A Short Introduction to Type Theory

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Introduction

The purpose of this note is to explain shortly some features and notations of Type Theory for a mathematician.

1 Analogy between implication and exponentiation

In set theory, if we write $A \to B$ the set of functions from the set A to the set B, one can find "uniform" inhabitants to the sets

$A \to A$: the identity function $x \longmapsto x$

$A \to (B \to A)$: the parametric constant functional $x \longmapsto (y \longmapsto x)$

$(A \to (A \to B)) \to (A \to B)$: the diagonalisation functional $f \longmapsto (x \longmapsto f(x)(x))$

$(B \to C) \to ((A \to B) \to (A \to C))$: the composition functional $f \longmapsto (g \longmapsto g \circ f)$

(We have actually listed some of Hilbert's axioms for the implication.) This suggests a strong analogy between the notion of *implication* in logic and the notion of *exponentiation* in set theory. A schematic formula, such as

$$A \to ((A \to X) \to X) \tag{1}$$

being provable if, and only if, it has an uniform inhabitant. (In this case, the functional $x \longmapsto (f \longmapsto f(x))$ which transforms an element into a functional on functions.)

This analogy goes deeper. For instance, the *false* proposition $\perp$ naturally corresponds to the empty set $\emptyset$. The fact that we have $\perp \to A$ corresponds then to the fact that there is a map $\emptyset \to A$. In propositional logic, one can define $\neg A$ to be $A \to \perp$ and the formula (1) for $X = \perp$ corresponds to the provable formula $A \to \neg\neg A$.

What about the converse? This would be the formula $\neg\neg A \to A$. This is a possible way to state the law of Excluded Middle. Now the set $\neg\neg A$ is empty if A is empty, and has only one element if A is not empty. To have the law of Excluded Middle would require to have an uniform inhabitant of $\neg\neg A \to A$, which would mean an uniform way to pick an element from any non empty set. This is a strong form of the axiom of choice. (Hence, this simple analogy hints to connections between the axiom of choice and the law of Excluded Middle, that are indeed confirmed in the form of Diaconescu's Theorem, see below.)

2 Dependent Products and Sums

So far, we have only looked at *propositional* logic. What about universal and existential quantification?

In set theory, we have the notion of *family of sets* A_i indexed over a set I. We have also the notion of *dependent product* $(\Pi i \in I)A_i$ which is the set of all family $(a_i)_{i\in I}$ and *dependent sum* $(\Sigma i \in I)A_i$, which is the set of pairs (i,a) with $i \in I$ and $a \in A_i$. The dependent product corresponds to universal quantification (and actually the notation $(\Pi x \in I)\phi(x)$ can be found instead of $\forall x \in I.\phi(x)$ in early work on first-order logic) and dependent sum corresponds to existential quantification.

When the family A_i is constant, $A_i = A$ for all $i \in I$, it is natural to write $I \to A$ instead of $\Pi i \in I.A_i$ and $I \times A$ instead of $\Sigma i \in I.A_i$. With this notation in mind, the type

$$((\Pi x \in A)B(x)) \times ((\Pi x \in A)C(x)) \to (\Pi x \in A)(B(x) \times C(x))$$

corresponds to the first-order tautology

$$(\forall x.\phi(x)) \wedge (\forall x.\psi(x)) \to \forall x.\phi(x) \wedge \psi(x)$$

and it is inhabited uniformely by the function $(g,h) \longmapsto (x \longmapsto (g(x), h(x)))$.

If $f \in (\Pi x \in A)B(x)$ and $a \in A$ then we have $f(a) \in B(a)$.

If $z \in (\Sigma x \in A)B(x)$ then z is a pair and we write $p(z) \in A$ its first component and $q(z) \in B(p(z))$ its second component.

3 The lambda calculus notation

This notation was introduced by Church around 1930 for denoting functions: we write $\lambda x.x^2$ for instance for the square function $x \longmapsto x^2$. The following equality then holds: $\lambda x.x^2(3) = 3^2 = 9$. In general, we have

$$(\lambda x.t(x))(a) = t(a)$$

which corresponds to the fact that if f is the function $x \longmapsto t(x)$ we have $f(a) = t(a)$. Another convenient notation is to write $f\ (a_1, \ldots, a_n)$ instead of $(\ldots((f(a_1))(a_2))\ldots)(a_n)$. We write also $(A_1, \ldots, A_n) \to A$ instead of $A_1 \to (A_2 \to (\ldots \to A_n)\ldots)$. For instance $\lambda f.\lambda x.f(x,x)$ is the diagonalisation operation, of type $(A, A \to B), A \to B$.

For a given family of type $B(x)$ over a type A, if we have $t(x)$ element of $B(x)$ for $x \in A$ then $\lambda x.t(x)$ is an element of $(\Pi x \in A)B(x)$.

The corresponding rule for dependent sum is: if $a \in A$ and $b \in B(a)$ then (a,b) is an element of $(\Sigma x \in A)B(x)$.

4 Context

Type Theory brings elegant new notations and new notions. One of them is the notion of *context*. This is a list of the form

$$x_1 \in A_1, x_2 \in A_2(x_1), \ldots, x_n \in A_n(x_1, \ldots, x_{n-1})$$

where A_1 is a type, $A_2(x_1)$ is a family of type over the type A_1, $A_3(x_1, x_2)$ is a family of types depending on $x_1 \in A_1$ and $x_2 \in A_2(x_1)$ and so on.

There is a natural notion of *vectors* $(a_1, a_2, \ldots, a_n)$ fitting such a context: we ask $a_1 \in A_1, a_2 \in A_2(a_1), \ldots, a_n \in A_n(a_1, \ldots, a_{n-1})$.

This can be seen as the formal representation of a notion of contexts of mathematical hypothesese. For instance, consider a statement of a proposition, starting with: "let x be a natural number, assume that $\phi(x)$ holds for x, and let y be a rational number, ...". This would be represented by the context $x \in N,\ h \in B(x),\ y \in N$ where N is the type of natural number, $B(x)$ the type representing the proposition $\phi(x)$, ... Compared to the usual mathematical notation, notice that we have an explicit name for the hypothesis that $\phi(x)$ holds.

5 The Intensional Axiom of Choice

Bishop made the following remark: "A choice function exists in constructive mathematics, because a choice is *implied by the very meaning of existence*" (more precisely, in his own terminology, he should have said "A choice *operation*"). This can be elegantly "explained" using type theoretic notions: we suppose given two sets A and B and a relation R between elements of A and elements of B. In Type Theory, this is represented as a dependent type $R(x, y),\ x \in A,\ y \in B$. The type theoretic formulation of the intensional axiom of choice is then

$$((\Pi x \in A)(\Sigma y \in B)R(x, y)) \rightarrow (\Sigma f \in A \rightarrow B)(\Pi x \in A)R(x, f(x)) \tag{2}$$

It is quite easy indeed to build an element of this type: take $\lambda F.(\lambda x.p(F(x)), \lambda x.q(F(x)))$. If $F \in (\Pi x \in A)(\Sigma y \in B)R(x, y)$ we have $F(x) \in (\Sigma y \in B)R(x, y)$ and so $p(F(x)) \in B$ and $q(F(x)) \in R(x, p(F(x)))$. We deduce that we have $f = \lambda x.p(F(x)) \in A \rightarrow B$ but also $\lambda x.q(F(x)) \in \Pi x \in A.R(x, f(x))$. Notice that we have used the equality

$$R(x, f(x)) = R(x, (\lambda x.p(F(x)))(x)) = R(x, p(F(x)))$$

6 Set theory and type theory

It is possible to represent Bishop set theory in a most natural way in type theory, where a Bishop set is represented as a type with an equivalence relation and *type theoretic functions* give a direct formal representation of Bishop's notion of *operations*.

Notice that the type theoretic formulation of the axiom of choice (2) does not correspond really to the set-theoretic form of the axiom of choice (hence the name "intensional"). If R_A is an equivalence relation on A and R_B an equivalence relation on B, the set-theoretic axiom of choice requires, given $(\Pi x \in A)(\Sigma y \in B)R(x, y)$ to build $f \in A \rightarrow B$ such that, not only $(\Pi x \in A)R(x, f(x))$ holds (is inhabited), but also

$$(\Pi x_1 \in A)(\Pi x_2 \in A)\ (R_A(x_1, x_2) \rightarrow R_B(f(x_1), f(x_2)))$$

Contrary to the form (2), there is no direct way to justify this axiom, which can be called the *extensional* axiom of choice.

This phenomenum is well-know in constructive mathematics. Let us consider the case where $A = B = N \rightarrow N_2$ where N is the type of natural numbers, corresponding to the set $\mathbb{N}$ and N_2 is a type with two elements, corresponding to the set $\{0, 1\}$. Any element of $N \rightarrow N_2$ defines a real number in $[0, 1]$. It is possible in type theory to define R_A such that $R_A(f_1, f_2)$ is inhabited iff f_1, f_2 define the same real. We consider also R_B such that $R_B(f_1, f_2)$ is inhabited iff f_1, f_2define the same function. We clearly have $(\Pi f \in A)(\Sigma g \in B)R_B(f, g)$. The extensional axiom of choice will require to have $F \in A \rightarrow B$ such that, not only $R_B(f, F(f))$ but also

$$(\Pi f_1 \in A)(\Pi f_2 \in A)\ (R_A(f_1, f_2) \rightarrow R_B(F(f_1), F(f_2)))$$

There is no computable such functional, and hence we cannot expect to have such a functional in Type Theory. More reflection on the example would indicate that the extensional axiom of choice implies Bishop's principle of omniscience. It can be shown that the extensional axiom of choice implies the law of Excluded-Middle $(\neg\neg A) \rightarrow A$.

If extensional choice is not validated in type theory, some weak form of the axiom of choice in set theory, namely the axiom of countable choices and dependent choices, are however validated when one represents Bishop set theory in type theory.

Conclusion

Type Theory is an attempt to build a foundation of mathematics where logic and basic set theoretical notions are developped simulatenously and where the analogy between implication and exponentiation is taken as a guiding principle.

Working in Type Theory is similar to working in set theory, but where all operations have a direct computational meaning [2]. It can be thought as a language similar to the ones used in computer algebra system, but with a way to formulate properties (and proofs) of algorithms, and with a rich collection of types (for instance one has a type of groups of order n, a type of ordered lists, ...)

It is interesting to build category theory directly in type theory, and some current works explore such a representation of higher-order category theory. In such works, it is actually more appropriate to think of *types* as a very general formal representation of *topological spaces* (and a family of types should be thought of as a *continuous family* of spaces over a given space).

References

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[2] P. Martin-Löf. Constructive Mathematics and Computer Programming. In *Logic, Methodology and Philosophy of Science VI*, p. 153-175. Amsterdam: North-Holland 1982.

[3] P. Martin-Löf. 100 years of Zermelo's axiom of choice: what was the problem with it? to appear, 2007.