MANAGING WITHOUT POINTS

(*a draft*)

ALEŠ PULTR
Department of Applied Mathematics
and
Institute of Theoretical Computer Science (ITI),
Charles University, Prague

Contents:

Introduction

The intuition behind pointfree topology is very natural. One simply prefers thinking of a space as constituted by "places", "spots" of nontrivial extent. The points are abstractions of limits of diminishing places and one typically forgets about them, if possible (and sometimes they in fact do not exist). If you view, say, $\sqrt{2}$ as "1.414-and-I-don't-know-what-comes-next", or take the result of a measurement, say, "5.3" as "5.3 $\pm$ error in some tolerance" you think in a point-free way.

Being not a specialist in history of mathematics I can only briefly outline the development in the field, hoping that my omitting, by ignorance, important milestones will be pardoned.

Modern topology originates, in principle, from Hausdorff's "Mengenlehre" [15] in 1914 (one year earlier there was a paper by Caratheodory [10] containing the idea of a point as an entity localized by a special system of diminishing sets; this is also of relevance for the modern point-free thinking). In the twenties and thirties the importance of (the lattice of) open sets became gradually more and more apparent (see e.g. Alexandroff [1] or Sierpinski [41]). In [43] and [44] (1934, 1936), M.H. Stone presented his famous duality theorem from which it followed that compact zero-dimensional spaces and continuous maps are well represented by the Boolean algebras of closed open sets and lattice homomorphisms. Although zero-dimensional spaces are rather special, and not very geometric, this was certainly an encouragement for those who endeavoured to treat topology other than as a structure on a given system of points (Wallman 1938 [48], Menger 1940 [34], McKinsey and Tarski 1944 [33]). In the Ehresmann seminar in the late fifties ([13], [8]), we encounter frame theory already in the form we know today (it should be noted that almost at the same time, independently, there appeared two important papers – Bruns [9], Thron [45] – on homeomorphism of spaces with isomorphic lattices of open sets, under weak separation axioms). After that, many authors got interested (C.H. Dowker, D. Papert (Strauss), J. Isbell, B. Banaschewski, etc.) and the field started to develop rapidly. The pioneering paper by J. Isbell [17], which opened several topics, merits particular mentioning. In 1983, P.T. Johnstone published his monograph [22] which is still a primary source of reference (also, I can warmly recommend his excellent surweys in [23] and [24]). Since the mid eighties, intensive research has been done in enriched point-free structures such as uniform and nearness frames, or metric frames. It should be noted that this also has its origins in [22]).

Pointfree spaces (frames, locales) are obtained as an abstraction of the properties of lattices of open sets of classical spaces. This way we have , in essence, more spaces than we had before (not quite: some badly behaved classical spaces cannot be treated this way, but this is not very important at the moment). An extension and generalization of a time-tested concept calls for a justification. We cannot ignore the questions that naturally arise:

(Q1) Is the broader range of "spaces" desirable at all?

(Q2) When abandoning points, do we not lose too much information?

(Q3) Is the theory in this context, in whatever sense, more satisfactory?

(Q4) Is it not so that the new techniques obscure the geometric contents?

The following text, besides trying to present some of the basic facts, is intended as an apologetics of the field. That is, I would like to answer (hopefully to satisfaction) the questions above.

In Section 2 the basic concepts are defined and a reconstruction of points is described and discussed in some detail. Thus the question Q2 is answered.

Section 3 is devoted to separation axioms (regularity, etc.) I try to illustrate the parallels of classical and point-free thinking. The reader will get some idea of the basic techniques, too; perhaps it will be apparent that although the intuition is somewhat different, the reasoning is transparent and sometimes very simple. Thus we will have something in the direction of Q4. The detailed proofs, also elsewhere, illustrate the simplicity of the point-free reasoning (albeit, admittedly, sometimes traded for loss of intuition).

Generalized subspaces, discussed in Section 4, represent a part of point-free reasoning with features somewhat different from the classical theory. It turns out that even a classical space can have subspaces that are not classical (for instance, rationals and irrationals in the locale of reals still meet in a dense subspace). The reader may or may not see it as an indication that the exra spaces are of interest. If not, never mind, more arguments will come later.

The next section concerns compactness. On the one hand we see that the concept is particularly easy to be treated without points, and there are facts parallel to the classical ones. On the other hand we present a variant of Stone-Čech compactification (from [5]). Here, first, the reader will appreciate that the construction is much simpler. Second, surprisingly, unlike in classical spaces, this compactification is fully constructive. This is a point in the direction of Q3 (it should be noted that point-free reasoning is often constructive where the classical is not).

In Section 6 we discuss local compactness, show its relation with continuous lattices, and present the Hofmann-Lawson duality ([16]). This last (heavily dependent on choice principles) show that there is a very important part of topology where we have the points granted.

In the last section we present several examples of results that do not hold classically, and are more satisfactory than their classical counterparts. This should answer Q1.

1. Preliminaries

1.1. As usual, for a subset M or an element x of a partially ordered set $(X, \leq)$ we write

$$\downarrow M = \{x \mid x \leq m \in M\},\ \downarrow x = \downarrow\{x\},\ \uparrow M = \{x \mid x \geq m \in M\},\ \uparrow x = \uparrow\{x\}.$$

1.2. Recall that monotone maps $f : (X, \leq) \to (Y \leq)$, $g : (Y, \leq) \to (X \leq)$ are (Galois) adjoint (f on the left and g on the right) if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y)$$

and that this is equivalent to $fg(y) \leq y$ and $x \leq gf(x)$.

Also recall the well-known fact that the left (resp. right) adjoints preserve suprema (resp. infima) and if the posets in question are complete lattices, each map preserving all suprema is a left adjoint and similarly the maps preserving infima are right adjoints.

1.3. If $(X, \leq)$, $(Y, \leq)$ are lattices then *lattice homomorphisms* $f : X \to Y$ preserve suprema and infima of couples of elements. *Complete lattice homomorphisms* between complete lattices preserve all suprema and all infima.

1.4. A *Heyting algebra* is a lattice with an extra operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

(hence each map $- \wedge b$ is a Galois adjoint).

A pseudocomplement of an element a in a lattice L is an $a^* \in L$ such that

$$x \wedge a = 0 \quad \text{iff} \quad x \leq a^*.$$

Thus, in a Heyting algebra every element has a pseudocomplement, namely $a^* = a \to 0$.

1.5. Only basics from category theory are assumed.

The reader should know that an adjoint situation

$$\varepsilon_{AB} : \mathcal{B}(L(A), B) \cong \mathcal{A}(A, R(B))$$

can be equivalently described by transformations $\lambda : LR \to \mathrm{Id}$, $\rho : \mathrm{Id} \to RL$ such that all the $\lambda_{L(A)} \cdot L(\rho_A)$ and $R(\lambda_B) \cdot \rho_{R(B)}$ are identities.To understand a remark in Section 5 it is helpful to realize that the fact that a reflective category inherits all the limits (in particular, products) *holds without any non-constructive principles.*

1.6. The reader wishing for more information on posets can consult [11], for category theory one can use any standard monograph, for instance [30]; more information about frames can be found in [22], [23], [24], [47], [38] or [37].

2. Frames, locales, and how to reconstruct points if we are so minded. Spectrum

2.1. A *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\} \qquad (*)$$

for each subset $A \subseteq L$ and $b \in L$.

A *frame homomorphism* $h : L \to M$ is a mapping preserving all suprema and all finite infima. The category of frames and frame homomorphisms will be denoted by

Frm.

Note that the distributivity law $(*)$ is precisely the same as stating that each of the maps $- \wedge b = (x \mapsto x \wedge b)$ preserves all suprema. Thus (recall 1.2 and 1.4), each $- \wedge b$ is a left adjoint and the respective right adjoints constitute a Heyting structure

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

on L. Hence, in a way, frames and complete Heyting algebras are the same. But this concerns the intrinsic structure only; frame homomorphisms and complete Heyting homomorphisms differ. It may be of interest that the latter also has a topological interpretation, modelling the open continuous maps ([26]).

2.2. Consider a topological space X with the topology given by the system of open sets which we will denote by $\Omega(X)$. This $\Omega(X)$ is obviously a frame, and if $f : X \to Y$ is a continuous map we have a frame homomorphism $\Omega(f) = (U \mapsto f^{-1}(U)) : \Omega(Y) \to \Omega(X)$.

> **Note.** The joins in $\Omega(X)$ coincide with the unions, and *finite* meets coincide with (finite) intersections. This results in the required distributivity and in $\Omega(f)$ being really frame homomorphisms. Note that due to the different nature of the infinite meets, the distributivity dual to $(*)$ typically does not hold, and the $\Omega(f)$ typically does not preserve them.

Thus we have a contravariant functor

$$\Omega : \mathbf{Top} \to \mathbf{Frm}.$$

We will see soon that for a substantial class of topological spaces this Ω constitutes an isomorphism with a full subcategory of the opposite (dual) of **Frm**. This dual category can then be viewed as an extension, that is, a class of generalized spaces. It is called the category of *locales* and usually denoted as **Loc**, and we have a (covariant) functor

$$\Omega : \mathbf{Top} \to \mathbf{Loc}.$$

We deal with frames resp. locales mostly as follows: when computing or proving facts we will adopt the frame (algebraic) point of view) while in interpreting the facts – for instance when dealing with "subspaces" – we often think as in **Loc**.

A frame L isomorphic to an $\Omega(X)$ is said to be *spatial*.

> **Note.** A complete Boolean algebras is spatial only if it is atomic (in other words, spatial locales and Boolean algebras intersect precisely in discrete spaces). Thus, a simple example of a non-spatial frame is any non-atomic Boolean algebra.

2.3. Given a spatial frame $L \cong \Omega(X)$ the eminent question to be answered is whether one can reconstruct th space X, and given a frame homomorphism $h : \Omega(Y) \to \Omega(X)$ we want to know whether it determines a continuous $f : X \to Y$ such that $h = \Omega(f)$. Obviously a restriction of the class of spaces is necessary: for instance the two element (indicrete) topology $\{\emptyset, X\}$ certainly does not determine the underlying set X; thus, we see that, at least, we have to assume that the spaces satisfy the separation axiom T_0.

The T_0 alone does not suffice, but we do not need much more. We will present a very natural approach involving sobriety.

In every space, the open sets of the form $X \smallsetminus \overline{\{x\}}$ behave with respect to the "multiplication" $\cap$ as primes. That is, they are distinct from the unit, and if $X \smallsetminus \overline{\{x\}} = U \cap V$ then either $U = X \smallsetminus \overline{\{x\}}$ or $V = X \smallsetminus \overline{\{x\}}$; instead of *prime* one usualy speaks of *meet irreducible* elements. A space is *sober* if it is T_0 and if there are no other meet irreducibles than the $X \smallsetminus \overline{\{x\}}$.

It is an easy exercise to show that for instance each Hausdorff space is sober. But a sober space does not have to be even T_1: for instance every finite T_0 space is sober (on the other hand, T_1 does not imply sobriety either).

2.3.1. Proposition. *Let Y be a sober space and X a general one. Then for each frame homomorphism $h : \Omega(Y) \to \Omega(X)$ there is exactly one continuous map $f : X \to Y$ such that $h = \Omega(f)$.*

Proof. The uniqueness immediately follows from Y being T_0. Now let $h : \Omega(Y) \to \Omega(X)$ be a frame homomorphism. For $x \in X$ set

$$\mathcal{F}_x = \{U \in \Omega(Y) \mid x \notin h(U)\} \quad \text{and} \quad F_x = \bigcup \mathcal{F}_x.$$

Since joins are preserved we have $x \notin h(F_x)$ and hence, for $U \in \Omega(Y)$,

$$x \notin h(U) \quad \text{iff} \quad U \subseteq F_x. \tag{$*$}$$

F_x is meet ireducible: Indeed, since $x \notin h(F_x)$, $F_x \neq X$; if $F_x = U \cap V$ we have $x \notin h(U) \cap h(V)$ and hence, say $x \notin h(U)$ and $U \subseteq F_x$. Thus, by sobriety, $F_x = Y \smallsetminus \overline{\{y\}}$ for a unique $y \in Y$ and if we chose such y for $f(x)$ we can rewrite $(*)$ to

$$x \notin h(U) \;\text{ iff }\; U \subseteq Y \smallsetminus \overline{\{y\}} \;\; (\text{iff } \; f(x) \notin U, \;\; \text{since } \; U \text{ is open})$$

and hence

$$x \in h(U) \;\text{ iff }\; f(x) \in U, \;\; \text{that is, } \; x \in f^{-1}[U].$$

Thus, $f^{-1}[U] = h(U) \in \Omega(X)$ and hence f is continuous and $h = \Omega(f)$. □

Now we can reconstruct a sober space X as follows:
Denote by P the one point space $\{\cdot\}$. The $x \in X$ are in the natural one-one correspondence with the (continuous) maps $f_x = (\cdot \mapsto x) : P \to X$, and hence with the frame homomorphisms

$$h : \Omega(X) \to \Omega(P) \cong \mathbf{2},$$

($\mathbf{2}$ is the two-element Boolean algebra $\{0, 1\}$). An element x belongs to an open set U iff $h(U) = f_x^{-1}[U] \neq \emptyset$. Thus, X is homeomorphic with

$(\{h \mid h : \Omega(X) \to \mathbf{2}\}, \{\widetilde{U} = \{h \mid h(U) = 1\} \mid U \in \Omega(X)\})$. Furthermore, we see that the restriction of Ω to the subcategory of sober spaces is a full embedding, and locales can be viewed as an extension of sober spaces.

2.4. Spectrum. The reconstruction above leads to the following definitions:

A *point* of a frame L is a frame homomorphism $h : L \to \mathbf{2}$. Denote by ΣL the set of all points of L. For $a \in L$ set $\Sigma_a = \{h : L \to \mathbf{2} \mid h(a) = 1\}$.

Observation $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$, $\Sigma_0 = \emptyset$, $\Sigma_1 = \Sigma L$ *and* $\Sigma_{\vee a_i} = \bigcup \Sigma_{a_i}$.

Consequently $\{\Sigma_a \mid a \in L\}$ is a topology on ΣL. From now on, ΣL will be always considered as thus obtained space, and called the *spectrum* of L.

For a frame homomorphism $h : L \to M$ consider the mapping

$$\Sigma h : \Sigma M \to \Sigma L$$

defined by $(\Sigma h)(\alpha) = \alpha h$.

Lemma. *For each $a \in L$,*

$$(\Sigma h)^{-1}(\Sigma_a) = \Sigma_[h(a)]. \qquad (2.4.1)$$

Proof. Indeed, $(\Sigma h)(\alpha) \in \Sigma_a$ iff $(\Sigma h)(\alpha))(a) = \alpha(h(a)) = 1$. $\square$

Corollary. *$\Sigma h : \Sigma M \to \Sigma L$ is a continuous mapping and we have obtained a (contravariant) functor*

$$\Sigma : \mathbf{Frm} \to \mathbf{Top}.$$

2.4.1. Proposition. *Each ΣL is a sober space.*

Proof. Use the standard representation of ΣL. We have

$$\alpha \in \overline{\{\beta\}} \quad \text{iff} \quad \alpha \le \beta$$

(indeed, the first formula says that if $\alpha \in \Sigma_a$, that is, $\alpha(a) = 1$ then $\beta \in \Sigma_a$, that is, $\beta(a) = 1$).

Let Σ_a be a meet irreducible open set in ΣL. Set $b = \bigvee\{c \mid \Sigma_c \subseteq \Sigma_a\}$; hence in particular $\Sigma_b = \Sigma_a$. If $x \wedge y \le b$ then $\Sigma_x \cap \Sigma_y \subseteq \Sigma_b = \Sigma_a$ and hence, say, $\Sigma_x \subseteq \Sigma_b$ so that $x \le b$. Thus, b is meet irreducible and the α defined by $\alpha(x) = 0$ iff $x \le b$ is easily seen to be a frame homomorphism. We have

$$\beta \notin \overline{\{\alpha\}} \text{ iff } (\exists c, \beta(c) = 1 \text{ and } \alpha(c) = 0) \text{ iff}$$
$$(\exists c, \beta(c) = 1 \text{ and } c \le b) \text{ iff } \beta(b) = 1 \text{ iff } \beta \in \Sigma_b = \Sigma_a.$$

Thus, $\Sigma_a = \Sigma L \smallsetminus \overline{\{\alpha\}}$. $\square$

2.5. Spectrum is adjoint to Ω.

Theorem. $\Sigma : \mathbf{Loc} \to \mathbf{Top}$ *is a right adjoint to* $\Omega : \mathbf{Top} \to \mathbf{Loc}$.

Proof. The covariant ("localic") formulation enables us to say which of the functors is to be viewed as the right adjoint and which one as the left adjoint. The proof will be done, however, using the algebraic (frame) reasoning.

For a topological space X define

$$\eta_X : X \to \Sigma\Omega X$$

by setting $\eta_X(x)(U) = 1$ iff $x \in U$ (checking that each $\eta_X(x)$ is a frame homomorphism is straightforward, and we have

$$\eta_X^{-1}(\Sigma_U) = \{x \mid \eta_X(x) \in \Sigma_U\} = U \;) \tag{2.5.1}$$

so that each η_X is continuous.

For a frame L define

$$\varepsilon_L : L \to \Omega\Sigma L \quad (\Omega\Sigma L \to L \text{ in } \mathbf{Loc})$$

by setting $\varepsilon_L(a) = \Sigma_a$ (by the Observation in 2.4 it is a frame homomorphism).

If $f : X \to Y$ is a continuous map ($h : L \to M$ a frame homomorphism), we have $(\Sigma\Omega f(\eta_X(x)))(U) = \eta_X(x)(\Omega f(U)) = \eta_X(x)(f^{-1}(U)) = 1$ iff $x \in f^{-1}[U]$ iff $f(x) \in U$ iff $\eta_Y(f(x))(U) = 1$ ($\Omega\Sigma h(\varepsilon_L(a) = (\Sigma h)^{-1}[\Sigma_a] = \Sigma_{h(a)} = \varepsilon_M(h(a))$, by (2.4.1)) so that we have natural transformations $\eta : \mathrm{Id} \to \Sigma\Omega$ and $\varepsilon : \mathrm{Id} \to \Omega\Sigma$. These transformations are adjunction units: We have

$$(\Sigma\varepsilon_L(\eta_{\Sigma L}(\alpha)))(U) = \eta_{\Sigma L}(\alpha)(\varepsilon_L(U))) = 1 \text{ iff } \alpha \in \Sigma_U \text{ iff } \alpha(U) = 1,$$

hence $\Sigma\varepsilon_L \cdot \eta_{\Sigma L} = \mathrm{id}$, and, by (2.5.1)

$$\Omega(\eta_X)(\varepsilon_{\Omega X}(U)) = \eta_X^{-1}(\Sigma U) = U. \quad \square \tag{2.5.2}$$

2.5. The units as spatiality and sobriety criteria.

Proposition. *L is spatial iff ε_L is an isomorphism.*

Proof. The implication $\Leftarrow$ is trivial.

$\Rightarrow$: Let $h : L \to \Omega(X)$ be an isomorphism. Then $\varepsilon_L = (\Omega\Sigma h)^{-1} \cdot \varepsilon_{\Omega X} \cdot h$, and $\varepsilon_{\Omega X}$ is one-one since $\Omega\eta_X \cdot \varepsilon_{\Omega X} = \mathrm{id}$. $\square$

Thus,

> *if ε_L is not an isomorphism, L is isomorphic to no $\Omega(X)$ whatsoever.*

Proposition. *X is sober iff η_X is a homeomorphism.*

Proof. $\Rightarrow$ by 2.3.1, η_X is invertible and by (2.5.1) $\eta_X[U] = \eta_X[\eta_X^{-1}(\Sigma_U)] = \Sigma_U$ and this map is also open. $\Leftarrow$ follows from 2.4.1. $\square$

3. Separation axioms. Some parallels with the classical case

In classical topology one typically does not need quite general spaces. One of the ways to reduce the generality is provided by the separation axioms. Since their standard formulations use points (with the exception of normality which, on the other hand is not very satisfactory anyway) one can think, on the first sight, that in the point-free context we will not have much in this department.

But this is not the case. We have regularity, complete regularity and normality precisely corresponding to the classical homonymous properties. T_0 is irrelevant, and instead of T_1 we can harness the weaker, but very satisfactory sibfitness. Only the Hausdorff property is really hard to fit into the picture (see 3.7 below).

3.1. Regularity. In classical spaces, regularity can be formulated by stating that

$$\forall \text{ open } U, \quad U = \bigcup\{V \text{ open} \mid \overline{V} \subseteq U\}.$$

The relation $\overline{V} \subseteq U$ can be extended to the general pointfree case by defining

$$x \prec y \quad \text{if} \quad x^* \vee y = 1$$

(x^* is the pseudocomplement, recall 1.4) or, which is the same,

$$x \prec y \quad \text{if} \quad \exists y,\ x \wedge y = 0,\ y \vee y = 1$$

(note that the pseudocomplement of V in $\Omega(X)$ is, obviously, $V^* = X \smallsetminus \overline{V}$).

The following easy observation will be useful later

3.1.1. Lemma. *If $a_i \prec b_i$ for $i = 1, 2$ then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.*
(Indeed, we have $(a_1 \vee a_2)^* \vee (b_1 \vee b_2) = (a_1^* \wedge a_2^*) \vee (b_1 \vee b_2) \geq (a_1^* \vee b_1) \wedge (a_2^* \vee b_2)$ and $(a_1 \wedge a_2)^* \vee (b_1 \wedge b_2) \geq (a_1^* \vee a_2^*) \vee (b_1 \wedge b_2) \geq (a_1^* \vee b_1) \wedge (a_2^* \vee b_2)$.)

A frame L is said to be *regular* if

$$\text{for each } \ a \in L, \quad a = \bigvee \{b \mid b \prec a\}.$$

and we have that

a space X is regular in the standard sense iff the frame $\Omega(X)$ is regular.

As an illustration of working with the concept let us prove an analogon of the classical fact that if two continuous maps between Hausdorff spaces coincide on a dense subset they are equal (we will use the stronger regularity instead of the Hausdorff property).

A frame homomorphism h is said to be *dense* if $h(x) = 0$ implies $x = 0$ (this corresponds to the classical notion: for a continuous map f, $h^{-1}[U] = \emptyset$ only for $U = \emptyset$). We have

3.2.1. Proposition. *Let L be a regular frame, let $f, g : L \to M$ be homomorphisms, and let $h : M \to N$ be dense. If $hf = hg$ then $f = g$.*

Proof. Let $x \prec a$ in L. Thus there is a y such that $x \wedge y = 0$ and $y \vee a = 1$. We have $h(g(x) \wedge f(y)) = h(f(x) \wedge f(y)) = h(f(x \wedge y)) = 0$ and hence $g(x) \wedge f(y) = 0$. Since $f(y) \vee f(a) = 1$ we have now $g(x) \wedge f(x) = g(x)$ and $g(x) \leq f(a)$. Thus,

$$g(a) = g(\bigvee \{x \mid x \prec a\}) \leq g(a),$$

and by symmetry also $f(a) \leq g(a)$. $\square$

Thus

in the category of regular frames the dense homomorphisms are monomorphic.

3.3. Complete regularity. Write

$$x \prec\!\!\prec y$$

if there are $x_r \in L$ for r dyadic rationals in the interval $[\![0,1]\!]$ such that

$$x_0 = x,\ x_1 = y \ \text{ and } \ x_r \prec x_s \ \text{ for } \ r < s.$$

(Call a relation R *interpolative* if

$$aRb \quad \Rightarrow \quad \exists c,\ aRcRb.$$

Obviously, $\prec\!\!\prec$ *is the largest interpolative* $R \subseteq \prec$.)

A frame L is said to be *completely regular* if

$$\text{for each } \ a \in L, \quad a = \bigvee\{b \mid b \prec\!\!\prec a\}. \tag{1}$$

We have, again, that

a space X is completely regular iff the frame $\Omega(X)$ is completely regular.

(The proof is not quite so straightforward as in 8.2.1. For the implication $\Leftarrow$ one has to construct suitable real functions; this can be done by a procedure imitating the standard proof of the well-known Urysohn lemma of classical topology.)

3.4. Normality. This is perhaps the most immediate translation of a classical separation notion: we say that a frame L is *normal* if

> whenever $a \vee b = 1$ for $a, b \in L$, there exist $u, v \in L$ such that $u \wedge v = 0$, $u \vee b = 1$ and $a \vee v = 1$.

Trivially,

space X is normal iff the frame $\Omega(X)$ is normal.

3.5. Subfitness. A frame L is said to be *subfit* ([17], *conjunctive* in [42]) if

$$a \nleq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \neq b \vee c. \tag{Sfit}$$

Note. In classical setting, subfitness is a separation axiom weaker than T_1. It combines to T_1 together with another separation axiom between T_0 and T_1, the T_D ([2],[9])

$$\forall x\ \exists U \text{ open, } x \in U \text{ such that } U \smallsetminus \{x\} \text{ is open.}$$

Note that

every regular frame is subfit.

(indeed, if $a \nleq b$ then there is an $x \prec a$, $x \nleq b$. Set $c = x^*$; then $a \vee c = 1$ and $b \vee c \neq 1$.)

An interesting feature of subfitness is that reflecting unit makes a homomorphism one-one.

Proposition. *Let L be subfit and let $h : L \to M$ be a homomorphism such that $h(a) = 1$ implies $a = 1$. Then h is one-one.*
(compare with 3.2.1)

Proof. Suppose $f(a) = f(b)$ and $a \vee c = 1$. Then $f(b \vee c) = f(a \vee c) = 1$, hence $b \vee c = 1$, and hence necessarily $b \leq a$. $\square$

3.6. The relation of regularity, complete regularity and normality.

Lemma. *The relation $\prec$ in a normal frame is interpolative.*

Proof. Let $a \prec b$ in a normal frame L. Then there are u, v such that $u \leq v^*$, $u \vee b = 1$ and $a^* \vee v = 1$. Thus, $a \prec v$ and $v^* \vee b \geq u \vee b = 1$ so that also $v \prec b$. $\square$

3.6.1. Proposition. *Let L be normal and subfit. Then it is completely regular.*

Proof. By Lemma, $\prec = \prec\!\!\prec$. Thus, it suffices to show that L is regular. For $a \in L$ set $b = \bigvee\{x \mid x \prec a\} = \bigvee\{x \mid x^* \vee a = 1\}$. Let $a \vee c = 1$. By normality we have a u such that $u \vee c = 1$ and $a \vee u^* = 1$. Thus, $u \leq b$ and $b \vee c = 1$. Hence $a \vee c = 1 \ \Rightarrow\ b \vee c = 1$ and by subfitness $a \leq b (\leq a)$. $\square$

3.6.2. Like in classical spaces, regular and Lindelöf implies normal (a cover of L is a subset $A \subseteq L$ such that $\bigvee A = 1$, and a frame is and Lindelöf if each cover contains an at most countable subcover).

Proposition. *Each regular Lindelöf frame is normal; consequently, it is completely regular.*

Proof. Let $a \vee b = 1$ in L. By regularity, $a = \bigvee\{x \mid x \prec a\}$ and $b = \bigvee\{y \mid y \prec b\}$. Thus, $\bigvee\{x \mid x \prec a\} \vee b = 1 = a \vee \bigvee\{y \mid y \prec b\}$ and by the Lindelöf property there are $x_1, x_2, \dots \prec a$, $y_1, y_2, \dots \prec b$ such that $\bigvee_{i=1}^{\infty} x_i \vee b = 1 = a \vee \bigvee_{i=1}^{\infty} y_i$. By 3.1.1 we can assume that

$$x_1 \le x_2 \le \cdots \quad \text{and} \quad y_1 \le y_2 \le \cdots$$

Set $u_i = x_i \wedge y_i^*$ and $v_i = y_i \wedge x_i^*$. Then we have $a \vee v_i = a \vee (x_i^* \wedge y_i) = (a \vee x_i^*) \wedge (a \vee y_i) = a \vee y_i$ and similarly $u_i \vee b = x_i \vee b$. Consequently, if we set $u = \bigvee u_i$ and $v = \bigvee v_i$ we have

$$a \vee v = \bigvee(a \vee y_i) = a \vee \bigvee y_i = 1 \quad \text{and} \quad u \vee b = 1.$$

Finally, $u_i \wedge v_j = x_i \wedge y_i^* \wedge x_j^* \wedge y_j = 0$ for any two i, j since if $i \le j$ then $x_i \wedge x_j^* = 0$ and if $i \ge j$ then $y_j \wedge y_i^* = 0$. Thus, $u \wedge v = \bigvee_{i,j} u_i \wedge v_j = 0$. □

3.7. Hausdorff type properties. Unlike in the separation principles above, in the Hausdorff case we have to do with substitutes. The first such requirement was suggested by Dowker and Strauss in [12]:

$$\begin{array}{l}\text{if } a \vee b = 1 \neq a, b \text{ then there are } u, v \\ \text{such that } u \not\le a,\ v \not\le b \text{ and } u \wedge b = 0.\end{array}$$

To make it hereditary at least for "open subspaces" it can be modified to

$$\begin{array}{l}\text{if } a \vee b = c \neq a, b \text{ then there are } u, v \le c \\ \text{such that } u \not\le a,\ v \not\le b \text{ and } u \wedge b = 0.\end{array}$$

Another requirement of this type, introduced by Isbell in [17] imitates the fact that in the classical case we can characterize Hausdorff spaces as those X for which the diagonal is closed in $X \times X$: a frame L is said to be Hausdorff if the codiagonal $\nabla : L \oplus L \to L$ is closed (we have to skip the definition of the coproduct $L \oplus L$).

None of the two definitions corresponds exactly to the classical property in the case of spatial frames (in the latter case because the coproducts $\Omega(X) \oplus \Omega(X)$ in frames do not always correspond to the products – that is, they are not necessarily isomorphic to $\Omega(X \times X)$, which seems to be unpleasant, but in fact has elsewhere nice consequences).

One has the implications

$$\text{regular} \quad \Rightarrow \quad \text{Isbell-Hausdorff} \quad \Rightarrow \quad \text{Dowker-Strauss-Hausdorff}.$$

Anyway, the subtitutes turned out to be fairly useful and it is not hard to reconcile ourselves with the lack of an exact equivalent.

4. The wonders of generalized subspaces

4.1. What is a natural definition of a sublocale, a "generalized subspace" of a "generalized space"? Think of some structure, say that of graph, or of spaces. A one-one structure preserving mapping does not represent well a subobject. Compare, say

an embedding of a subgraph vs. an embedding of an induced subgraph,

or

a one-one continuous map $f : X \to Y$ vs. an $f : X \to Y$ that restricts to a homeomorphism $f' : X \to f[X]$.

In categories, the one-one maps are (very roughly speaking) modelled by *monomorphisms*, that is, morphisms m such that $mf = mg$ implies $f = g$. There are various extra conditions making a monomorphism something like an embedding of a subobject. A very natural one is in the concept of an *extremal monomorphism*, a monomorphism m that cannot be decomposed as $m = m'e$ with e an epimorphism unless e is an isomorphism ("one cannot fit into the one-one map a non-trivial adjustment of the structure, the e").

Now since we have in mind such subspace embeddings in the category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$, we have to think of the *extremal epimorphisms* instead, namely of the epimorphisms e such that in each decomposition $e = me'$ with m monomorphic, m is an isomorphism). Now although general epimorphisms in **Frm** have a rather non-trivial structure, the *extremal epimorphisms*, luckily enough, are transparent: they are precisely the onto frame homomorphisms.

This leads to the following definition: a *sublocale* (more precisely, *sublocale map*) of L is an onto frame homomorphism $h : L \to M$.

> Let Y be a subspace of X. Then we have the sublocale $U \mapsto U|Y) : \Omega(X) \to \Omega(Y)$ of $\Omega(X)$. Note that, of course, $U \mapsto U|Y)$ is nothing else but the $\Omega(j)$ of the embedding map $j : Y \subseteq X$.

4.2. The sublocale map approach is, from the point of view of the motivation, a lucid representation of a subspace. It is not always easy to compute with, though. Therefore one uses, according to the needs of the moment, several others.

I. Congruences. The onto maps $h : L \to M$ can be, up to isomorphism, represented by the congruences

$$E_h = \{(x, y) \mid h(x) = h(y)\}$$

(having in mind the natural projection $p_E = (x \mapsto xE) : L \to L/E$). In a sense, it is a better representation (albeit not quite so intuitive): if there is an isomorphism i such that $ih = h'$, the sublocale maps $h : L \to M$ and $h' : L \to M$ represent the same subspace although they are not identical; the congruences E_h and $E_{h'}$ are.

Furthermore, one immediately sees that the sublocales thus represented constitute a complete lattice; let us denote the congruence lattice of L by

$$\mathbf{C}L.$$

One only has to keep in mind that the natural order of the sublocales is opposite to that of $\mathbf{C}L$: a bigger congruence represents a smaller subspace.

> **Note,** The complete lattice $\mathbf{C}L$ is a frame. The proof of the distributive law is not hard, it just needs some Heying computation.

II. Nuclei. This is a representation one often finds in the literature. A *nucleus* on a frame L is a mapping $\nu : L \to L$ such that

(N1) $a \leq \nu(a)$,

(N2) $\nu\nu(a) = \nu(a)$, and

(N3) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

The translation of congruences to nuclei and back is given by the formula

$$\begin{aligned} &E \mapsto \nu_E, \quad \nu_E(x) = \bigvee xE, \\ &\nu \mapsto E_\nu, \quad xE_\nu y \ \text{ iff } \ \nu(x) = \nu(y) \end{aligned}$$

(the reader can check the properties as an exercise).

III. Sublocale sets. A subset S of a frame L is said to be a *sublocale set* if

(S1) for each $A \subseteq S$, $\bigwedge A \in S$ (in particular, $1 = \bigwedge \emptyset \in S$), and

(S2) for each $a \in L$ and $b \in S$, $a \to b \in S$.

Nuclei can be translated to sublocale sets and back by the formulas

$$\nu \mapsto \nu[L]; \quad S \mapsto \nu_S, \ \nu_S(a) = \bigwedge \{s \mid s \in S, \ a \leq s\}.$$

This is a very handy representation; we will see some of its advantages in 4.4 and 4.5 below.

4.3. Open and closed sublocales. These are represented as follows. Let a be an element of L. We have the sublocale maps

$\widehat{a} = (x \mapsto a \wedge x) : L \to {\downarrow}a$, the open sublocales, and

$\breve{a} = (x \mapsto a \vee x) : L \to {\uparrow}a$, the closed sublocales.

the respective *open and closed congruences* are

$$\begin{aligned} \Delta_a &= \{(x,y) \mid x \wedge a = y \wedge a\}, \quad \text{and} \\ \nabla_a &= \{(x,y) \mid x \vee a = y \vee a\}. \end{aligned}$$

In the sublocale set representation we obtain the open resp. closed sublocale sets

$$\mathbf{o}(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\} \quad \text{resp.} \quad \mathbf{c}(a) = {\uparrow}a.$$

We have

4.3.1. Proposition. *1. ∇_0 is the minimal congruence and ∇_1 the maximal one; $\nabla_a \cap \nabla_b = \nabla_{a \wedge b}$ and $\bigvee_{i \in J} \nabla_{a_i} = \nabla_{\vee a_i}$. Thus, $\nabla = (a \mapsto \nabla_a) : L \to \mathbf{C}L$ is a frame homomorphism.*

2. ∇_a and Δ_a ($\breve{a}$ and $\widehat{a}$, $\mathbf{c}(a)$ and $\mathbf{o}(a)$) are complements in $\mathbf{C}L$.

Proof. 1. Set $a = \bigvee a_i$. Obviously $\nabla_{a_i} \subseteq \nabla_a$ for all i; if $\nabla_{a_i} \subseteq E$ for all i, we have $a_i E 0$ for all i and hence $a = (\bigvee a_i)E0$, and hence if $x \vee a = y \vee a$ we have $xE(x \vee a) = (y \vee a)Ey$ so that $\nabla_a \subseteq E$. If $x \vee a = y \vee a$ and $x \vee b = y \vee b$ then $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) = (y \vee a) \wedge (y \vee b) = y \vee (a \wedge b)$, and $x \vee (a \wedge b) = y \vee (a \wedge b)$ obviously implies $x \vee a = y \vee a$ and $x \vee b = y \vee b$.

2. If $(x,y) \in \Delta_a \cap \nabla_a$ then $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$. Hence $x = x \vee (x \wedge a) = x \vee (y \wedge a) = (x \vee y) \wedge (x \vee a) = (x \vee y) \wedge (y \vee a) = (x \wedge (y \vee a)) \vee y = x \wedge y$ and by symmetry also $y = x \wedge y$, so that $x = y$.

Now let $E \supseteq \Delta_a, \nabla_a$. Then, since $1\Delta_a a$ and $0\nabla_a a$, $1EaE0$, and $1E0$ makes $E = L \times L$. □

Each sublocale can be constructed from open and closed ones as follows.

4.3.2. Proposition. *For every $C \in \mathbf{C}L$ we have*

$$C = \bigvee \{\nabla_a \cap \Delta_b \mid aCb\}.$$

Proof. If aCb and $(x,y) \in \nabla_a \cap \Delta_b$ we have

$$x = x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a) C (x \wedge y) \vee (x \wedge b) =$$
$$= (x \wedge y) \vee (y \wedge b) = (x \vee b) \wedge y C (x \vee a) \wedge y = (y \vee a) \wedge y = y$$

so that $\nabla_a \cap \Delta_b \subseteq C$. On the other hand, let $E \supseteq \nabla_a \cap \Delta_b$ for all $(a,b) \in C$. Let $(a,b) \in C$. We have $(b, a \vee b) \in \nabla_a \cap \Delta_b \subseteq E$ and since also $(b,a) \in C$, we have $(a, a \vee b) \in E$ as well. Thus, $aE(a \vee b)Eb$, and $C \subseteq E$. $\square$

Note. By 3.1, $\nabla^L = (a \mapsto \nabla_a$ is a one-one homomorphism $L \to \mathbf{C}L$. From 3.2 we obtain

Corollary. *The embedding $\nabla^L : L \to \mathbf{C}L$ is an epimorphism.* (Indeed, if $f, g : \mathbf{C}L \to M$ coincide in the ∇_a's, they coincide, by complement, also in the Δ_a's, and hence by the Proposition above in every $C \in \mathbf{C}L$.)

This shows how weird the structure of epimorphisms in **Frm** is: these embeddings can be combined to

$$L \xrightarrow{\nabla^L} \mathbf{C}L \xrightarrow{\nabla^{\mathbf{C}L}} \mathbf{C}^2 L \longrightarrow \dots \longrightarrow \mathbf{C}^n L,$$

and also the transfinite step is easy. Thus, one has epimorphisms $\nabla^\alpha : L \to \mathbf{C}^\alpha L$ for all ordinals α. For some frames L the $\mathbf{C}^\alpha L$ never stop growing (in fact, in all the known cases it is either this, or the growth stops before the fourth step; whether it can be otherwise is an open problem). Thus, for such a fixed frame L one has epimorphisms $\varepsilon : L \to M$ with arbitrarily large M.

4.4. Closure. The closure of h (the smallest closed sublocale $\check{a}$ such that contains h) is easily seen to be obtained as the $\check{c}$ with $c = \bigvee\{x \mid h(x) = 0\}$.

A particularly handy representation of the closure is, however, obtained using sublocale sets. Obviously, $\overline{S} = \uparrow \bigwedge S$ is the least closed sublocale set containing S.

We have

Proposition. *We have $\overline{\{1\}} = \{1\}$, $\overline{\overline{S}} = \overline{S}$ and $\overline{S \vee T} = \overline{S} \vee \overline{T}$.*

Proof. The first two facts are trivial, and the third one very easy: Set $a = \bigwedge S$, $b = \bigwedge T$. Then $\overline{S} \vee \overline{T} = \uparrow a \vee \uparrow b = \{x \wedge y \mid x \geq a, y \geq b\} = \uparrow (a \wedge b) = \uparrow \bigwedge (S \vee T) = \overline{S \vee T}$. $\square$

4.5. Isbell's density theorem. A sublocale (more generally, a frame homomorphism) is said to be *dense* if $a \neq 0 \;\Rightarrow\; h(a) \neq 0$ (this agrees with the homonymous notion concerning spaces: $Y \subseteq X$ is dense iff $Y \cap U \neq \emptyset$ for each non-void open U in X). In the language of sublocale sets this translates to the condition that

$$S \textit{ is dense iff } 0 \in S$$

(indeed, the congruence class of 0 should be $\{0\}$, and hence $\nu(0) = 0$ for the corresponding nucleus). Thus

$$S \textit{ is dense iff } \overline{S} = L.$$

Now consider a dense sublocale set $S \subseteq L$. As $0 \in S$, we have, by (S2) $x^* = x \to 0$ in X for any $x \in L$. On the other hand, $B_L = \{x^* \mid x \in L\}$ is easily seen to be a sublocale set. Thus we have a somewhat surprising

Theorem. *Each frame has a smallest dense sublocale, namely B_L.*

(By the way, B_L is the well known Booleanization of the Heyting algebra L.)

> **Note.** One of the facts one sees from this last theorem is that a sublocale of a space is not necessarily a subspace. In fact, it is seldom the case that all the sublocales of a space are subspaces. By a result of Niefield and Rosenthal ([36]) this is the case only in the so called scattered spaces.
> Another phenomenon that can be observed is that typically not every sublocale has a complement. For more about the complementation in $\mathbf{C}L$ see Isbell, [19].

5. Compactness: similarities, and a rather important difference

5.1. As we have already defined above, a *cover* of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$, and a *subcover* B of A is a subset $B \subseteq A$ which is still a cover. These notions obviously correspond to those of open covers and subcovers in classical topology. Hence we also have the following immediate extensions of the classical notion:

A frame is said to be *compact* if each cover A of L has a finite subcover.

5.2. Proposition. *1. Each subframe of a compact frame is compact.*

2. Each closed sublocale $\uparrow c$ of a compact) frame is compact.

3. Each regular compact frame is normal. Consequently (recall 3.5. and 3.6) it is completely regular.

Proof. 1 is trivial (note that this statement is related to the classical fact that quotients of compact spaces are compact).

2 follows from the fact that if A is a cover of $\uparrow a$ then $\{a\} \cup A$ is a cover of L.

3: this follows from 3.6, but for a compact L it is much easier: If $a \vee b = 1$ we have $\bigvee\{x \mid x \prec a\} \vee b = 1$ and hence by compactness and 3.1.1 there is an $x \prec a$ (and hence $x^* \vee a = 1$) such that $x \vee b = 1$. $\square$

5.3. Two counterparts of statements on Hausdorff compact spaces.

5.3.1. Proposition. *Let L be regular and let M be compact. Then each dense $h : L \to M$ is one-one.*

Proof. In view of 3.5 it suffices to show that h is co-dense. Suppose $h(a) = 1$. Since $a = \bigvee\{x \mid x \prec a\}$, the set $\{h(x) \mid x \prec a\}$ is a cover of M and hence there are $x_1, \dots, x_n \prec a$ such that $\bigvee h(x_i) = 1$. By 3.1.1 $x = x_1 \vee \cdots \vee x_n \prec a$ and we have

$$h(x) = 1 \quad \text{and} \quad x^* \vee a = 1.$$

Since $h(x^*) \leq h(x)^* = 0$, $x^* = 0$ and finally $a = 1$. $\square$

5.3.2. Proposition. *Let L be regular and $h : L \to M$ a sublocale with compact M. Then h is closed.*

Proof. Use the closure in the form of the $\check{c} : L \to \uparrow c$ of h from 4.4, that is, $c = \bigvee\{x \mid h(x) = 0\}$. The homomorphism $g : \uparrow c \to M$ such that $g \cdot \check{c} = h$ is dense onto. Hence, by 5.3.1, it is an isomorphism. $\square$

5.4. Compactification. First we will present an easy construction that is not a compactification in the strict sense. It will give us, however, a good picture of what will happen next.

An *ideal* in a frame L is a non-void subset $J \subseteq L$ such that

(I1) $b \leq a \in J \;\Rightarrow\; b \in J$, and

(I2) $a, b \in J \;\Rightarrow\; a \vee b \in J$.

Denote by

$$\mathfrak{J}L$$

the set of all ideals in L ordered by inclusion.

5.4.1. Proposition. *$\mathfrak{J}L$ is a compact frame.*

Proof. Obviously, intersection of ideals is an ideal. For the supremum we have the formula

$$\bigvee J_i = \{\bigvee X \mid X \text{ finite}, \;\; X \subseteq \bigcup J_i\}$$

(obviously the set is an ideal, and each ideal J containing all J_i has to contain all the $\bigvee X$). If J_i, K are ideals and $x = x_1 \vee \cdots \vee x_n \in (\bigvee J_i) \cap K$, $x_j \in J_{i_j}$, then by (I1) $x_j \in J_{i_j} \cap K$ and $x \in \bigvee (J_i \cap K)$. The inclusion $\bigvee (J_i \cap K) \subseteq (\bigvee J_i) \cap K$ is trivial and hence $\mathfrak{J}L$ is a frame. Now let $\{J_i \mid i \in I\}$ be a cover. Thus, $1 \in L = \bigvee J_i$ and there are $x_j \in J_{i_j}$ such that $1 = x_1 \vee \cdots \vee x_n$. Then we have $1 \in \bigvee_{j=1}^n J_{i_j}$ and by (I1) $L = \bigvee_{j=1}^n J_{i_j}$. □

5.4.2. The homomorphisms v_L. Define a mapping $v_L : \mathfrak{J}L \to L$ by setting $v_L(J) = \bigvee J$.

Lemma. *v_L is a dense sublocale homomorphism.*

Proof. Define a mapping $\alpha : L \to \mathfrak{J}L$ by setting $\alpha(a) = {\downarrow} a$. Obviously, $v_L \alpha(a) = a \quad \text{and} \quad \alpha v(J) = {\downarrow}\bigvee J \supseteq J$. Thus, first, v is onto and, second, v is a left Galois adjoint and hence it preserves all suprema. We have $v(L) = 1$ and, by (I1),

$$\begin{aligned} v(J_1) \wedge v(J_2) = \bigvee \{x \wedge y \mid x \in J_1, y \in J_2\} \leq \\ \leq \bigvee \{z \mid z \in J_1 \cap J_2\} = v(J_1 \cap J_2) (\leq v(J_1) \wedge v(J_2)) \end{aligned}$$

so that v preserves finite meets.

Finally, if $v(J) = \bigvee J = 0$ then necessarily $J = \{0\}$, the bottom of $\mathfrak{J}L$ (ideals are non-void). □

5.4.3. For a frame homorphism $h : L \to M$ define $\mathfrak{J}h : \mathfrak{J}L \to \mathfrak{J}M$ by setting

$$\mathfrak{J}h(J) = {\downarrow} h[J].$$

By a trivial immediate checking we obtain

Proposition. *$\mathfrak{J}$ is a functor* **Frm** $\to$ **Frm** *and $v = (v_L)_L$ is a natural transformation $\mathfrak{J} \to$* Id.

5.4.4. The real thing: Stone-Čech compactification. Now, following Banaschewski and Mulvey, [5], we will obtain an extension of the Stone-Čech compactification for general completely regular locales by an easy modification of the previous construction.

An ideal $J \subseteq L$ is said to be *regular* if

(IR) for each $a \in J$ there is a $b \in J$ such that $a \prec\!\!\prec b$.

The set of all regular ideals in L will be denoted by

$$\mathfrak{R}L.$$

Lemma. *$\mathfrak{R}L$ is a subframe of $\mathfrak{J}L$. In particular, it is compact.*

Proof. Intersection of regular ideals is obviously regular. Now let J_i be regular and let $a \in \bigvee J_i$; then $a = x_1 \vee \cdots \vee x_n$ with some $x_j \in J_{i_j}$. There are $y_j \in J_{i_j}$ such that $x_j \prec\!\!\prec y_j$ and hence $b = y_1 \vee \cdots \vee y_n \in \bigvee J_i$, and $a \prec\!\!\prec b$ by 3.1.1. □

For an element a of a frame L set

$$\sigma(a) = \{x \mid x \prec\!\!\prec a\}.$$

Using the interpolativity of $\prec\!\!\prec$ we immediately obtain

Fact. *$\sigma(a)$ is a regular ideal.* □

Proposition. *$\mathfrak{R}L$ is a completely regular compact frame.*

Proof. By Lemma and by 5.2.3 it suffices to show that $\mathfrak{R}L$ is regular. Since, further, for a regular ideal J obviously $J = \bigcup\{\sigma(a) \mid a \in J\} = \bigvee\{\sigma(a) \mid a \in J\}$ it suffices to show that

$$b \prec\!\!\prec a \text{ in } L \quad \Rightarrow \quad \sigma(b) \prec \sigma(a) \text{ in } \mathfrak{R}L.$$

Interpolate $b \prec\!\!\prec x \prec\!\!\prec y \prec\!\!\prec a$. Since $\sigma(b^*) \cap \sigma(b) = \{0\}$ we have $\sigma(b^*) \subseteq \sigma(b)^*$, and since obviously if $b \prec\!\!\prec x$ then $x^* \prec\!\!\prec b^*$, $x^* \in \sigma(b^*) \subseteq \sigma(b)^*$. Thus, $1 = x^* \vee y \in \sigma(b)^* \vee \sigma(a)$ and hence $\sigma(b)^* \vee \sigma(a) = L$, the top of $\mathfrak{R}L$. □

Theorem. (Stone-Čech compactification) *Define $\mathfrak{R}h = \mathfrak{J}h$ for homomorphisms $h : L \to M$ and $v_L : \mathfrak{R}L \to L$ by $v_L(J) = \bigvee J$. These formulas yield a functor $\mathfrak{R}$:* **CRegFrm** $\to$ **CRegFrm** *and a natural transformation $v : \mathfrak{R} \to \mathrm{Id}$ such that*

(1) *each $\mathfrak{R}L$ is (regular and) compact,*

(2) *each* v_L *is a dense sublocale homomorphism, and*

(3) v_L *is an isomorphism iff* L *is compact.*

Proof. If L is completely regular we have $v_L\sigma(a) = a$ for each a, and obviously $\sigma(v_L(J)) \supseteq J$. Thus (again) v_L is a left Galois adjoint and hence it preserves suprema. Preserving finite meets is seen by the same procedure as in 5.4.2. Obviously v_L is dense.

If J is a regular ideal in L, $\mathfrak{J}h(L) = \downarrow h[L]$ is obvioucly regular as well.

Thus, the only statement left to be proved is that if L is compact then v_L is an isomorphism.

> (At this moment, the reader may wonder why we do not simply use 5.3.1. This would indeed yield the result, but we wish to have everything very explicitly constructive. Therefore we prefer to show directly that σ is the inverse of v_L.)

We already know that $J \subseteq \sigma v_L(J)$. Now let L be compact and let $x \in \sigma v_L(J)$. Then $x \prec \bigvee J$, hence $x^* \vee \bigvee J = 1$ and by compactness there are $a_1, \dots a_n \in J$ such that $x^* \vee a_1 \vee \dots \vee a_n = 1$. Then $a = a_1 \vee \dots \vee a_n \in J$, and $x \prec a$ and hence $x \in J$; thus also $\sigma v_L(J) \subseteq J$ and σ is the inverse of v_L.
□

5.5. Scrutinizing the proof you will find no use of choice or excluded middle. One feature of the pointfree approach is that results that in classical topology have to use such principles are fully constructive. See also 7.1 and 7.2 below.

6. Local compactness: meeting an old acquaintance

6.1. Continuous lattices. The relation "well below", usually denoted by

$$x \ll y,$$

is defined (typically in a complete lattice, but it makes sense more generally) by requiring that

$$\forall D \text{ directed in } L, \quad y \leq \bigvee D \ \Rightarrow\ \exists d \in D,\ x \leq d.$$

A lattice L is said to be *continuous* if

$$\forall a \in L, \quad a = \{x \mid x \ll a\}. \tag{$*$}$$

We immediately see that

$0 \ll a$ for each a, and if $x \leq a \ll b \leq y$ then $x \ll y$, and that
if $a_1, a_2 \ll b$ then $a_1 \vee a_2 \ll b$.

Consequently, the set $\{x \mid x \ll a\}$ is directed from which we readily infer

Proposition. *In a continuous lattice the relation $\ll$ interpolates.*

(Indeed, let $a \ll b$. We have the directed unions $b = \bigvee\{x \mid x \ll b\} = \bigvee\{\bigvee\{y \mid y \ll x\} \mid x \ll b\} = \bigvee\{y \mid \exists x,\ y \ll x \ll b\}$. and hence there is a y and an x such that $a \leq y \ll x \ll b$.)

6.2. Scott topology. Let L be a continuous lattice (this definition makes sense for any poset, though). A subset $U \subseteq L$ is *Scott-open* if $\uparrow U = U$ and if $U \cap D \neq \emptyset$ whenever D is directed and $\bigvee D \in U$. (Roughly speaking, Scott topology is the topology in which suprema of directed sets appear as limits.)

Note. In [40], Scott proved that continuous lattices L endowed with the Scott topology are precisely the injective topological spaces. The L in this statement are not assumed to satisfy any distributivity requirement. We will be interested in the special case of *continuous frames.* It may be of some interest that

for a continuous lattice to be a frame it suffices to be distributive.

Indeed, suppose $x \ll (\bigvee_{i \in J} a_i) \wedge b$. Then in particular $x \ll (\bigvee_{i \in J} a_i) = \bigvee\{\bigvee_{j=1}^{n} a_{i_j} \mid \{i_1, \ldots, i_n\} \subseteq J\}$ The second join is directed and hence we have some $\{i_1, \ldots, i_n\} \subseteq J$ such that $x \leq \bigvee_{j=1}^{n} a_{i_j}$ and hence $x \leq (\bigvee_{j=1}^{n} a_{i_j}) \wedge b = \bigvee_{j=1}^{n}(a_{i_j} \wedge b) \leq \bigvee(a_i \wedge b)$. Thus, $(\bigvee a_i) \wedge b \leq \bigvee(a_i \wedge b)$, and $(\bigvee a_i) \wedge b \geq \bigvee(a_i \wedge b)$ is trivial. $\square$

6.3. Locally compact spaces. If X is a locally compact space then $\Omega(X)$ is a continuous frames. Indeed, since every neighbourhood contains a compact neighbourhood we have for any open $U \subseteq X$ and every $x \in U$ an open V and a compact K such that

$$x \in V \subseteq K \subseteq U.$$

In this case then $V \ll U$ (if $\mathcal{D}$ is a directed system of open sets and $K \subseteq \bigcup \mathcal{D}$ then there is a $D \in \mathcal{D}$ such that $K \subseteq D$). Hence $U = \bigcup\{V \mid V \ll U\}$.

We will see that (up to isomorphism) each continuous frame is such an $\Omega(X)$. Thus, for locally compact spaces, in a sense, the pointfree and classical topologies coincide.

6.4. Convention It will be of advantage to modify the representation of points and spectra from Section 2. If $h : L \to \mathbf{2}$ is a (frame) homomorphism, we have a filter

$$F_h = \{x \mid h(x) = 1\}.$$

This filter is *complete*, that is,

$$\text{if } \bigvee_{i \in J} a_i \in F_h \text{ then } a_j \in F_j \text{ for some } j$$

which is much more than being *prime* (that is, such that $a \vee b \in F$ implies $a \in F$ or $b \in F$).

On the other hand, for each *complete filter* F we have the homomorphism h_F defined by

$$h_F(x) = 1 \quad \text{iff} \quad x \in F,$$

and we have the one-one correspondence

$$h \mapsto F_h, \quad F \mapsto h_f.$$

Thus, we can represent points in L by the complete filters and work with the spectrum as with

$$\Sigma L = (\{F \mid F \text{ complete filter in } L\}, \{\Sigma_a \mid a \in L\}), \quad \Sigma_a = \{F \mid a \in F\},$$
and $\Sigma h(F) = h^{-1}[F]$ for $h : L \to M$.

6.5. Scott open filters.

6.5.1. Lemma. *A filter F in a frame is complete iff it is prime and Scott open.*

Proof. The implication $\Rightarrow$ is trivial, and so is the other as well: Let F be prime and Scott open and let $\bigvee_{i \in J} a_i \in F$. Since F is open, there are $a_{i_1}, \dots, a_{i_n}$ with $a_{i_1} \vee \cdots \vee a_{i_n} \in F$. Since F is prime, some of the a_{i_j} is in F. $\square$

6.5.2. Proposition. *Let F be a Scott open filter in a frame L such that $a \in F$ and $b \notin F$. Then there is a complete filter $P \supseteq F$ such that $a \in P$ and $b \notin P$.*

Proof. This is just the famous Birkhoff theorem with the openness added. Using the Zorn's lemma in the standard way (taking into account that unions of open sets are open) we obtain an open filter $P \supseteq F$ maximal with respect to the condition that $b \notin P \ni a$. We will prove that it is prime (and hence, by Lemma, complete). Suppose it is not; then there are $u, v \notin P$ such that $u \vee v \in P$. Set

$$G = \{x \mid x \vee v \in P\}.$$

G is obviously a Scott open filter and, because of the u, $P \subsetneq G$. Thus, $b \in G$, $b \vee v \in P$ and we can repeat the procedure with $v, b \notin P$, $v \vee b \in P$ and $H = \{x \mid x \vee b \in P\}$ to obtain the contradiction $b = b \vee b \in P$. $\square$

6.5.3. Corollary. *Each Scott open filter in a frame is an intersection of complete filters.*

6.6.1. Proposition. *Each continuous frame is spatial.*

Proof. Recall 2.5.1. Since all the ε_L are onto it suffices to show that our ε_L is one-one. In the representation in 6.4 this reduces to showing that if $a \nleq b$ then there is a complete filter P such that $b \notin P \ni a$. By 6.5.2 it suffices to find a Scott open filter F such that $b \notin F \ni a$. Since $a \nleq b$ and L is continuous, there is a c such that

$$c \ll a \quad \text{and} \quad c \nleq b.$$

Interpolate (recall 6.1) inductively

$$a \gg x_1 \gg x_2 \gg \cdots \gg x_n \gg \cdots \gg c \tag{6.6.1.1}$$

and set

$$F = \{x \mid x \geq x_k \;\text{ for some }\; k\}. \tag{6.6.1.2}$$

Then F is obviously a Scott open filter, and $b \notin F \ni a$. $\square$

6.6.2. Lemma. *Let L be a frame. A subset $K \subseteq \Sigma L$ is compact iff $\bigcap\{P \mid P \in K\}$ is Scott open.*

Proof. Let $\bigcap\{P \mid P \in K\}$ be Scott open and let $K \subseteq \bigcup\{\Sigma_a \mid a \in A\}$. Then $\bigvee A \in \bigcap\{P \mid P \in K\}$ since for each $P \in K$ there is an $a \in A$ such that $a \in P$, and hence $\bigvee A \in P$. Thus, there are $a_1, \ldots, a_n \in A$ with

$a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$ and hence $K \subseteq \Sigma_{a_1\vee\cdots\vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$. If K is compact and $\bigvee A \in \bigcap\{P \mid P \in K\}$ then $K \subseteq \Sigma_{\vee A} = \bigcup\{\Sigma_a \mid a \in A\}$ and there are $a_1, \ldots, a_n \in A$ such that $K \subseteq \Sigma_{a_1\vee\cdots\vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$ and finally $a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$. □

6.6.3. Theorem. (Hofmann-Lawson duality) *The functors Ω and Σ (recall section 2) restrict to a dual equivalence of the category of sober locally compact spaces and the category of continuous frames.*

Proof. After 5.6.1 it remains to be proved that if L is a continuous frame then ΣL is locally compact.

Let $P \in \Sigma L$ and $P \in \Sigma_a$, that is, $a \in P$. Since $a = \bigvee\{x \mid x \ll a\}$ there is a $c \ll a$ such that $c \in P$. Consider the F constructed as in (6.1.1) and (6.1.2) and set

$$K = \{Q \in \Sigma L \mid F \subseteq Q\}.$$

By 6.5.3, $F = \bigcap K$ and hence, by 6.6.2, K is compact. Now if $c \in Q$ we have $F \subseteq Q$, and if $F \subseteq Q$ then $a \in Q$. Thus, $P \in \Sigma_c \subseteq K \subseteq \Sigma_a$. □

6.7. Note. The results of this section include regular compact frames since

every regular compact frame is continuous.

Indeed, it suffices to prove that in a compact frame $x \prec y$ implies $x \ll y$. This is easy: if $y \leq \bigvee D$, and $x \prec y$ then $x^* \vee \bigvee D = 1$, and if D is directed we can choose a $d \in D$ such that $\{x^*, d\}$ is a cover, and hence $x \leq d$.

7. Notes

7.1. Recall 5.5. In fact, by a much more involved reasoning one can prove ([21]) that there is no need of choice (and other non-constructive principles) even for the proof that a product of a system of any compact locales is compact, contrasting the fact that the Tychonoff theorem in classical topology is equivalent to the axiom of choice. The reader may, for a moment, wonder how to reconcile this with the fact of Section 6: at least the regular compact locales are spaces! But of course this is no contradiction: this last fact is choice dependent. Thus, in Tychonoff theorem the choice is not needed for the compactness, but for having points at all (which for infinite products is no surprise at all).

7.2. In this text we have not spoken about the point-free counterparts of enriched topology (uniformity, nearness, metric). Let us just mention that similarly like the compactification, also the completion is in the point-free context fully constructive ([6],[4]).

7.3. In classical topology and its applications, paracompact spaces are very naturally defined and very useful. But they are badly behaved: even a product of a paracompact space with a metric one is not necessarily paracompact. Not so in the pointfree context: here the subcategory of paracompact locales is reflective in **Loc** ([17].

7.4. Furthermore, one has a beautiful characteristics of paracompact locales that has no classical counterpart.

a locale is paracompact if and only if it admits a complete uniformity.

7.5. Unlike in classical spaces, (one form of) realcompactness is equivalent to the Lindelöf property (Madden and Vermeer [32], see also [3],[39]).

7.6. Connected locally connected frames are in a well-defined sense pathwise connected (Moerdijk and Wraith [35]), and the rift between locally connected connectedness; also, plain connectedness is more pronounced, which corresponds to the geometrical intuition (connected but not locally connected spaces do not look really connected, do they?) ([27], [28]).

7.7. One can develop the theory of locallic groups (analogues of topological groups). It may be of interest that subgroups of localic groups are always closed ([20], [25]); moreover, localic groups are always complete in the natural uniformity ([7]).

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