. The construction of the target category is based on the theory of ind-sheaves by Kashiwara-Schapira and uses Tamarkin’s work on symplectic topology. Among the main ingredients of the proof is the description of the structure of flat meromorphic connections due to Mochizuki and Kedlaya.

The theory of arithmetic $D$-modules was introduced by Berthelot around 80s. The goal of this theory was to establish the six functor formalism containing rigid cohomology theory. Thanks to effort of many people, most of the expected properties has been proven. I will report on the current status of the theory.
Workshop’s plenary talks
(in speakers’ alphabetical order)

The \( p \)-adic Simpson correspondence
Ahmed Abbes
IHES, France

The \( p \)-adic Simpson correspondence, initiated by Gerd Faltings in 2005, aims at describing all \( p \)-adic representations of the fundamental group of a proper smooth variety over a \( p \)-adic field in terms of linear algebra — namely Higgs bundles. My lecture will be an introduction to this topic. I will focus on the approach that I developed with Michel Gros relying on a new family of period rings built from the torsor of deformations of the variety over a universal \( p \)-adic thickening defined by J.-M. Fontaine.

Trace formalism and \( l-p \) independence in arithmetic
\( D \)-modules
Tomoyuki Abe
Kavli IPMU, Japan

A main goal of this talk is to prove that the trace of a correspondence acting on the \( p \)-adic cohomology coincides with that of \( l \)-adic cohomology. This was done using the theory of arithmetic \( D \)-modules of Berthelot. I will recall the theory briefly, and using the theory, I will establish the trace formalism in the style of SGA 4 in the theory. If time permits, I will discuss some application of the independence.

Complex rigid connections and \( F \)-isocrystals
Hélène Esnault
Free University, Berlin, Germany

We explain the relation (and prove Simpson’s conjecture on the integrality of cohomologically rigid connections). Joint with Michael Groechenig.

t.b.a.
Javier Fresán
École Polytechnique, France

abstract
A $q$-deformation of the local Ogus-Vologodsky correspondance

Michel Gros
Université de Rennes I, France

We will discuss some Azumaya properties of a ring of twisted differential operators naturally appearing in $p$-adic Hodge theory and draw some consequences (joint work with A. Quiros and B. Le Stum)

**t.b.a.**

Marco Hien
Augsburg University, Germany

abstract

**Fourier transform on hyperplane arrangements**

Michael Kapranov
Kavli IPMU, Japan

The category of perverse sheaves on a complex vector space smooth with respect to a hyperplane arrangement with real equations, has an algebraic description. The corresponding linear algebra data can be called hyperbolic sheaves. The talk, based on a joint work with M. Finkelberg and V. Schechtman, describes, in these terms, several basic functors of microlocal sheaf theory such as forming vanishing cycles, specialization and Fourier-Sato transform. The answers have the form of some polyhedral ("tropical") interpretation of the functors.

**Periodic monopoles and difference modules**

Takuro Mochizuki
RIMS, Japan

One of the main themes in complex geometry is to pursue correspondences between differential geometric objects and algebro-geometric objects. In this talk, we shall explain a kind of Kobayashi-Hitchin correspondence between periodic monopoles of GCK type and difference modules with parabolic structure.
An overview over the relative Riemann-Hilbert correspondence

Teresa Monteiro Fernandes
Universidade de Lisboa, Portugal

Let $X \times S$ be a product of complex manifolds where $S$ is a complex line, and let $p: X \times S \to S$ be the projection. In this talk we present an overview of the construction of the relative Riemann-Hilbert functor as a right quasi-inverse to the solution functor for regular relative holonomic modules, in recent joint work with Claude Sabbah. We will explain, if times permits, how these functors behave with regard to the associated natural $t$-structures (joint work with Luisa Fiorot). If the dimension of $X$ is 1, myself and Claude Sabbah proved that, in a generic sense, any relative holonomic module admits a coherent restriction to any point, which allowed to prove that, in a generic sense, the Riemann-Hilbert functor is also a left exact functor. As an application we can prove that, generically, a complex is regular holonomic iff it is regular at each point of $X$.

\textit{t.b.a.}

Claude Sabbah
\textit{Эcole Polytechnique, France}

abstract

\textit{t.b.a.}

Vivek Shende
\textit{UC Berkeley, USA}

abstract

Rapid decay homologies vs enhanced ind-sheaves

Kiyoshi Takeuchi
University of Tsukuba, Japan

The theory of rapid decay homology was introduced by Hien. It is very useful to construct integral representations of holomorphic solutions of $D$-modules. On the other hand, D’Agnolo and Kashiwara introduced enhanced ind-sheaves to establish irregular Riemann-Hilbert correspondence. In this talk, we discuss the relationship between them and give some applications.
The axiomatic microlocal category

Dmitri Tamarkin
Northwestern University, USA

I will talk on my work in progress on associating an \((\infty, 1)\)-category to a compact symplectic manifold via imposing a natural list of axioms.

Mirror symmetry for minuscule flag varieties

Nicolas Templier
Cornell University, USA

We prove cases of Rietsch mirror conjecture that the quantum connection for projective homogeneous varieties is isomorphic to the pushforward \(D\)-module attached to Berenstein-Kazhdan geometric crystals. We show that the quantum differential equations have simple irregular singularities, and then establish an isomorphism with a space of Hecke transformations. At the heart of the proof, we link the de Rham cohomology of crystals to the Ramanujan property of eigen-\(D\)-modules. Work with Thomas Lam.

Moduli of Stokes torsors and singularities of differential equations

Jean-Baptiste Teyssier
KU Leuven, Belgium

The aim of this talk is to explain how the geometry of the Stokes phenomenon in any dimension sheds light on the interplay between the turning point locus of a meromorphic connection and the singularities of its solutions.

Microlocal categories and two-functors

Boris Tsygan
Northwestern University, USA

For a symplectic manifold, there are (at least) two ways to define a category by microlocal methods. One, due to D. Tamarkin, starts with a Darboux chart \(U\) and defines a category \(C(U)\) as the quotient category of a certain subcategory of sheaves on \(\mathbb{R}^n \times \mathbb{R}\). For a general symplectic manifold \(M\), the category \(C(M)\) is constructed by a gluing construction which is not a naive local-to-global procedure but involves Hamiltonian isotopies between different Darboux charts. The other way to construct a category, outlined in arXiv:1512.02747, starts with deformation quantization of our manifold \(M\) and produces a sheaf of dg algebras \(A\) on \(M\); the fundamental groupoid of \(M\) acts on the sheaf of categories \(A\mod\). The category is obtained by a gluing procedure using this action. I will outline both constructions and, time permitting, discuss a conjectural Riemann-Hilbert functor between the two.
Workshop’s short communications
(in speakers’ alphabetical order)

Monodromy groups of $F$-isocrystals

Marco D’Addezio
Free University, Berlin, Germany

Thanks to the Tannakian formalism, one can associate to every $F$-isocrystal an algebraic group. This group behaves quite similarly to the algebraic monodromy group of a lisse sheaf. I will present some of its properties and I will explain how to deduce from them the constancy of the Newton polygons of $F$-isocrystals defined on abelian varieties over finite fields.

Holomorphic cohomological convolution and Hadamard product

Christophe Dubussy
Université de Liège, Belgium

In his thesis, T. Pohlen succeeded in defining a Hadamard product between holomorphic functions defined on star eligible open sets of the Riemann Sphere. We show how this theory is actually a particular case of the holomorphic cohomological convolution, defined in a general way thanks to the integration map on a complex Lie group.

The derived period map

Julian Holstein
Lancaster University, UK

Griffiths’ period map expresses how the Hodge filtration on cohomology varies in a family of smooth projective varieties. Thus one can linearize moduli problems of varieties and study them using the moduli of Hodge structure. In this talk I will describe how to extend the period map to the realm of derived geometry and construct a period map for a smooth projective map of derived stacks. This generalises both Griffiths classical period map and the infinitesimal derived period map that was introduced by Fiorenza and Manetti. This is joint work with Carmelo Di Natale.
Sheaf quantization of the great circle
Tatsuki Kuwagaki
Kavli IPMU, Japan

Sheaf quantization is a way to assign a sheaf to a Lagrangian submanifold. For exact Lagrangian submanifolds in cotangent bundles, sheaf quantization was done by Guillermou and Jin-Treumann. In this talk, I will give some observations for sheaf quantization of the great circle.

On the motive of the stack of vector bundles on a curve
Simon Pepin Lehalleur
Free University, Berlin, Germany

Following Grothendieck’s vision that many cohomolgical invariants of of an algebraic variety should be captured by a common motive, Voevodsky introduced a triangulated category of mixed motives which partially realises this idea. After describing this category, I will explain how to define the motive of certain algebraic stacks in this context. I will then report on joint work in progress with Victoria Hoskins, in which we study the motive of the moduli stack of vector bundles on a smooth projective curve and show that this motive can be described in terms of the motive of this curve and its symmetric powers.

Milnor monodromies and mixed Hodge structures for non-isolated hypersurface singularities
Takahiro Saito
University of Tsukuba, Japan

We study the Milnor monodromies of non-isolated hypersurface singularities and show that the reduced cohomology groups of the Milnor fibers are concentrated in the middle degree for some eigenvalues of the monodromies. As an application of this result, we give a formula for some parts of their Jordan normal forms.
Let $A$ be a $d \times n$ integer matrix. Gel’fand et al. proved that most $A$-hypergeometric systems have an interpretation as a Fourier–Laplace transform of a direct image. The set of parameters for which this happens was later identified by Schulze and Walther as the set of not strongly resonant parameters of $A$. A similar statement relating $A$-hypergeometric systems to exceptional direct images was proved by Reichelt. In this talk, we will discuss a hybrid approach involving neighborhoods $U$ of the torus of $A$ and consider compositions of direct and exceptional direct images. Our aim will be to present a characterization for which parameters the associated $A$-hypergeometric system is the inverse Fourier–Laplace transform of such a “mixed Gauss–Manin” system. Special emphasis will be placed on the normal case, where every $A$-hypergeometric system turns out to be of this form. In order to describe which $U$ work for such a parameter, we will introduce the notions of fiber support and cofiber support of a $D$-module.

**Arithmetic aspects of open de Rham spaces**

Dimitri Wyss
FSM Paris, France

In this join work with Tamas Hausel and Michael Wong we determine the motivic classes of open de Rham spaces as defined by Boalch using a motivic Fourier transform. Through the purity conjecture and the wild Riemann-Hilbert correspondence our computations give numerical evidence for the recent conjecture of Hausel, Mereb and Wong on the mixed Hodge polynomial of wild character varieties. Furthermore we give a construction of open de Rham spaces in terms of quivers with multiplicities in the spirit of Crawley-Boevey and Yamakawa.

**Lifting the Cartier transform of Ogus-Vologodsky modulo $p^n$**

Daxin Xu
California Institute of Technology, USA

Let $W$ be the ring of the Witt vectors of a perfect field of characteristic $p$, $\mathfrak{X}$ a smooth formal scheme over $W$, $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by the Frobenius morphism of $W$, $\mathfrak{X}_2'$ the reduction modulo $p^2$ of $\mathfrak{X}'$ and $X$ the special fiber of $\mathfrak{X}$. We lift the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}_2'$ modulo $p^n$. More precisely, we construct a functor from the category of $p^n$-torsion $O_{\mathfrak{X}'}$-modules with integrable $p$-connection to the category of $p^n$-torsion $O_{\mathfrak{X}}$-modules with integrable connection, each subject to suitable nilpotence conditions. If there exists a lifting $F : \mathfrak{X} \rightarrow \mathfrak{X}'$ of the relative Frobenius morphism of $X$, our functor is compatible with a functor constructed by Shiho from $F$. As an application, we give a new interpretation of Faltings’ relative Fontaine modules and of the computation of their cohomology.