# A tutorial on formal topology and the basic picture

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## Contents



## 1 Part 1. Origins and ideas of formal topology

My aims in this first part are:

> to recall briefly the motivations for an intuitionistic and predicative foundation of mathematics, type theory in particular;

*This is reading material for my tutorial at the Third Workshop on Formal Topology, Padua, 7-8 May 2007. Most of it is taken from my paper [24], with some adjustments (mainly in the first and last parts). The two parts correspond very roughly to the two lectures. In the actual tutorials, I will give a more updated version.

to discuss the general motivations of formal topology;

to give the definition and first results in formal topology;

to show how the notion of the continuum (real numbers) is expressed in formal topology;

to show how some important fields of research, like the theory of domains and Stone representation theory for lattices, can be embedded in formal topology.

The new approach to formal topology, which I call the basic picture, is treated in part 2.

## 1.1 Some ideas about type theory and the minimalist foundation

Constructive type theory aims at a constructive foundation of mathematics, which is alternative to the standard classical foundation, that is axiomatic set theory ZFC. However, it is also different from other constructive foundations, like constructive set theory CZF (see Rathjen's lectures) and topos theory. I recall here very briefly some general facts about type theory, in particular those which make it different from other common foundations of mathematics.

Using an axiomatic theory of sets as a foundation means assuming that it has a model, and this model is taken as a universe of all sets, in which mathematics is done using only the properties of sets as specified by the axioms. But such universe is considered as given, and thus there is no information on how the sets, and hence all other mathematical entitites, are built up. When topos theory is assumed as a foundation, the universe is assumed to be a topos, and in this sense the situation is similar.

In type theory, the universe in which mathematics is done is built up in the same time as mathematics is built up. In practice, this means that whenever we use anything, we have total information about it, or total knowledge of what it is and by which ingredients it has been built up. The name "type theory" is due to the fact that any entity goes together with its (logical) type, and the distinctions between different types are carefully preserved (while in set theory everything is reduced to only one ingredient, viz. sets, and only one relation, viz. membership, and in category theory to only two ingredients, viz. objects and arrows).

This methodological request is not at as difficult to fulfill as an education inside the ZF tradition might lead one to believe. On the contrary, it is very natural when any entity must be constructed: indeed, it is enough to keep information about it in the same moment it is constructed. In type theory, the control of information is so strict that any proof of any statement is automatically also an algorithm, or computer program fulfilling that statement (the proofs-as-programs principle). This is the main source of strong interest in type theory by the computer science community.

To know that $S$ is a set (or data type, or small type) means that we know by which rules all its elements are formed; these rules must be in front of us, hence in finite number, and cannot change with time. If we know that $S$ is a set, clearly to say that $a$ is an element of $S$ means that $a$ is produced by the rules; we write $a \in S$.

We can be in the position of knowing that $S(i)$ is a set, on the assumption that $i$ is an element of a set $I$; this is called a family of sets indexed by $I$, written $S(i)$ *set* $(i \in I)$. Similarly, we can be in the position of knowing that $f(i)$ is an element of a set $S$, on the assumption that $i \in I$; this is just the type theoretic notion of function from $I$ to $S$, and is written $f(i) \in S$ $(i \in I)$. More generally, if $S$ is not a set but a family of sets depending on an index set $i$, $f(i) \in S(i)$ $(i \in I)$ means that we know $f(i)$ to be an element of $S(i)$ whenever $i \in S$. By abstracting on the variable $i$, we obtain the elements of a new set $(\Pi i \in I)S(i)$, called the direct product. If $f \in (\Pi i \in I)S(i)$ and $i \in I$, by applying $f$ to $i$, we obtain an element of $S(i)$, which we here denote again by $f(i)$. In the special case in which $S$ does not depend on $I$, the direct product is denoted by $I \to S$; if $f \in I \to S$ and $i \in I$, then $f(i) \in S$.

All other definitions of sets in type theory are given in a similar way. In particular, given a set $I$ and a family $S(i)$ *set* $(i \in I)$, we can form the disjoint union $(\Sigma i \in I)S(i)$, whose canonical elements are pairs $\langle i, a\rangle$ with $i \in I$ and $a \in S(i)$. The special case in which $S$ does not depend on $I$ gives the cartesian product $I \times S$, the set of ordered pairs from $I$ and $S$.

In type theory one considers also objects which are not elements of a set, but rather belong to a collection, or logical type (or category in [16]). Two typical and important examples are the collection of subsets of a given set and the collection of all sets (or propositions).

The standard formulation of type theory by Per Martin-Löf ([16], [19]), includes the interpretation of propositions as sets. This means that the judgement that $P$ is a proposition obeys formally the same rules as the judgement that $P$ is a set, reading $p \in P$ as $p$ is a verification, or proof of $P$. The logical rules of inference are then exactly the same as the rules for sets, the only difference being their "reading".

A consequence of the propositions-as-sets interpretation is that the axiom of choice holds, that is the proposition $(\forall i \in I)(\exists a \in S)R(i, a) \to (\exists f \in I \to S)(\forall i \in I)R(i, f\,i)$ is provable. This makes Martin-Löf's type theory constructively incompatible with topos theory (because axiom of choice and powerset axiom together allow one to derive the principle of excluded third).

For this reason, ground theory adopted here is a variant of Martin-Löf's type theory, which is characterized by the fact that sets and propositions are kept distinct (that is, the principle propositions-as-sets is not followed). This means that intuitionistic logic must be given independently of sets. As a consequence, the axiom of choice does not hold (so that any use of the axiom of choice is fully under control).

Moreover, to be able to do mathematics, besides the ground type theory, which is intensional, one has to provide also with a foundational theory in which one can deal with mathematical concepts, which are extensional (for instance, equality of functions means equality of their graphs). As first discussed in [14], the present attitude is to have a foundation with two different levels of abstraction: the extensional level is obtained from the intensional one by abstraction, that is by forgetting some information, which is not considered in mathematics (typically, the proof which says that a certain proposition is true). The test for correctness of such abstraction is that the converse should work, that is, one should always have a metatheorem saying that one can implement in the ground theory whatever is done in the extensional theory (that is, one can restore the information which was abstracted).

A precise specification of the extensional theory satisfying these requests is still to be given in all details. In [26] it was shown that one can include subsets with all their standard (intuitionistic) properties. Recent work of Milly Maietti shows that one can include also sets with an extensional notion of equality (quotient sets).

In practice, the result is that one works with standard notions and their notation, except for a distinction on membership: while for an element $a$ of a set $S$ one writes $a \in S$, for an a element $a$ of a subset $U$ one writes $a \epsilon U$. In fact, since a subset $U$ of $S$ is here just a propositional function with an argument in $S$, $a \epsilon U$ is just an abbreviation for $a \in S$ and $U(a)$ true.

Sets can be thought of as given (leaving the rules to build them implicit). They are meant to be equipped with an extensional equality. A function or a relations must respect such equality to be well defined. The power of a set is never a set.

The logic used is higher order intuitionistic logic. To keep predicativity, one must be careful to apply comprehension only on elementary formulae, that is those in which quantifiers range over sets and subsets. In practice, one can use quantification on subsets to express a certain property of some given entities, but one is not allowed to introduce a new entity by a quantification over subsets.

An important improvement with respect to standard notation is the use of $\between$. Since the quantifier $\exists$ is primitive, and not definable by means of $\forall$, it is convenient to introduce a notation for the notion which is dual to that of inclusion. That is, for any $U, V \subseteq S$ we put

$$U \between V \equiv (\exists a \in S)(a \epsilon U \;\&\; a \epsilon V)$$

and we read "$U$ overlaps $V$". Note that $U \between V$ is intuitionistically much stronger than $U \cap V \neq \emptyset$.

## 1.2 The point of formal topology

A topological space is classically defined (cf. e.g. [13], [9]) as a pair $(X, \mathcal{O}X)$ where $X$ is a set, whose elements $x, y, \ldots$ are called points, and $\mathcal{O}X$ is a family of subsets of $X$, which contains $\emptyset, X$ and is closed under finite intersections and arbitrary unions. The family $\mathcal{O}X$ is called a topology on the space $X$ and the subsets in $\mathcal{O}X$ are said to be open.

The conditions on $\mathcal{O}X$ are written more precisely as:

$\mathcal{O}1$ $\quad \emptyset, X \in \mathcal{O}X$

$\mathcal{O}2$ $\quad$ for any $E, F, \subseteq X$, if $E, F \in \mathcal{O}X$ then $E \cap F \in \mathcal{O}X$

$\mathcal{O}3$ $\quad$ for any family of subsets $\mathcal{F}$, if $\mathcal{F} \subseteq \mathcal{O}X$, then $\bigcup \mathcal{F} \in \mathcal{O}X$

This formulation of the notion of topological space is unacceptable, as it stands, from a predicative point of view, since apparently a quantification not only over subsets, but over families of subsets (hence of the third order) is to be used. Though usually this is given meaning by conceiving the collection of subsets as a completed totality, we now see that actually no intrinsic impredicativity is involved, and that one can easily find a definition of topological space which is fully acceptable also predicatively.

A collection of subsets, and $\mathcal{O}X$ is one such, is most simply given in type theory as a set-indexed family, that is a function, which we call $\mathsf{ext}$, from some set, which we call $S$, into $\mathcal{P}X$. In this way a quantification over open subsets - we cannot dispense with it in topology - can be reduced to a quantification over the set $S$.

However, one cannot expect $\mathsf{ext}$ to give all open subsets as values; the special case in which $\mathcal{O}X$ is the whole of $\mathcal{P}X$ - the discrete topology - would require $\mathcal{P}X$ to become indexed by the set $S$, and this is not welcome in type theory.[1] Moreover, the expression of $\mathcal{O}3$ would still require an impredicative quantification.

These difficulties are solved by asking the family $\mathsf{ext}\,(a) \subseteq X$ $(a \in S)$ to be a base for the topology. Thus subsets $\mathsf{ext}\,(a)$ are called (basic) neighbourhoods, and open subsets are defined as arbitrary unions of neighbourhoods. That is, a subset $D$ of $X$ is open if and only if $D = \mathsf{ext}\,(U)$ for some $U \subseteq S$, where $\mathsf{ext}\,(U) \equiv \bigcup_{b \epsilon U} \mathsf{ext}\,(b)$. We may assume that the usual conditions on bases are satisfied in the sense that $S$ is provided with a binary operation $\cdot$ and with a distinguished element 1 such that

$$\mathsf{ext}\,(1) = X \quad \text{and} \quad \mathsf{ext}\,(a) \cap \mathsf{ext}\,(b) = \mathsf{ext}\,(a \cdot b)$$

The resulting structure is called a *concrete space* (in [21], example 2.1), or a concrete presentation of a topological space.

From our constructive point of view, this definition is certainly acceptable, but not sufficient to develop topology. One has to add two further notions, that of formal topology and that of formal point, and hence also that of formal space, as the collection of all formal points. In fact, in many interesting examples, the collection $X$ of points of a classical topological space is not given directly as a set, in the constructive sense. And this may happen also when basic neighbourhoods of $X$ can be given as a family $\mathsf{ext}$ indexed on a set $S$. The reason for this is that an infinite amount of information, which means infinitely many basic neighbourhoods, may be necessary to determine a point. The idea is then to consider elements $a, b, c...$ of $S$ as formal neighbourhoods, and hence subsets $U, V, ...$ of $S$ as formal substitutes of open subsets. One has to define, however, when two subsets of $S$ are topologically equivalent, that is when they produce the same open subset. This leads to the definition of formal topology, which thus is a specific structure on the set of formal neighbourhoods. Then an infinite amount of information can be given by a subset $\alpha$ of $S$, and when $\alpha$ has some properties which make it formally similar to a point, it will be called a formal point.

[1] By adapting to type theory the well-known argument due to Diaconescu, it is shown in [15] that this would bring to classical logic.

The method to obtain the definition of a formal notion, those of formal topology and formal point to begin with, is always the same, and it can be described as formed by three steps:

1. Study the notion to be defined in the presentable case, in which both a set of points $X$ and a set of formal neigbourhoods $S$ are present. This allows to choose some new primitives to be added to the formal side, in view of step 2.

2. Analyse the structure induced on the formal side, and write down all those properties of the primitives on $S$ which can be expressed without mentioning the points of $X$. Of course, the best choice of primitives is that which allows to describe the original concrete notion in the best possible way.

3. Abandon points altogether, and retain those properties of formal primitives as an axiomatic definition.

We apply this method first of all to obtain the definition of formal topology itself. In the concretely presentable case, two subsets $U$, $V$ of $S$ correspond to the same open subset of $X$ when $\mathsf{ext}\,V = \mathsf{ext}\,U$. To express this in pointfree terms, it is enough to express $\mathsf{ext}\,V \subseteq \mathsf{ext}\,U$, and this in turn, by the definition $\mathsf{ext}\,V = \cup_{a \epsilon V} \mathsf{ext}\,a$, reduces to $(\forall a \;\epsilon\; V)(\mathsf{ext}\,a \subseteq \mathsf{ext}\,U)$. So we add an infinitary relation $a \lhd U$ as primitive, with the idea that it corresponds to $\mathsf{ext}\,a \subseteq \mathsf{ext}\,U$; using it one can define $V \lhd U \equiv (\forall a \;\epsilon\; V)(a \lhd U)$, which then corresponds to $\mathsf{ext}\,V \subseteq \mathsf{ext}\,U$, and finally $V =_{\lhd} U \equiv V \lhd U \,\&\, U \lhd V$ will correspond to $\mathsf{ext}\,V = \mathsf{ext}\,U$.

The distinguished element 1 and the operation $\cdot$ are also kept, and the idea is that $\mathsf{ext}\,1 = X$ and that $\mathsf{ext}\,(a \cdot b) = \mathsf{ext}\,a \cap \mathsf{ext}\,b$. Finally, we also add a unary predicate $Pos(a)$ *prop* $(a \in S)$, whose meaning in the concrete case is that $\mathsf{ext}\,a$ is inhabited; in fact, this is constructively not reducible to $\mathsf{ext}\,a \neq \emptyset$. The result of applying now steps 2 and 3 of the method above is the definition of formal topology given in next section.

The method by which we reached the definition says that any concrete space gives a formal topology, which is then called (*concretely*) *presentable*. But note that not all formal topologies are presentable;[2] if it were so, their introduction would be much less motivated.

## 1.3 A formal topology is...

The first result of the method described in the previous section is the definition of formal topology itself. The following is a minor variant (but equivalent from many aspects)[3] of the original in [21]:

**Definition 1.1** *A formal topology $\mathcal{S}$ consists of:*

*a set $S$,*

*a distinguished element* 1 *and a binary operation* $\cdot$ *on $S$,*

*a relation $\lhd$ between elements and subsets of $S$, called (formal) cover, which for arbitrary $a, b \in S$, $U, V \subseteq S$ satisfies:*

*reflexivity* $$\frac{a \;\epsilon\; U}{a \lhd U}$$

*transitivity* $$\frac{a \lhd U \qquad U \lhd V}{a \lhd V} \qquad \textit{where } U \lhd V \equiv (\forall b \;\epsilon\; U) b \lhd V$$

*$\cdot$-Left* $$\frac{a \lhd U}{a \cdot b \lhd U} \qquad \frac{a \lhd U}{b \cdot a \lhd U}$$

[2] An example of non-presentable formal topology is given in [6], but simpler, finite examples can be built up.

[3] The aim of this variant is to avoid problems connected with equality; usually $(S, \cdot, 1)$ is assumed to be a monoid or a semilattice, which is expressible only using equality of $S$. Equivalence holds in the sense that putting $a =_{\lhd} b \equiv (a \lhd \{b\} \;\&\; b \lhd \{a\})$, one can show that $(S, \cdot, 1, =_{\lhd})$ is a semilattice.

$\cdot$*-Right* $\dfrac{a \lhd U \qquad a \lhd V}{a \lhd U \cdot V}$ *where* $U \cdot V \equiv \{a \cdot b : a \,\epsilon\, U, b \,\epsilon\, V\}$

*top* $a \lhd 1$

*a predicate* $\mathsf{Pos}(a)$ *on* $S$*, called positivity predicate, which for arbitrary* $a \in S$*,* $U \subseteq S$ *satisfies:*

*monotonicity* $\dfrac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(b)}$

*positivity* $\dfrac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$

A frequently asked question about formal topology is whether quantification over subsets is really avoided; the claim is that the very definition of formal topology involves a quantification over subsets. The crucial point is of course the use of "for arbitrary $U$", especially in a formalistic reading of the definition. The answer is that we use subset variables as arguments of (higher order) functions, that is we do *not* use them to build up new propositions (that is, new subsets) of the form $\forall U \ldots$, but keep always the quantification at the meta-level (formally: subset variables remain free). So the definition with all details should read:

**Definition 1.2** *A formal topology* $\mathcal{S}$ *consists of:*

*a set* $S$*, which is determined by specifying its introduction and elimination rules;*

*a distinguished element* $1$ *and a binary operation* $\cdot$ *on* $S$*, that is* $a \cdot b \in S$ $(a \in S, b \in S)$*;*

*a relation* $\lhd$ *between elements and subsets of* $S$*, that is* $a \lhd U$ *prop* $(a \in S, U \subseteq S)$ *(which will be defined as usual for any proposition by furnishing introduction and elimination rules, either directly or indirectly by means of an expression like a logical formula, of which we already know that it produces propositions), and six functions* $refl$*,* $trans$*,* $l_1$*,* $l_2$*,* $t$ *and* $r$ *of the convenient types which satisfy:*

*reflexivity* $refl(a, U) \in U(a) \to a \lhd U \ (a \in S, U \subseteq S)$,

*transitivity* $trans(a, U, V) \in a \lhd U \,\&\, U \lhd V \to a \lhd V \ (a \in S, U, V \subseteq S)$,

$\cdot$*-Left* $l_1(a, b, U, V) \in a \lhd U \to a \cdot b \lhd V \ (a \in S, U \subseteq S)$
$l_2(a, b, U, V) \in a \lhd U \to b \cdot a \lhd V \ (a \in S, U \subseteq S)$,

$\cdot$*-Right* $r(a, U, V) \in a \lhd U \,\&\, a \lhd V \to a \lhd U \cdot V \ (a \in S, U, V \subseteq S)$,

*top* $t(a) \in a \lhd 1$*;*

*a predicate* $\mathsf{Pos}(a)$ *on* $S$*, that is* $\mathsf{Pos}(a)$ *prop* $(a \in S)$*, and two functions* $m$ *and* $p$ *which satisfy:*

*monotonicity* $m(a, U) \in \mathsf{Pos}(a) \,\&\, a \lhd U \to (\exists b \,\epsilon\, U)\mathsf{Pos}(b) \ (a \in S, U \subseteq S)$,

*positivity* $p(a, U) \in (\mathsf{Pos}(a) \to a \lhd U) \to a \lhd U \ (a \in S, U \subseteq S)$.

This formalistic definition (with proof-terms spelled out to please a computer language) has absolutely no quantification over subsets. I never wrote it explicitly before, because I assumed it was understood.[4] The notation with hidden proof-terms is more suitable to human mathematicians. Keeping explicit track of all the proof-terms, that is of computational content, would impede a

[4] Definition 1.2 was given for the first time explicitly in my talk at TYPES'98, Kloster Irsee, together with the comments given in this section.

more abstract understanding, or at least would make it much harder.[5] In any specific example, of course, one has to produce, at least in principle, *all* the required information, so including the functions in variant 1.3, simply to be sure that one has actually *given* an example of formal topology.

The apparent quantification over subsets needed in the definition of formal topology is of the same kind as the quantification over propositions $A$, $B$ which is needed to understand a simple inference rule such as

$$\frac{A}{A \vee B}$$

In fact, one understands here that the rule applies to any propositions $A$ and $B$, but nobody has ever questioned whether a second-order quantification is here involved, since it is clear that the quantification involved remains at the metalevel.

However, one must be *extremely* careful on this topic, since not all quantifications at the metalevel are equally innocent. Let me first recall one aspect of the intuitionistic meaning of quantifiers. The meaning of a statement of the form $(\forall x \in S)A(x)$ in intuitionistic terms is that we have a method proving $A(a)$ for every $a \in S$. So the meaning of $(\forall x \in S)(\exists y \in S)A(x, y)$ is that we have a method which applies to any $a \in S$ and produces a proof of $(\exists y \in S)A(a, y)$, that is an element $c$, depending on $a$, such that $A(a, c)$ holds.

It seems to me that there is no other way to give constructive meaning to a universal-existential statement, also when quantifiers are meant to be kept at the metalevel. So I am able to grasp that "for every $U \subseteq S$, there exists $b$ such that $A(U, b)$" holds only when I *have* a function $F$ such that $A(U, F(U))$ holds for every $U \subseteq S$. One can debate whether this function should always be expressible *within* the language. But assuming that the meaning of "for every $U$ there exists $b$" is always predicatively clear (which is implicit when such a combination of quantifiers is used to define an object, like a subset) amounts to assuming that the function $F$ can be obtained always, and that it is expressible in the language, which means that a second-order axiom of choice of the kind $\forall U \exists b A(U, b) \to \exists F \forall U A(U, F(U))$ must hold. But then this brings us immediately to classical logic (see [15] for the precise statement and proof of this fact).

This is an example of a "powerful" principle which actually destroys the quality of information or equivalently, at least in my own case, which destroys the possibility of an intuitive grasping. A consequence is that in formal topology one will always find directly the function $F$, and never the combination "for every $U$, there exists $b$" (or "for every $U$, there exists $W$") to which it gives meaning (see for instance the case of the definition of $U{\downarrow}V$ in section 2.1).

Another critique to the definition of formal topology is that... there are too many different definitions. I would just like to recall that even what now looks as the most stable definition of (usual) topology, namely that of topological space, is actually the result of a long historical process, which stabilized relatively recently. One advantage of the variant given above is explained in footnote 3. Two further variants will be introduced in section 2.1 and in section 2.5, together with some good reasons to do it.

## 1.4 Development of formal topology

Building on the definition of formal topology, one introduces the notions of formal open, formal point, formal space, continuous function,.... They will be treated in part 2, where a deeper justification is possible. Here we briefly review some of the first developments and ideas.

### 1.4.1 Predicative presentation of frames

An infinitary relation $\lhd$ satisfying only the properties of reflexivity and transitivity, as in the definition of covers, is called an infinitary preorder. It was discovered long ago (see [21]) that

[5] To be pedantic, this is an example of the forget-restore principle: one should make sure that hiding the proof-terms of all the propositions does not prevent us from obtaining them back when wished. This is possible because all proofs will be intuitionistic, and thus preserve proof-terms.

infinitary preorders on a set $S$ correspond biunivocally to closure operators on $S$ (that is, functions $\mathcal{C} : \mathcal{P}S \to \mathcal{P}S$ such that $U \subseteq \mathcal{C}U$, $U \subseteq V \to \mathcal{C}U \subseteq \mathcal{C}V$ and $\mathcal{C}\mathcal{C}U \subseteq \mathcal{C}U$). In fact, by setting $\mathcal{A}U \equiv \{a \in S : a \lhd U\}$ one has that $a \lhd U$ is the same as $a \,\epsilon\, \mathcal{A}U$, so that reflexivity can be rewritten as $U \subseteq \mathcal{A}U$ and transitivity as $V \subseteq \mathcal{A}U \to \mathcal{A}V \subseteq \mathcal{A}U$; one can then easily check that these two conditions on $\mathcal{A}$ are equivalent to those in the definition of closure operator. Moreover, it is well known that closure operators correspond to complete lattices (given a closure operator $\mathcal{A}$, the collection of saturated subsets $Sat(\mathcal{A}) \equiv \{U \subseteq S : U = \mathcal{A}U\}$ is a complete lattice, in which meet is given by intersection and join by the saturation of union, and conversely, given a complete lattice, putting $a \,\epsilon\, \mathcal{A}U \equiv a \leq \bigvee U$ gives a closure operator).

Building on these remarks, one can obtain a modular presentation of sup-lattices (that is, lattices with arbitrary joins - and hence also meets - but in which only joins are preserved by morphisms), quantales and frames by generators and relations. The sup-lattice freely generated by a set $S$ of generators is just $\mathcal{P}S$. So the idea is to describe the ordering of *any* sup-lattice generated by $S$ by adding conditions, or axioms $R(a, U)$, to be satisfied if $a \leq \bigvee_{b \epsilon U} b$. The main result (which generalizes a similar result in [12]) is that the least infinitary preorder $\lhd_R$ containing $R$ gives exactly the free sup-lattice satisfying the axioms given by $R$. The same result for quantales and frames is obtained in a modular way, by adding suitable extra conditions.

This line of research was begun very early, see [4], and several earlier versions of the final paper [3] circulated privately. In fact, it took a long time to understand properly how it is possible to generate $\lhd_R$ above in a predicative way, and for which $R$ this is possible (see [7]). One must be very careful here: when one says that formal topologies (without $\mathsf{Pos}$) form a category which is equivalent to that of frames, one must realize that the proof cannot be predicative, unless one previously restricts to a predicative definition of frames. The point here is that a predicative notion of frame... is nothing but the notion of formal topology.

### 1.4.2 Inductive generation of formal topologies and proof-theoretic methods

A formal topology, one could say, is just a way to present a frame (the frame $Sat(\mathcal{A})$) by generators (the set $S$) and relations (the cover $\lhd$, or equivalently the closure operator $\mathcal{A}$). The choices taken when defining formal topologies are actually linked with the choice for predicative methods. But whatever the reason is, the introduction of formal topologies has opened the way to the use of inductive methods in topology. Actually, all the axioms or conditions are preferably written in the form of inference rules exactly for the purpose of applying proof-theoretic methods or ideas. This appears as a conceptual novelty in the field of topology, and gives to formal topology its distinctive character: formal topology, which happened to begin as a theory of locales developed over a different foundation (namely, type theory rather than topos theory), has later developed a specific identity also from a strictly mathematical point of view. One typical result in this sense is the normal form theorem for covers on real numbers, and the problem it leads to (see section 1.4.3 below). Another one is that the finitary content of a formal cover generated by axioms $\Sigma$ is just the cover generated by the finitary part of $\Sigma$, that is, by those axioms of $\Sigma$ in which only finite subsets are involved.

The importance of the inductive generation of formal topologies is clear, for a predicative approach, when one observes that, for instance, the product of two formal topologies cannot be defined predicatively, unless they are inductively generated (see [7]).

Any other information about the inductive generation of formal topologies can be found in [7]; in particular, the readers will discover there that almost all the examples of formal topologies which can be found “in nature” do fall under the scope of the theorem on inductive generation. This gives a solid argument in favour of formal topology, since it automatically means that all those examples can be formalized into a computer language.

Generation of formal topologies can be extended also to the new definition of formal topology, which includes a (binary) positivity predicate $\ltimes$, see section 2.10 below. It is to be remarked that $\ltimes$ is generated by coinduction, a fact which considerably widens the fields of applications of formal topology.

### 1.4.3 The continuum as a formal space

In [21] it was suggested that the continuum could be presented via formal topology essentially as in [12]. This idea was later worked out by my student Daniele Soravia in [31], where also the beginning of real analysis is developed (all this appeared subsequently in [18]). The main idea is that a real number is a formal point on a suitable formal topology where basic neighbourhoods are pairs of rational numbers, $(p,q)$ with $p,q \in \mathbf{Q}$. The positivity predicate is defined by $\mathsf{Pos}((p,q)) \equiv p < q$, and the cover $\lhd$ is defined inductively by the following rules (which are a formulation in our context of Joyal axioms, cf. [12], pp. 123-124):

$$\frac{q \le p}{(p,q) \lhd U} \qquad \frac{(p,q) \in U}{(p,q) \lhd U}$$

$$\frac{(p' \le p < q \le q') \qquad (p',q') \lhd U}{(p,q) \lhd U} \qquad\qquad \frac{p \le r < s \le q \qquad (p,s) \lhd U \quad (r,q) \lhd U}{(p,q) \lhd U}$$

$$\infty \qquad \frac{wc((p,q)) \lhd U}{(p,q) \lhd U}$$

where in the last axiom we have used the abbreviation $wc((p,q)) \equiv \{(p',q') : p < p', q' < q\}$. (where $wc$ stands for 'well-covered'). This presentation of the cover is essentially due to Coquand. The *formal reals* are just the formal points of such a formal topology.

We have then the following normal form theorem, by which the 'infinitary' rule $\infty$ is isolated:

*Theorem of canonical form.* Any derivation of a statement $a \lhd U$ can be brought to a form where the only application of the rule $\infty$ is the last one, just above the conclusion.

In this way the finitary part of the cover is distinguished from its infinitary component, and the logical tool we make use of is limited to a *finitary* inductive definition. The proof is by induction on the derivation of $a \lhd U$, as standard in proof theory. If $\lhd_\omega$ is the (finitary) compactification of $\lhd$, which by the remarks in the previous section coincides with the cover generated by the rules above except $\infty$, this amounts to have proved that

$$(p,q) \lhd U \text{ if and only if } wc((p,q)) \lhd_\omega U,$$

providing thus a definition of $(p,q) \lhd U$ as $wc((p,q)) \lhd_\omega U$, that is an elementary definition over a finitary inductive definition.

I express here the expectation that a similar (proof-theoretic) procedure can be used to separate the infinitary content of a cover from its finite part for a wider class of topologies (which presumably should be compact in some sense; cf. for instance [5]). This is still an open problem.

The above notion of well-covered elements can be generalized to an arbitrary formal topology, by setting

$$wc(a) \equiv \{b : S \lhd b^* \cup \{a\}\}$$

where $b^* \equiv \{c : c{\downarrow}b \lhd \emptyset\}$ is the subset of neighbourhoods which are apart from $b$. Then $b \,\epsilon\, wc(a)$ is classically equivalent to saying that $\mathsf{ext}\,(b)$ is well covered by $\mathsf{ext}\,(a)$ if the closure of $\mathsf{ext}\,(b)$ is contained in $\mathsf{ext}\,(a)$. This brings us to define regular formal topologies as those topologies in which $a \lhd wc(a)$ for any $a$. It can be shown that such definition has some of the properties one would expect. For instance, one can prove that for any two formal points $\alpha$ and $\beta$, if $\alpha \subseteq \beta$ then $\alpha = \beta$, that is, the ordering on formal points is discrete.

### 1.4.4 Domain theory as a branch of formal topology

The notion of formal point is defined and justified in section 2.4, and thus I do not repeat it here. For any formal topology $\mathcal{S}$, the collection of its formal points $Pt(\mathcal{S})$ is said to be a *formal space*. This is a genetic characterization of formal spaces. In general, an axiomatic definition is not available; one can only define as usual the specialization ordering on formal points $\alpha$, $\beta$ by setting $\alpha \leq \beta \equiv \beta \subseteq \alpha$ ($\alpha$ is less than $\beta$ if it is more informative, i.e. contains more elements of $S$) and observe that $Pt(\mathcal{S})$ thus becomes a complete partial order. But if we restrict our attention to the class of unary formal topologies, which are those in which the cover is unary, or 1-compact,

$$a \lhd U \quad \text{iff} \quad \mathsf{Pos}(a) \rightarrow (\exists b \,\epsilon\, U)(a \lhd \{b\}),$$

then the associated class of formal spaces admits of an axiomatization, and actually a well-known one, since it turns out to be exactly the class of Scott domains (the link with Scott domains was present from the beginning, see [21], section 7, but it was spelled out only later in [27]). In fact, a unary cover is intuitively one in which no two neighbourhoods do cooperate to produce coverings. So one can see that in any unary $\mathcal{S}$ all subsets of the form $\uparrow a \equiv \{b : a \lhd b\}$, for any positive $a$, are formal points of $\mathcal{S}$, and all formal points are obtained by forming unions of these. In other words, positive elements of $\mathcal{S}$ correspond to compact elements of the Scott domain $Pt(\mathcal{S})$. Then one can read both Scott's definition of information systems [28] and the even simpler definition of information bases in [27] either as an axiomatization of the structure of compact elements of a domain or as a simplified characterization of unary formal topologies. This is to say that the category of unary formal topologies, that of information systems, and that of information bases are mutually equivalent, and Scott domains are obtained by applying the functor $Pt$ bringing to formal spaces. So the definitions of domain theory become special cases of notions having a general topological meaning, and in the end this has produced a simplified approach to the theory of domains, which moreover is fully predicative.

The connection between formal topology and domain theory is clear also in the approach to formal topology via the basic picture, which is described in part 2 below. A curious fact is that, while the categories of (arbitrary) formal topologies, in the old and in the new sense, are equivalent, this is no longer true for unary formal topologies. So unary formal topologies, in the new sense, are equivalent to algebraic domains, and the extra condition characterizing $Pt(\mathcal{S})$ as a Scott domain is *not* independent of the way $\mathcal{S}$ is given (see [23]). The next natural step is to extend the connection with domain theory by finding predicative definitions of the way-below relation and of continuous domains; a common expectation is that the right idea should be that of bases with the interpolation property (see [1]).

A nice topic for research is to reveal which of the results for unary formal topologies extend to the case of finitary (or Stone) formal topologies, that is those in which the cover satisfies

$$a \lhd U \quad \text{iff} \quad \text{there is a finite subset } K \text{ of } U \text{ such that } a \lhd K.$$

In particular, it is still unknown to me whether it is possible to find an axiomatization of formal spaces corresponding to finitary formal topologies.

### 1.4.5 Classical theorems constructivized

A natural and reasonable question is of course how many of the classical definitions and theorems of (classical) topology can be obtained in the framework of formal topology. I am firmly convinced that, as with any form of constructive mathematics, the fact that relatively few results have been found up to now is not due to intrinsic obstacles, but mainly to the relatively little research energy which has been put in finding them. This state of facts has been rapidly changing in the last few years. A bibliography is available at: `www.math.unipd.it/~sambin/?????????`.

### 1.4.6 Predicative completeness proofs

An important insight about intuitionistic logic, which goes back to the 30s, is that its propositions (or formulae) can be interpreted mathematically as the open subsets of a topological space. As shown in [22], also formal topology provides with a complete semantics, by interpreting formulae as formal open subsets (and by suppressing the predicate $\mathsf{Pos}$). Since the notion of formal topology is fully predicative, the result is a proof of completeness of topological semantics which is also fully predicative. As in the original proof by Henkin, the key step is the construction of a generic model from the syntax itself; in our case, a suitable cover on the set of formulae must be introduced. Two such covers are studied in detail in [6], where it is shown that formal points over one of them are exactly the same thing as Henkin sets. This gives a precise form to the idea that points correspond to models [10]; for some other comments and references, see the introduction of [6].

The completeness proof in [22] is actually given in a modular way for a variety of logics, which all are extensions of intuitionistic linear logic. To this aim, the notion of cover is generalized to that of precover, in which the two assumptions on $\cdot$, namely $\cdot$-Left and $\cdot$-Right, are replaced by the single one:

$$\text{stability} \qquad \frac{a \lhd U \qquad b \lhd V}{a \cdot b \lhd U \cdot V}$$

or its equivalent

$$\text{localization} \qquad \frac{a \lhd U}{a \cdot b \lhd U \cdot b} \qquad \text{where } U \cdot b \equiv U \cdot \{b\}$$

A pretopology is a commutative monoid equipped with a precover. A cover becomes exactly the same as a precover satisfying the conditions corresponding to the structural rules of weakening and contraction, which can be seen to be $a \cdot b \lhd a$ and $a \lhd a \cdot a$, respectively (or some other equivalents). On the other hand, pretopologies in which the double negation law is valid turn out to coincide literally with phase spaces, that is the semantics of linear logic given by Jean-Yves Girard in [11].

# 2 Part 2. The basic picture, a future for formal topology?

When I first gave a full course on formal topology, in Padua in 1990, I also first conceived the idea of writing a book on it. My manuscripts, lecture notes and typescripts have increased in number and length since then. However, at the end of 1995 it happened that I discovered a clear, very simple structure underlying topology and consisting of symmetries and logical dualities. I now call it the basic picture.

Mathematically, the basic picture is a natural generalization of topology, obtained by considering relations rather than functions as transformations. For the debate on foundations, it should be interesting that it is well visible only over an intuitionistic and predicative foundation, and this probably explains why it was not noticed before.

The discovery of the basic picture caused a radical and extensive reformulation of formal topology itself, in particular by the introduction of a positivity relation, the presence of coinduction and the generalization to nondistributive topology (that is why the book [25] is still on its way...).

So my aims in this second part are:

- to give an introduction to the ideas and main results of the basic picture;
- to show how the basic picture serves as the starting point for a new conception and a new technical development of constructive topology, formal topology in particular;
- to illustrate the advantages with respect to the previous approach;
- to give an overview of open problems and expected developments.

## 2.1 From concrete spaces to basic pairs

Though successful, the definition of formal topology, as given in [21] or here in section 1.3, still leaves something to be desired. One desire is a convincing definition of formal closed subsets. Another is to avoid the operation $\cdot$ of formal intersection, which makes the treatment of some important examples, like $\mathcal{P}X$ and upper subsets in a preorder, a bit artificial. A positive solution to both requests has come from a deeper analysis of the notion of topological space. This has actually brought up much more than that. In fact, a whole new ground structure has emerged, which I have called the basic picture, since it shows how the main definitions of topology are deeply rooted to very basic ingredients, such as symmetry and logical duality. Topology, either with or without points, turns out to be obtainable simply by adding a principle of additivity of approximations (expressed by B1, in section 2.1, and by $\downarrow$-Right, in section 2.4), that is adding a notion of convergence. This in my opinion gives a very satisfactory explanation of the ground concepts of topology, which is independent of any foundational theory.

Let us resume our analysis of the notion of topological space, in section 1.2, and more precisely at the moment in which we assumed the base to be closed under intersection. We now see that this is not necessary, and that actually relaxing that assumption allows one to see a simpler and deeper structure.

So assume, as before, that $X$ is a set (of points), $S$ is a set of indexes, and $\mathsf{ext}$ is a function from $S$ into subsets of $X$. We consider all the subsets obtained by union, that is, all subsets of $X$ of the form $\mathsf{ext}\,(U) \equiv \cup_{a \epsilon U} \mathsf{ext}\,(a)$ for some $U \subseteq S$. Then we want to find out under which conditions on $\mathsf{ext}$ the subsets $\mathsf{ext}\,(U), U \subseteq S$, form a topology, that is, satisfy $\mathcal{O}1 - \mathcal{O}3$.

To this end, it is convenient to adopt a notation better suited than $\mathsf{ext}$, as we now explain. Since a subset of $X$ is nothing but a propositional function over $X$, a family of subsets $\mathsf{ext}\,(a) \subseteq S$ $(a \in S)$ is nothing but a propositional function with two arguments, one in $X$ and one in $S$, in other words a binary relation between $X$ and $S$. Then it is better to write such a relation as

$$x \Vdash a \; prop \; (x \in X, a \in S)$$

and to define $\mathsf{ext}$ in terms of it, by setting

$$\mathsf{ext}\,(a) \equiv \{x \in X : x \Vdash a\} \qquad \text{for any } a \in S.$$

In this way the abstraction is kept at a lower level, both intuitively and formally (since $\mathsf{ext}\, a$ is obtained from $x \Vdash a$ by abstraction on $x$). Elements of $S$ are called formal basic neighbourhoods, or more briefly observables, and $x \Vdash a$ is read as "$x$ lies in $a$", or "$x$ satisfies $a$", or more neutrally "$x$ forces $a$". The choice of the name $\mathsf{ext}$ should then be clear: $\mathsf{ext}\,(a)$ is called the *extension* of the observable $a$. The notation with $\Vdash$ is extended to subsets by setting

$$x \Vdash U \equiv (\exists b \in S)(x \Vdash b \;\&\; b \;\epsilon\; U) \equiv x \;\epsilon \cup_{b \epsilon U} \mathsf{ext}\,(b)$$

which agrees with the reading "$x$ lies in $U$" since $\mathsf{ext}\,(U) \equiv \cup_{b \epsilon U} \mathsf{ext}\,(b) \equiv \{x : x \Vdash U\}$. It is easy to check, at any desired level of formal details, that the family of subsets $\mathsf{ext}\,(U) \subseteq S$ $(U \subseteq S)$ is closed under unions. By this we mean, of course, that for any family of subsets $U_i \subseteq S$ $(i \in I)$ indexed by a set $I$, it holds that $\cup_{i \in I} \mathsf{ext}\,(U_i) = \mathsf{ext}\,(\cup_{i \in I} U_i)$. In fact, $x \;\epsilon \cup_{i \in I} \mathsf{ext}\,(U_i) \equiv (\exists i \in I)(\exists b \;\epsilon\; U_i)(x \Vdash b)$ is equivalent to $(\exists b \;\epsilon \cup_{i \in I} U_i)(x \Vdash b) \equiv x \;\epsilon\; \mathsf{ext}\,(\cup_{i \in I} U_i)$. So $\mathcal{O}3$ is automatically satisfied.

Condition $\mathcal{O}1$ also is easily expressed. In fact, $\emptyset = \mathsf{ext}\,(\emptyset)$ because $a \;\epsilon\; \emptyset$ holds for no $a$, and $X = \mathsf{ext}\,(U)$ for some $U \subseteq S$ is equivalent to $X = \mathsf{ext}\,(S)$, that is:

$$\text{B2} \qquad x \Vdash S \text{ for any } x \in X.$$

We thus can concentrate on $\mathcal{O}2$. If we express it without care, writing

$$(\forall U, V \subseteq S)(\exists W \subseteq S)(\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, W),$$

again an impredicative quantification comes up. However, this luckily is not really necessary. The quantification of the form $\forall U, V\, \exists W$ is solved if we find a uniform method which associates a subset $W$ satisfying $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, W$ with any pair of subsets $U, V$. The simplest such method is to pick the largest among the open subsets contained in $\mathsf{ext}\, U \cap \mathsf{ext}\, V$. That is, if $\mathsf{ext}\, U \cap \mathsf{ext}\, V$ is open, which means that it is equal to $\mathsf{ext}\, W$ for some $W$, then it is bound to be equal to $\mathsf{ext}\, Z$ where $Z$ is formed by all $c \in S$ whose extension is contained in $\mathsf{ext}\, U \cap \mathsf{ext}\, V$, in symbols $Z \equiv \{c \in S : \mathsf{ext}\,(c) \subseteq \mathsf{ext}\, U \cap \mathsf{ext}\, V\}$. So $\mathcal{O}2$ is equivalent to $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, Z$. However, we can do much better than this. If we apply the same idea to open subsets of the form $\mathsf{ext}\, a$ with $a \in S$, we obtain

$$\text{B1} \qquad \mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}\,(a \downarrow b)$$

where $a \downarrow b \equiv \{c \in S : \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}$ is the largest subset whose extension is contained in $\mathsf{ext}\, a \cap \mathsf{ext}\, b$. It is now easy to see, by the distributivity property of $\mathcal{P}X$, that B1 is the right condition. In fact for any $U, V \subseteq S$ we have

$$\begin{aligned}
\mathsf{ext}\, U \cap \mathsf{ext}\, V &\equiv (\cup_{a\epsilon U}\, \mathsf{ext}\, a) \cap (\cup_{b\epsilon V}\, \mathsf{ext}\, b) && \text{by definition of } \mathsf{ext} \text{ on subsets,}\\
&= \cup_{a\epsilon U} \cup_{b\epsilon V}\, (\mathsf{ext}\, a \cap \mathsf{ext}\, b) && \text{by distributivity of } \mathcal{P}X,\\
&= \cup_{a\epsilon U} \cup_{b\epsilon V}\, \mathsf{ext}\,(a \downarrow b) && \text{by B1,}\\
&= \mathsf{ext}\,(\cup_{a\epsilon U} \cup_{b\epsilon V}\, a \downarrow b) && \text{because } \mathsf{ext} \text{ distributes over unions.}
\end{aligned}$$

So we put

$$U \downarrow V \equiv \cup_{a\epsilon U} \cup_{b\epsilon V}\, a \downarrow b.$$

If B1 holds, then also $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\,(U \downarrow V)$ holds, and hence $\mathcal{O}2$ is satisfied. Note that now $U \downarrow V$ is not necessarily the largest subset $Z$ as defined above. But this is irrelevant. In fact, if $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\,(U \downarrow V)$, then also $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, Z$ because $\mathsf{ext}\,(U \downarrow V) \subseteq \mathsf{ext}\, Z \subseteq \mathsf{ext}\, U \cap \mathsf{ext}\, V$.

The reason for names B1 and B2 is that they are just a compound expression, in our language, of the standard conditions for bases for a topology (see e.g. [9], p. 38). B2 is clear: it says that the whole $X$ is open. The inclusion $\mathsf{ext}\,(a \downarrow b) \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b$ of B1 always holds, and the other can be written as

$$\forall x (x \Vdash a \;\&\; x \Vdash b \to (\exists c)(x \Vdash c \;\&\; \mathsf{ext}\,(c) \subseteq \mathsf{ext}\,(a) \cap \mathsf{ext}\,(b)),$$

that is, for any point $x$ lying in the basic neighbourhoods $\mathsf{ext}\,(a)$ and $\mathsf{ext}\,(b)$, there is a neighbourhood $\mathsf{ext}\,(c)$ of $x$ which is contained both in $\mathsf{ext}\,(a)$ and in $\mathsf{ext}\,(b)$.

So we have proved that the collection of subsets $\mathsf{ext}\, U \subseteq X$ $(U \subseteq S)$, where $\mathsf{ext}\, U \equiv \cup_{a\epsilon U}\, \mathsf{ext}\,(a)$, is a topology on $X$, that is, it satisfies $\mathcal{O}1 - \mathcal{O}3$, if and only if $\mathsf{ext}$ is a base, that is, it satisfies B1 and B2 above.

We have thus reached a definition of concrete space (see section 1.2) which is free of the operation $\cdot$ of formal intersection, as we wished. To help the intuition, we express B1 and B2 in the notation with $\Vdash$.

**Definition 2.1** *A concrete space is a structure $\mathcal{X} = (X, \Vdash, S)$ where*

*$X$ is a set, whose elements $x, y, z, \ldots$ are called concrete points;*

*$S$ is a set, whose elements $a, b, c, \ldots$ are called observables, or formal basic neighbourhoods;*

*$\Vdash$ is a binary relation from $X$ to $S$, called forcing, which satisfies*

$$\text{B1} \quad \frac{x \Vdash a \qquad x \Vdash b}{x \Vdash a \downarrow b} \quad \textit{for any } a, b \in S \textit{ and } x \in X;$$

$$\text{B2} \quad x \Vdash S \quad \textit{for any } x \in X.$$

This brings us easily to a new formulation of the notion of formal topology, which is obtained from definition 1.1 by suppressing $\cdot$ , 1 and their axioms, and by replacing them with the single condition (which expresses B1 in formal terms)

$$\downarrow\text{-Right} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}$$

where $U \downarrow V \equiv \{d : (\exists b \mathrel{\epsilon} U)(d \lhd \{b\})\ \&\ (\exists c \mathrel{\epsilon} V)(d \lhd \{c\})\}$. This variant on the definition (see also [29] for a similar approach) has been adopted in [7] since it allows a smoother approach to the topic of inductive generation. Note also that now both $\mathcal{P}S$ and the collection of upper subsets of a preordered set $(S, \leq)$ fall easily and naturally under the definition of formal topology. Moreover, it can be proved that for any formal topology $\mathcal{S}$ with $\downarrow$ there is a formal topology $\mathcal{S}'$ with $\cdot$ (as in section 1.3) such that $\mathcal{S}$ and $\mathcal{S}'$ produce the same frame of formal open subsets. The condition $\downarrow$-Right is present also in the new definition of formal topology which will be given in section 2.4.

These are useful technical improvements. However, the most important consequence of the analysis which led to definition 2.1 above is conceptual, rather than technical. At an impredicative reading, the above definition of concrete space is just a cumbersome formulation, but perfectly equivalent to the usual definition of topological space. Predicatively, the notion of set is much stricter, and hence many examples of spaces do not fall under definition 2.1 simply because the collection of points $X$ is not a set: this is a good reason to develop formal topology. Nevertheless, although keeping this crucial remark in mind, one can see that the framework provided by definition 2.1 is fully sufficient to define the notions of open and closed subset in a way which is perfectly acceptable also constructively. In fact, as we will see, the way to dispense with the powerset axiom and second-order quantifications is to reduce systematically to quantifications over basic neighbourhoods, that is over the set $S$. Thus the set $S$ is an essential ingredient of the definition, and it should not be forgotten, contrary to the common approach which tends to avoid any reference to bases.

The usual definition can be rephrased by saying that a subset $E \subseteq X$ is *open* if: whenever $x \mathrel{\epsilon} E$, then this is true in a continuous way, that is not only $x$, but also a whole neigbourhood of $x$ is contained in $E$. In our notation this becomes

$$x \mathrel{\epsilon} E \to (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\, a \subseteq E).$$

We put as usual $\mathsf{int}\, E \equiv \{x \in X : (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\, a \subseteq E)\}$. Such operator $\mathsf{int}$, for interior, can be thought of intuitively as a *rejector*, or *thinner*, which makes $E$ as thin as possible, that is, which throws away from $E$ all isolated points, but is unable to throw away from $E$ a whole neighbourhood $\mathsf{ext}\, a$. So $E$ is open if $E \subseteq \mathsf{int}\, E$, which is equivalent to saying that the rejector operator has no effect on $E$.

The definition of closed subset can be put in perfectly dual terms. In fact, the usual definition can be expressed by saying that $D \subseteq X$ is closed if whenever it is continuously satisfiable for $x$ to be in D, then actually $x \mathrel{\epsilon} D$. I here say that $x \mathrel{\epsilon} D$ is continuously satisfiable if any neighbourhood of $x$ touches $D$. We now can see that the notion of meet $\between$ begins to be useful. In fact, the above intuitive definition is formally expressed by

$$(\forall a \in S)(x \Vdash a \to \mathsf{ext}\, a \between D) \;\to\; x \mathrel{\epsilon} D.$$

The subset $\mathsf{cl}\, D \equiv \{x : (\forall a \in S)(x \Vdash a \to \mathsf{ext}\, a \between D)\}$ is the *closure* of $D$, and one can intuitively think of $\mathsf{cl}$ as an *attractor*, or *fattener* operator: it adds to $D$ all points $x$ which “continuously touch” $D$, in the sense that any neighbourhood of $x$ meets $D$. Note that this is a positive way of affirming that $x$ cannot be continuously separated from $D$, which would be $\neg\exists a(x \Vdash a \,\&\, \mathsf{ext}\,(a) \cap D = \emptyset)$ and which is equivalent to $\forall a(x \Vdash a \to \mathsf{ext}\,(a) \between D)$ only classically. So $D$ is closed if the attractor operator $\mathsf{cl}$ has no effect on $D$, that is, $D$ is already as big as it is consistent to be.

The notation we adopted, together with explicit expression of the logical formalism involved, allows one to see immediately the strong logical relation between interior and closure. The definition of closure is logically dual to that of interior, in the sense that $\exists$ is replaced by $\forall$, & is replaced by $\rightarrow$ (which in type theory are special cases of $\exists$ and $\forall$, respectively) and $\subseteq$ is replaced by $\between$ (whose definitions are in turn obtained one from the other by interchanging $\forall$ with $\exists$). We want to keep this duality, and actually build on it and make it clearer. Adopting classical logic here would immediately reduce it to the much simpler duality between a subset and its complement. In fact, by classical logic we would have: $D$ *closed* $\equiv$ $(\forall a \in S)(x \Vdash a \rightarrow \mathsf{ext}\, a \between D) \rightarrow x \,\epsilon\, D$ if and only if $\neg(\exists a \in S)(x \Vdash a \,\&\, \neg(\mathsf{ext}\, a \between D)) \rightarrow x \,\epsilon\, D$ if and only if $x \,\epsilon\, -D \rightarrow (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\, a \subseteq -D) \equiv -D$ *open*.

So, in the same way as classical logic reduces the meaning of existential quantification to a negation of a universal quantification, here it would reduce the definition of closed subset, which in the essence is a quantification of the form $\forall\exists$, to that of open subset, which is of the form $\exists\forall$.

An obvious remark, which however is of crucial importance for what follows, is that the conditions B1 and B2 have no role in the definitions of open and of closed subsets. Then it is worthwile to analyse the logical duality between closure and interior in the more general structure given simply by two sets $X$, $S$ and any binary relation $\Vdash$ between them. I call it a *basic pair*. Moreover, the simple remark that the notion of basic pair is perfectly self-symmetric, will lead to the discovery of the role of symmetry in topology.

## 2.2 A structure for topology

From now on, we keep the sets $X$ and $S$ always fixed, also in the sense that we think of $X$ as situated at the left, and of $S$ as situated at the right as in Picture 1 below.

$$\begin{array}{ccc} X & \Vdash & S \\ \textit{points} & & \textit{observables} \end{array}$$

Picture 1

We systematically use $x, y, \ldots$ for elements and $D, E, \ldots$ for subsets of $X$, $a, b, \ldots$ for elements and $U, V, \ldots$ for subsets of $S$. In this way we can avoid to mention the domain of quantifications, and we shall do so from now on. One can think intuitively of $x, y, \ldots$ as points and of $a, b, \ldots$ as observables (cf. [30]), so that $x \Vdash a$ means that the observable $a$ applies to the point $x$.

The relation $x \Vdash a$ is expressed at the left by the synonym $x \,\epsilon\, \mathsf{ext}\, a$, where $\mathsf{ext}\, a \equiv \{x : x \Vdash a\}$ is the extension of $a$, and at the right by the synonym $a \,\epsilon\, \Diamond x$, where $\Diamond x \equiv \{a : x \Vdash a\}$. The relation $\Vdash$ induces four monotone operators on subsets; in the language of categories, these are just functors from $\mathcal{P}X$ into $\mathcal{P}S$ or conversely, when both $\mathcal{P}X$ and $\mathcal{P}S$ are seen as preordered categories. First we define the functors $\mathsf{ext}$ and $\mathsf{rest}$ from $\mathcal{P}S$ into $\mathcal{P}X$ by setting:

$$x \,\epsilon\, \mathsf{ext}\,(U) \equiv \Diamond x \between U$$

$$x \,\epsilon\, \mathsf{rest}\,(U) \equiv \Diamond x \subseteq U$$

These are, respectively, just the definitions of weak, or existential, and of strong, or universal, anti-image of the subset $U$ along the relation $\Vdash$. The name $\mathsf{rest}$ is due to the idea of conceiving $\mathsf{rest}\, U$ as the *restriction* to those points of $X$ which live in $U$, in the sense that all their observables belong to $U$. If the relation from $X$ to $S$ is denoted more simply by $R$, or even better by a small $r$ (because we will think of it also as a function from $X$ into $\mathcal{P}S$, and not only as a binary propositional function), a good notation is $r^-$ for the weak, and $r^*$ for the strong anti-image. That is, using the notation $rx \equiv \{a : xra\}$ for the $r$-image of $x$ (which is $\Diamond x$ when $r$ is denoted by $\Vdash$), we put

$$x \,\epsilon\, r^-(U) \equiv rx \between U$$

$$x \,\epsilon\, r^*(U) \equiv rx \subseteq U$$

An important little observation, which will often be used tacitly, is that the existential anti-image is just the union of anti-images of elements, that is $\mathsf{ext}\,(U) \equiv \cup_{b \epsilon U} \mathsf{ext}\, b$; note also that this gives in particular $\mathsf{ext}\,(\{b\}) = \mathsf{ext}\, b$, and this is why we can use the same letter $\mathsf{ext}$ without confusion both for the operator on elements and for that on subsets. Note also that $r^-$ and $r^*$ coincide when $r$ is the graph of a function, because $x \between U$ if and only if $rx \subseteq U$ when $rx$ is a singleton.

The same definitions apply also to the inverse relations. So we have two functors $\Diamond$ and $\Box$ from $\mathcal{P}X$ into $\mathcal{P}S$ which are defined by[6]

$$a \,\epsilon\, \Diamond D \equiv \mathsf{ext}\, a \between D$$

$$a \,\epsilon\, \Box D \equiv \mathsf{ext}\, a \subseteq D$$

Note that, as for $\mathsf{ext}$, $\Diamond(\{b\}) = \Diamond b$. In the abstract notation with $r$, we write $r^-$ for the relation which is inverse of $r$, that is, which is defined by $ar^-x \equiv xra$ and also extend to $r^-$ the notation for images of elements, so that $r^-a \equiv \{x : xra\}$; this notation is justified since the $r^-$-image of the element $a$ coincides with the weak anti-image of the singleton subset $\{a\}$ along $r$ as defined before, that is $r^-a = r^-\{a\}$. Then we can put:

$$a \,\epsilon\, rD \equiv r^-a \between D$$

$$a \,\epsilon\, r^{-*}D \equiv r^-a \subseteq D$$

Note that the weak anti-image of $U$ along $r$, as defined before, coincides with the (direct) image of $U$ along the inverse relation $r^-$, and so both are denoted by $r^-U$. As before for $\mathsf{ext}$ and $r^-$, one can see also that $r\{x\} = rx$ and that $rD = \cup_{x \epsilon D} rx$.

The starting point of the basic picture is the discovery that the operators $\mathsf{int}$ and $\mathsf{cl}$ as defined in the preceding section are nothing but the composition of the operators just defined:

$$\mathsf{int} \equiv \mathsf{ext}\,\Box \qquad \text{and} \qquad \mathsf{cl} \equiv \mathsf{rest}\,\Diamond$$

In fact, one can easily see that $x \,\epsilon\, \mathsf{int}\, D \equiv \exists a(x \Vdash a \;\&\; \mathsf{ext}\, a \subseteq D) \equiv \Diamond x \between \Box D$ and that $x \,\epsilon\, \mathsf{cl}\, D \equiv \forall a(x \Vdash a \to \mathsf{ext}\, a \between D) \equiv \Diamond x \subseteq \Diamond D$. So one can see that the duality between $\mathsf{int}$ and $\mathsf{cl}$ is the result of a deeper duality between $\Diamond$ and $\Box$, and between $\mathsf{ext}$ and $\mathsf{rest}$.

This is a good point to repeat that the structure consisting of $X$, $\Vdash$ and $S$ is absolutely symmetric. Maybe it takes some effort to abandon the intuition of $X$ as points and of $S$ as observables, but the plain mathematical content is only that they are two sets linked by an arbitrary relation. So, in a fully symmetric way we can define two operators on $\mathcal{P}S$, which are symmetric of $\mathsf{int}$ and of $\mathsf{cl}$ respectively:

$$\mathcal{J} \equiv \Diamond\, \mathsf{rest} \qquad \text{and} \qquad \mathcal{A} \equiv \Box\, \mathsf{ext}$$

In fact, they are obtained by replacing $\mathsf{ext}$, $\Box$, $\mathsf{rest}$, $\Diamond$ with their symmetric $\Diamond$, $\mathsf{rest}$, $\Box$, $\mathsf{ext}$, respectively. The meaning of such operators[7] becomes clearer by making definitions explicit. Since $a \,\epsilon\, \mathcal{A}U \equiv \mathsf{ext}\, a \subseteq \mathsf{ext}\, U \equiv \forall x(x \Vdash a \to x \Vdash U)$, then $a \,\epsilon\, \mathcal{A}U$ means that all points lying in $a$ also

[6] Clearly, the signs $\Diamond$ and $\Box$ are taken from modal logic; if $S = X$ and $\Vdash$ is the accessibility relation, then $\Diamond D$ and $\Box D$ are the valuations of formulae $\Diamond\phi$ and $\Box\phi$, repectively, if $D$ is the valuation of $\phi$. The operators $\mathsf{ext}$ and $\mathsf{rest}$ then correspond to possibility and necessity in the past, respectively, as in temporal logic.

[7] The choice of the letter $\mathcal{J}$ is due to the fact that I had no other available, and it should not be connected with the so called $j$-operators of locale theory, see [12].

lie in $U$. So, $a \,\epsilon\, \mathcal{A}U$ is something we know already, since it expresses the intuition of the formal cover $a \lhd U$, as in section 1.2.

Let us turn to $a \,\epsilon\, \mathcal{J}U \equiv \mathsf{ext}\, a \between \mathsf{rest}\,(U)$. The explicit definition is $\exists x(x \Vdash a \,\&\, \Diamond x \subseteq U)$, which means that $a$ is inhabited by some point, about which we know in addition that all its neighbourhoods are in $U$. Informally, $a \,\epsilon\, \mathcal{J}U$ says that there is a point in $\mathsf{ext}\, a$, and $U$ gives positive information on where inside $\mathsf{ext}\, a$ it is. In the special case $U = S$, $a \,\epsilon\, \mathcal{J}S$ means simply that $\mathsf{ext}\, a$ is inhabited; we met this in section 1.2 as the intuitive explanation of the predicate $\mathsf{Pos}$.

So $a \,\epsilon\, \mathcal{J}U$ is the pointwise definition of a new formal relation between an element $a$ and a subset $U$ of $S$; we denote it by $a \ltimes U$, and call it a positivity relation. As it is evident from the preceding explanation, the idea of introducing $\mathcal{J}$ or $\ltimes$ is quite natural by *structural* reasons: symmetry, since $\mathcal{J}$ is symmetric to $\mathsf{int}$, and logical duality, since $\mathcal{J}$ is dual to $\mathcal{A}$. Whatever is the way to reach it, however, it gives a new possible choice of primitive relation on $S$, namely $\ltimes$, to be added to $\lhd$. So following the method in section 1.2 one is lead to a new definition of formal topology, with a positivity relation, see section 2.4. This is one of the main conceptual novelties of the present approach. Also, since $\mathcal{A}$ and $\mathcal{J}$ can be defined on *any* basic pair, one can apply the same method on an arbitrary basic pair and obtain a weaker notion than that of formal topology, see section 2.4. This is another important conceptual novelty.

Since the operators are defined in terms of a relation, through existential and universal quantifications, it follows that there is an adjunction between each existential operator and the universal operator in the opposite direction. So $\mathsf{ext}$ is left adjoint of $\Box$ and $\Diamond$ is left adjoint of $\mathsf{rest}$:

$$\mathsf{ext} \dashv \Box \qquad \text{that is } \mathsf{ext}\, U \subseteq D \text{ if and only if } U \subseteq \Box D, \text{ for any } D, U,$$

$$\Diamond \dashv \mathsf{rest} \qquad \text{that is } \Diamond D \subseteq U \text{ if and only if } D \subseteq \mathsf{rest}\, U, \text{ for any } D, U.$$

A formal proof is based on the equivalence between $\exists x Ax \to B$ and $\forall x(Ax \to B)$, in intuitionistic logic. In the notation with $r$, these are just the adjunctions:

$$r^- \dashv r^{-*} \qquad \text{that is } r^- U \subseteq D \text{ if and only if } U \subseteq r^{-*} D, \text{ for any } D, U,$$

$$r \dashv r^* \qquad \text{that is } rD \subseteq U \text{ if and only if } D \subseteq r^* U, \text{ for any } D, U,$$

respectively. I call these the two *fundamental adjunctions* determined by the relation $r$.

It is a general well known fact that the composition of the left adjoint (existential) after the right adjoint (universal) operator gives an interior operator. So $\mathcal{J} \equiv \Diamond\, \mathsf{rest}$ is an interior operator; this means that $\mathcal{J}$ satisfies $\mathcal{J}U \subseteq U$, $U \subseteq V \to \mathcal{J}U \subseteq \mathcal{J}V$ and $\mathcal{J}U \subseteq \mathcal{J}\mathcal{J}U$, or equivalently $\mathcal{J}U \subseteq U$ and $\mathcal{J}U \subseteq V \to \mathcal{J}U \subseteq \mathcal{J}V$. By symmetry, $\mathsf{int} \equiv \mathsf{ext}\,\Box$ also is an interior operator. Note that *i.* $\mathsf{int}$ is proved to be an interior operator on any basic pair (thus also when B1 and B2 are *not* assumed) and hence *ii.* $\mathsf{int}$ does not in general preserve finite intersections (one can prove that this is actually equivalent to B1), that is, it is *not* what is sometimes called a *topological* interior operator (see e.g. [30]).

Similarly, $\Box$ after $\mathsf{ext}$, namely $\mathcal{A}$, and of $\mathsf{rest}$ after $\Diamond$, namely $\mathsf{cl}$, are closure operators. This means that $U \subseteq \mathcal{A}U$, $U \subseteq V \to \mathcal{A}U \subseteq \mathcal{A}V$ and $\mathcal{A}\mathcal{A}U \subseteq \mathcal{A}U$ hold, or equivalently $U \subseteq \mathcal{A}U$ and $U \subseteq \mathcal{A}V \to \mathcal{A}U \subseteq \mathcal{A}V$. Similarly for $\mathsf{cl}$; of course, two remarks analogous to those on $\mathsf{int}$ apply to $\mathsf{cl}$.

For a closure operator, such as $\mathcal{A}$, we say that a subset $U \subseteq S$ is $\mathcal{A}$-saturated if $U = \mathcal{A}U$. So $D \subseteq X$ is $\mathsf{cl}$-saturated if $D = \mathsf{cl}\, D$, that is when $D$ is closed. We denote by $Sat(\mathcal{A})$, and $Sat(\mathsf{cl})$, the collection of saturated subsets.

Similarly, for an interior operator, such as $\mathcal{J}$, we say that a subset $U \subseteq S$ is $\mathcal{J}$-reduced if $U = \mathcal{J}U$. So $D \subseteq X$ is $\mathsf{int}$-reduced if $D = \mathsf{int}\, D$, that is when $D$ is open. The collections of reduced subsets are denoted by $Red(\mathcal{J})$ and $Red(\mathsf{int})$.

For any operator $\mathcal{C}$, either a closure or an interior operator, one can define suprema and infima by putting

$$\vee_{i \in I} \mathcal{C}U_i \equiv \mathcal{C}(\cup_{i \in I} \mathcal{C}U_i) \quad \text{and} \quad \wedge_{i \in I} \mathcal{C}U_i \equiv \mathcal{C}(\cap_{i \in I} \mathcal{C}U_i).$$

So $Sat(\mathcal{A})$, $Sat(\,\mathsf{cl}\,)$, $Red(\mathcal{J})$ and $Red(\,\mathsf{int}\,)$ are all complete lattices. It is not difficult to prove (by making systematic use of the two fundamental adjunctions) that actually $Red(\,\mathsf{int}\,)$ is isomorphic to $Sat(\mathcal{A})$, via the isomorphism $\Box : Red(\,\mathsf{int}\,) \to Sat(\mathcal{A})$ with inverse $\mathsf{ext} : Red(\,\mathsf{int}\,) \leftarrow Sat(\mathcal{A})$. This is why $\mathcal{A}$-saturated subsets are called *formal open*, and $\mathsf{int}$-reduced subsets, viz. open subsets of $X$, are called concrete open when there is danger of confusion.

The isomorphism between formal open and concrete open subsets was somehow expected, see the ideas in section 1.2. What come as a surprise is the fact that to be able to obtain a similar isomorphism for concrete closed subsets one has to introduce a new primitive, namely $\mathcal{J}$ or $\ltimes$, and define a subset of $S$ to be *formal closed* if it is $\mathcal{J}$-reduced.

Picture 2 sums up the situation. Note that in the top line we have two closure operators, which are of the form $\forall\exists$, while in the bottom line we have two interior operators, of the form $\exists\forall$. The choice of names is due to the fact that we want the two lattices of (concrete and formal) open subsets, and equally for closed subsets, to be isomorphic. This has the consequence that formal open subsets are described by a closure operator and formal closed subsets by an interior operator.

| | | |
|---|---|---|
| *concrete closed* | | *formal open* |
| $\mathsf{cl}\,D = D$ | *symmetric* | $\mathcal{A}U = U$ |
| $\mathsf{int}\,D = D$ | *symmetric* | $\mathcal{J}U = U$ |
| *concrete open* | | *formal closed* |

Picture 2

This concludes the first chapter of the basic picture. We are now going to see that similar structural characterizations can be obtained also for other notions of topology.

## 2.3 The essence of continuity

A common definition says that a function $f : X \to Y$ is continuous if, for any $x \in X$, whatever neighbourhood $E$ of $fx$ one considers, there is a whole neighbourhood $D$ of $x$ which is all sent "close" to $fx$, that is inside $E$. In our framework, assume $X$ and $Y$ are the sets of points of two basic pairs or concrete spaces $X \xrightarrow{\Vdash_1} S$ and $Y \xrightarrow{\Vdash_2} T$ (we will omit subscripts unless strictly necessary). Then the definition of continuity for $f$ is formally expressed by:

(1) $$\forall b(f\,x \Vdash b \to \exists a(x \Vdash a \;\&\; \forall z(z \Vdash a \to f\,z \Vdash b))$$

As it is well known, $f$ is continuous if and only if the inverse-images along $f$ of open subsets of $Y$ are open in $X$. In our framework, this means that for each $b \in T$, $f^- \,\mathsf{ext}\, b = \mathsf{ext}\,(\{a \in S : \mathsf{ext}\, a \subseteq f^- \,\mathsf{ext}\, b\})$. If we define a relation $s : S \to T$ by putting $a\,s\,b \equiv \mathsf{ext}\, a \subseteq f^- \,\mathsf{ext}\, b$, this equation means that $f^- \circ {\Vdash_2}^- = {\Vdash_1}^- \circ s^-$. But then, to restore symmetry, one is lead to generalize the treatment to a *relation* $r$ also from $X$ to $Y$. This move is of crucial importance, since it allows to make the structure underlying continuity more clearly visible, and simpler, than with functions.

Let us first find a suitable extension of (1) to relations. First, rewrite (1) as $\forall b(fx \,\epsilon\, \mathsf{ext}\, b \to \exists a$ $(x \,\epsilon\, \mathsf{ext}\, a \;\&\; \mathsf{ext}\, a \subseteq f^- \,\mathsf{ext}\, b))$. At this point, recall that $fx$ is an element, while $r\,x$ is a subset of $Y$. So we can think of $f\,x \,\epsilon\, \mathsf{ext}\, b$ as $\{fx\} \subseteq \mathsf{ext}\, b$ or as $\{fx\} \between \mathsf{ext}\, b$; the second choice works better. So we say that $r : X \to Y$ is continuous if

$$(2) \qquad r\, x \between \mathsf{ext}\, b \to \exists a(x \epsilon\ \mathsf{ext}\, a\ \&\ \mathsf{ext}\, a \subseteq r^- \,\mathsf{ext}\, b)$$

holds for any $x \in X$, $b \in T$.

An important discovery is the equivalence of the following conditions:

a. $r$ is continuous,

b. $r^-$ is open, that is $r^- \,\mathsf{ext}\, b$ is open in $X$ for any $b \in T$,

c. there exists $s : S \to T$ such that $r\, x \between \mathsf{ext}\, b \leftrightarrow \Diamond x \between s^- b$, for any $x \in X$, $b \in T$.

Note that the equivalence in *c.* is nothing but a way to express that $\Vdash \circ r = s \circ \Vdash$, that is commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Vdash} & S \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle s} \\ Y & \xrightarrow{\Vdash} & T \end{array}$$

So we define a morphism from $X \xrightarrow{\Vdash} S$ to $Y \xrightarrow{\Vdash} T$ to be a pair of relations $r : X \to Y$ and $s : S \to T$ which make the diagram commute. $(r, s)$ is called a *relation-pair*. The presence of $s$ in the definition has the purpose of keeping the information which otherwise is restored only by a quantification over relations, as in *c.* above.

Commutativity of a diagram is the clearest structural description one can find. The framework of basic pairs shows that the essence of continuity is just a commutative square. In the case of functions, we obtain the usual definition as a special case.

Several other equivalent formulations of continuity are possible. Since commutativity of the diagram is equivalently expressed by $r^- \circ \Vdash^- = \Vdash^- \circ s^-$ and $r^* \circ \Vdash^* = \Vdash^* \circ s^*$, the notion of relation-pair is equivalently presented by each of the equations

$$r^- \,\mathsf{ext}\, V = \mathsf{ext}\, s^- V \ \text{ for any } V \subseteq T,$$

$$r^* \,\mathsf{rest}\, V = \mathsf{rest}\, s^* V \ \text{ for any } V \subseteq T.$$

The first says that $r^-$ is open, and $s^-$ is a method by which we determine the open subsets of $X$ which are the existential anti-images of open subsets of $Y$ along $r$. The second says that $r^*$ is closed, and $s^*$ gives a method by which we determine the closed subsets of $X$ which are universal anti-images of closed subsets of $Y$ along $r$. In other words, $s$ gives the method by which we know that $r^-$ is open and $r^*$ is closed. Even if, given $r$, one can define a relation $s$ such that $(r, s)$ is a relation-pair by putting $asb \equiv \mathsf{ext}\, a \subseteq r^- \,\mathsf{ext}\, b$, to "forget" $s$ thinking that it can always be restored is not safe. For instance, only keeping the information $s$, some of the common equivalent characterizations of continuity, like $r^-$ is open if and only if $r^*$ is closed, can be proved constructively. If $s$ is lost, by knowing that $r^*$ is closed there is no way, not even impredicatively, neither to restore $s$, nor to prove that $r^-$ is open; in fact, two finite counterexamples in [25] show that the two conditions are no longer equivalent when the formal side is forgotten.

The category **BP** with basic pairs as objects and relation-pairs as arrows differs from the well-known category $\mathbf{Rel}^2$, of relations and commutative square diagrams, only in the fact that equality between relation-pairs is explicitly defined. Two relation-pairs $(r, s)$ and $(r', s')$ are declared to be equal if they behave in the same way with respect to open and to closed subsets, both concrete and formal. This too turns out to be equivalent to a fully structural condition, namely that their top-left bottom-right diagonals coincide,

$$
\begin{tikzcd}
X \arrow[r, "\Vdash_1"] \arrow[d, shift right, "r"'] \arrow[d, shift left, "r'"] & S \arrow[d, shift right, "s'"'] \arrow[d, shift left, "s"] \\
Y \arrow[r, "\Vdash_2"] & T
\end{tikzcd}
$$

that is $\Vdash_2 \circ r = \Vdash_2 \circ r' = s \circ \Vdash_1 = s' \circ \Vdash_1$. The category **BP** is also different from the category of boolean Chu spaces (see [20]), since morphisms of Chu spaces are functions (one in the reverse direction of the other).

Finally, the intrinsic symmetry of basic pairs and of relation-pairs is formally expressed by the fact that the functor $(\,)^-$, defined by $(X, \Vdash, S)^- \equiv (S, \Vdash^-, X)$ and by $(r, s)^- \equiv (s^-, r^-)$, is a self-duality of **BP**.

## 2.4 Basic topologies

The methodology to obtain the definition of a formal notion is always the same, and it has been described in section 1.2. The difference is that now for this task we can make use of the preceding analysis of the structure induced on a basic pair, and hence also on a concrete space. So on one hand it is easier, and on the other hand it produces a richer structure. First we introduce a new notion, namely that of basic topology, which is obtained by describing the structure induced on the formal side of a basic pair, and by taking the result as an axiomatic definition. The new definition of formal topology is then obtained simply by adding a formal condition expressing that the basic pair is actually a concrete space. Finally, the notion of formal point is obtained as an axiomatic description of the subset $\Diamond x$ determined by a concrete point $x$ on the formal side.

There are a few good reasons to do all this, that is, to study formal topology: the first is that it is a natural way, and often the only one, to be able to deal predicatively with certain spaces. After all, this is just how the real numbers are obtained from the topology of rational intervals. The second reason is that it provides more general tools to topology. A third good reason to do it is simply that it can be done, and that nice new structures emerge in this way. Thus it contributes to expand the territory of mathematical thought.

We have already seen that any basic pair $X \xrightarrow{\Vdash} S$ induces a closure operator $\mathcal{A} \equiv \Box\,\mathsf{ext}$ and an interior operator $\mathcal{J} \equiv \Diamond\,\mathsf{rest}$ on the formal side, namely on the set $S$. This is all we can say given that $\mathsf{ext}$ is left adjoint to $\Box$, and that $\Diamond$ is left adjoint to $\mathsf{rest}$, respectively. What we have to add now is a condition linking $\mathcal{A}$ with $\mathcal{J}$ and expressing the fact that the two adjunctions $\mathsf{ext} \dashv \Box$ and $\Diamond \dashv \mathsf{rest}$ are induced by the *same* relation $\Vdash$. For any $a \in S$ and $U, V \subseteq S$, the rule

$$
\frac{\mathsf{ext}\,a \between \mathsf{rest}\,V \qquad \mathsf{ext}\,a \subseteq \mathsf{ext}\,U}{\mathsf{ext}\,U \between \mathsf{rest}\,V}
$$

clearly holds. Since $\mathsf{ext}\,a \between \mathsf{rest}\,V \equiv a \,\epsilon\, \mathcal{J}V$, $\mathsf{ext}\,a \subseteq \mathsf{ext}\,U \equiv a \,\epsilon\, \mathcal{A}U$ and $\mathsf{ext}\,U \between \mathsf{rest}\,V$ iff $\exists b(b \,\epsilon\, U \;\&\; b \,\epsilon\, \mathcal{J}V)$, it says that $a \,\epsilon\, \mathcal{A}U$ and $a \,\epsilon\, \mathcal{J}V$ imply $U \between \mathcal{J}V$. Since the element $a$ does not appear in the conclusion, the conclusion is valid simply if such an element exists. So we have the rule

$$
\text{compatibility} \qquad \frac{\mathcal{A}U \between \mathcal{J}V}{U \between \mathcal{J}V}
$$

Thus the first definition is simply that a *basic topology* is a triple $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ where $S$ is a set, $\mathcal{A}$ is a closure operator, $\mathcal{J}$ is an interior operator, and they are linked by compatibility (note that compatibility is the same as the equivalence $\mathcal{A}U \between \mathcal{J}V \leftrightarrow U \between \mathcal{J}V$, since the direction $\leftarrow$ holds trivially). In the notation with $a \lhd U$ for $a \,\epsilon\, \mathcal{A}U$ and $a \ltimes V$ for $a \,\epsilon\, \mathcal{J}V$, this amounts to:

| | | | |
|---|---|---|---|
| reflexivity | $\dfrac{a \,\epsilon\, V}{a \lhd V}$ | transitivity | $\dfrac{a \lhd U \qquad U \lhd V}{a \lhd V}$ |
| coreflexivity | $\dfrac{a \ltimes V}{a \,\epsilon\, V}$ | cotransitivity | $\dfrac{a \ltimes U \qquad (\forall b)(b \ltimes U \to b \,\epsilon\, V)}{a \ltimes V}$ |
| compatibility | $\dfrac{a \ltimes V \quad a \lhd U}{U \ltimes V}$ | | |

where we now add the shorthand $U \ltimes V$ for $(\exists b \,\epsilon\, U)(b \ltimes V)$. It is just natural to carry over the terminology from basic pairs and say that $U$ is formal closed if $U = \mathcal{J}U$ and formal open if $U = \mathcal{A}U$.

The intuitive meaning of compatibility is that any formal closed subset $V = \mathcal{J}V$ must split any cover, in the sense that if $a \lhd U$ and if $a \,\epsilon\, V$, then $V$ must proceed and meet $U$. This is nothing but the symmetric of the usual condition defining the concrete closure. To see this, first note that, if we apply the same methodology to the concrete side, since a basic pair is fully symmetric we obtain a fully symmetric definition: a triple $(X, \mathsf{int}, \mathsf{cl})$ where $X$ is a set, $\mathsf{int}$ is an interior operator, $\mathsf{cl}$ is a closure operator, and they are linked by

$$\text{compatibility} \qquad \mathsf{cl}\, D \between \mathsf{int}\, E \leftrightarrow D \between \mathsf{int}\, E.$$

The role of compatibility perhaps gets clearer by examining its link with the impredicative definition of closure. In fact, if one defines closure $\mathsf{CL}_{\mathsf{int}}$ (impredicatively) in terms of all open subsets, rather than subbasic neighbourhoods (as in section 2.1), one puts:

$$x \,\epsilon\, \mathsf{CL}_{\mathsf{int}}(D) \equiv \forall E (x \,\epsilon\, \mathsf{int}\, E \to D \between \mathsf{int}\, E)$$

(here for convenience we quantify over an arbitrary subset $E$, and obtain the same effect as quantifying over open subsets by prefixing $E$ with $\mathsf{int}$ inside the quantifier). Then one can easily see that compatibility is equivalent to: $\mathsf{cl}\, D \subseteq \mathsf{CL}_{\mathsf{int}}\, D$ for every $D \subseteq X$. One can say that a basic topology is *dense* if the equation $\mathsf{cl} = \mathsf{CL}_{\mathsf{int}}$ holds. This equation is not part of the definition of basic topology for two reasons: 1. the definition of $\mathsf{CL}_{\mathsf{int}}$ is impredicative, 2. even admitting impredicative reasoning, the equation $\mathsf{cl} = \mathsf{CL}_{\mathsf{int}}$ is *not* preserved under morphisms of basic topologies, viz. continuous relations.

The definition of basic topology is very simple, and should find its place together with other definitions weaker than that of topological space which were given long ago by Kuratowski, Frechet, Čech, and others. Its peculiarity is that it has a purely structural justification, and that it is meaningful only by assuming intuitionistic logic and a primitive notion of closed subset.

It is to be noted, however, that also assuming classical reasoning, the definition of basic topology does *not* collapse. Given any interior operator $\mathsf{int}$ on a set $X$, one can classically prove that $(X, \mathsf{int}, -\mathsf{int}-)$ is a basic topology (actually, it is a dense basic topology). But again, also reasoning classically, not all basic topologies are of this form. Compatibility is classically equivalent to $\mathsf{cl} \subseteq -\mathsf{int}-$, but even classically not all basic topologies satisfy the equality $\mathsf{cl} = -\mathsf{int}-$. One can prove that a basic topology $(X, \mathsf{int}, \mathsf{cl})$ is classically the same as a structure $(X, \mathsf{int}, \mathsf{int}')$, where $\mathsf{int} \subseteq \mathsf{int}'$.

The fact that the definition of basic topology is not too weak is confirmed by some initial results on the structure of possible combinations of the operators $\mathsf{int}$, $\mathsf{cl}$ and opposite $-$. First, one can easily prove that the different combinations of $\mathsf{int}$ and $\mathsf{cl}$ are exactly seven, and that the mutual inclusions are only[8] those shown in the picture (in which inclusion appears as an edge upwards):

[8] The method to find counterexamples for the other inclusions is interesting: one can choose a suitable basic pair, and use the logical expressions for $\mathsf{int}$ and $\mathsf{cl}$ to show that they would give some implications which are not valid intuitionistically.

$$
\begin{tikzcd}[column sep=small]
 & & \mathsf{cl} \arrow[dl, dash] \arrow[ddr, dash] & \\
 & \mathsf{cl\ int\ cl} \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathsf{int\ cl} \arrow[dr, dash] & & \mathsf{cl\ int} \arrow[dl, dash] & \mathit{identity} \arrow[ddl, dash] \\
 & \mathsf{int\ cl\ int} \arrow[dr, dash] & & \\
 & & \mathsf{int} &
\end{tikzcd}
$$

Adding the equation $\mathsf{cl} = -\mathsf{int}-$ and classical logic, one can then easily obtain the well known result by Kuratowski telling that there are at most 14 different combinations of $-$, $\mathsf{int}$ and $\mathsf{cl}$.

In the general case, from compatibility one can obtain that $\mathsf{cl}\,D \cap \mathsf{int}\,E = \emptyset \leftrightarrow D \cap \mathsf{int}\,E = \emptyset$, and using this one can derive that the equations linking $-$, $\mathsf{int}$ and $\mathsf{cl}$ are:

$$
\begin{aligned}
\mathsf{cl}\,-D \;&\subseteq -\mathsf{int}\,D = \mathsf{cl}\,-\mathsf{int}\,D \\
\mathsf{int}\,-\mathsf{cl}\,D = \mathsf{int}\,-D \;&\subseteq -\mathsf{cl}\,D
\end{aligned}
$$

It is easy to find out that the inclusions $-\mathsf{int}\,D \subseteq \mathsf{cl}\,-D$ and $-\mathsf{cl}\,D \subseteq \mathsf{int}\,-D$ do not hold in general. The above equations seem to express the basic properties of closure, interior and opposite in the intuitionistic case. However, it is still not known whether other inclusions or equations involving more occurrences of $-$, $\mathsf{int}$ and $\mathsf{cl}$ hold. An initial study has shown that all the different combinations with only one occurrence of $-$ do not exceed the number of 22. With two occurrences of $-$, the number seems to get much higher. In general, it is apparently still an open problem even to decide whether the total number of combinations is finite or infinite.

## 2.5 Formal topologies and formal spaces

It must be emphasized that all definitions and results so far have been obtained starting from an arbitrary basic pair. It is now a relatively easy matter to find a formal condition corresponding to the property we called B1, that is $\mathsf{ext}\,U \cap \mathsf{ext}\,V = \mathsf{ext}\,(U \downarrow V)$. In fact, if we express it in the equivalent form $\forall x(x \Vdash U \;\&\; x \Vdash V \rightarrow x \Vdash U \downarrow V)$ we see that, by replacing an arbitrary concrete point $x$ with an arbitrary observable $a$ and the relation $\Vdash$ with the cover $\lhd$, we obtain $\forall a(a \lhd U \;\&\; a \lhd V \rightarrow a \lhd U \downarrow V)$. This is the same as the rule

$$
\downarrow\text{-Right} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}
$$

Note that the formal expression of $\mathsf{ext}\,b \cap \mathsf{ext}\,c = \mathsf{ext}\,(b \downarrow c)$, which by distributivity is equivalent to $\mathsf{ext}\,U \cap \mathsf{ext}\,V = \mathsf{ext}\,(U \downarrow V)$, would bring to $a \lhd b \;\&\; a \lhd c \rightarrow a \lhd b \downarrow c$, which is trivial since $a \lhd b \;\&\; a \lhd c$ gives $a \,\epsilon\, b \downarrow c$ by definition. In fact, the purpose of $\downarrow$-Right is exactly to express distributivity formally, and that is why we must start from $\mathsf{ext}\,U \cap \mathsf{ext}\,V = \mathsf{ext}\,(U \downarrow V)$.

In this way we have obtained yet another definition of formal topology, simply as a basic topology in which $\downarrow$-Right holds. To distinguish it from that given in section 2.1, one could call it a *convergent basic topology* or a *balanced formal topology*, because the difference is the presence of a positivity relation and the absence of the positivity axiom (see section 1.3).

As I hinted at in section 1.3, the variety of possible definitions is a richness which one should not be afraid of. In fact, at this stage of development it is hard to see which one will become the standard one. The different requests on the positivity predicate $\ltimes$ seem to be the analogue of different separation principles in pointwise topology. Like in pointwise topology, it will take time to find out virtues and defects of each assumption.

Some of the advantages of the definition given above are already clearly visible. The first is that it has a solid structural motivation. In fact, the new predicate $\ltimes$ is the result of the isomorphism between concrete closed and formal closed subsets, and at the same time it is the symmetric of the interior operator $\mathsf{int}$ and the dual of the operator $\mathcal{A}$, that is of the cover $\lhd$. So $\ltimes$ seems to be exactly what is necessary to make the definition fully balanced.

The second is that in this way it allows to introduce a natural notion of formal closed subset; recall that a subset $U$ is said to be formal closed if $\mathcal{J}U = U$ or equivalently if $a \,\epsilon\, U \to a \ltimes U$.

The third is that the richness of the structure allows to see that it is better to get rid of the condition called positivity (and study it as an extra assumption, if wished). In this way one can obtain both the theory of locales (or frames) and the previous version of formal topology as special cases. In fact, we say that $\mathcal{J}$ is improper when $\mathcal{J}U = \emptyset$ for any $U$; then one is left essentially only with the cover $\lhd$, which amounts to a predicative formulation of frames. We say that $\mathcal{J}$ is trivial when $\mathcal{J}S$ and $\emptyset$ are the only two formal closed subsets. One can prove constructively that $\mathcal{J}$ is trivial exactly when it satisfies $a \,\epsilon\, \mathcal{J}U \to a \,\epsilon\, H \;\&\; H \subseteq U$ for some monotone subset $H$ ($H$ is monotone if $a \,\epsilon\, H \;\&\; a \lhd U \to H \between U$; when $\mathcal{J}$ is trivial, put $H \equiv \{a : a \ltimes S\}$). So a formal topology in the sense of definition 1.1 is obtained as a special case by defining $\mathsf{Pos}$ to be $H$ and by requiring the condition of positivity.

I expect also other advantages, or applications, to become visible after learning how the new expressive power - due to the presence of $\ltimes$ - can be exploited. Before that, one has to adjust all the definitions and results of formal topology to take care also of the positivity relation $\ltimes$. This is not a routine task. As an example, I give here the new definition of formal point. Another example is given in section 2.10.

The definition of formal point of a formal topology $\mathcal{S} = (S, \lhd, \ltimes)$ is obtained as usual by considering the case in which $\mathcal{S}$ is presentable. So assume that $\mathcal{S}$ is the structure induced by a concrete space $(X, \Vdash, S)$ on the set $S$. The idea is first to describe the formal properties of a subset $\Diamond x$ traced on $\mathcal{S}$ by a concrete point $x$, and take them as abstract conditions for a subset $\alpha \subseteq S$ to be called a formal point. Recalling that $\downarrow$, $\lhd$ and $\ltimes$ in the presentable case are *defined* by means of concrete points, we see that the properties we need are simply

$$\frac{x \Vdash a \quad x \Vdash b}{x \Vdash a \downarrow b} \qquad \frac{x \Vdash a \quad a \lhd U}{x \Vdash U} \qquad \frac{x \Vdash a \quad \Diamond x \subseteq U}{a \ltimes U}\,.$$

The first says that $(X, \Vdash, S)$ satisfies B1, the second and third are just a re-formulation of the definitions $a \lhd U \equiv \mathsf{ext}\, a \subseteq \mathsf{ext}\, U$ and $a \ltimes U \equiv \mathsf{ext}\, a \between \mathsf{rest}\, U$. We also add $\exists b(x \Vdash b)$, which corresponds to B2. Now we can transform such properties into properties of $\Diamond x$ by writing $a \,\epsilon\, \Diamond x$ in place of $x \Vdash a$, and of course $\Diamond x \between U$ in place of $x \Vdash U$, and then take them as properties of an arbitrary $\alpha \subseteq S$. But if we now write $\alpha \Vdash a$ for $a \,\epsilon\, \alpha$, we see that the definition we look for is obtained by literally replacing $\alpha$ for $x$ in the properties above. So we have that $\alpha \subseteq S$ is a formal point if

$$\begin{array}{lll}
\alpha \text{ is inhabited:} & \alpha \between S & \\[2ex]
\alpha \text{ is convergent:} & \dfrac{\alpha \Vdash a \quad \alpha \Vdash b}{\alpha \Vdash a \downarrow b} & \\[2ex]
\alpha \text{ splits } \lhd\text{:} & \dfrac{\alpha \Vdash a \quad a \lhd U}{\alpha \Vdash U} & (\text{where } \alpha \Vdash U \equiv \alpha \between U) \\[2ex]
\alpha \text{ enters } \ltimes\text{:} & \dfrac{\alpha \Vdash a \quad \alpha \subseteq U}{a \ltimes U} &
\end{array}$$

The condition that $\alpha$ splits $\lhd$ is actually redundant (in fact, it can be deduced from $\alpha$ enters $\mathsf{Pos}$ and compatibility), but it helps to see that, when $\ltimes$ is trivial, the above definition gives back the definition of formal point previously given in [21].

As a last remark, note that the definition becomes much shorter in the notation with $\mathcal{A}$, $\mathcal{J}$ and $\between$. I leave it to readers to check that it is equivalent to $\alpha \between S$, $\alpha \between U \;\&\; \alpha \between V \to \alpha \between U \downarrow V$, $\alpha \between \mathcal{A}U \to \alpha \between U$, $\alpha \between U \;\&\; \alpha \subseteq F \to U \between \mathcal{J}F$. Or, even more simply, a subset $\alpha$ is a formal point if it is inhabited, convergent and formal closed.

## 2.6 Formal continuity and convergence

A notion of morphisms between basic topologies is introduced by following the same methodology which led us to the notion of basic topology. That is, we consider the notion of morphism between basic pairs, alias relation-pair, and we look for the properties which are enjoyed by its component on the formal side, with respect to formal open and formal closed subsets. These will be the properties we require to characterize morphisms between basic topologies.

It can be shown that a relation-pair $(r, s)$ is equivalently presented by each of the following two properties, symmetric to those mentioned in section 2.3:

$$s\Diamond D = \Diamond r\, D \text{ for any } D \subseteq X,$$

which means that $s$ is *formal closed*, and $r$ is a method to determine the formal closed subsets of $T$, which are the existential image of formal closed subsets of $S$ along $s$;

$$s^{-*}\Box D = \Box r^{-*} D, \text{ for any } D \subseteq X,$$

which means that $s^{-*}$ is *formal open*, and $r^{-*}$ is a method to determine the formal open subsets, which are the universal image of formal open subsets of $S$ along $s$.

When only the formal side is considered, the relations $r$ and $r^{-*}$ are lost, and the properties characterizing morphisms between basic topologies are then just the properties enjoyed by $s$. However, once $r$ and $r^{-*}$ are forgotten, it is no longer possible to prove the two conditions, that $s$ is formal closed and that $s^{-*}$ is formal open, to be equivalent to each other (counterexamples are given in [25]). Hence both of them are required.

Thus a morphism between two basic topologies $\mathcal{S} \equiv (S, \mathcal{A}, \mathcal{J})$ and $\mathcal{T} \equiv (T, \mathcal{B}, \mathcal{H})$ is a relation $s : S \to T$ such that

*i.* $s$ is formal closed, that is $U = \mathcal{J}U \to s\,U = \mathcal{H}s\,U$,

*ii.* $s^{-*}$ is formal open, that is $U = \mathcal{A}U \to s^{-*}U = \mathcal{B}s^{-*}U$.

We call it a *continuous relation*, and we denote it by $s : \mathcal{S} \to \mathcal{T}$. One can prove that, in the notation with $\ltimes$ and $\lhd$, the two conditions on $s$ are equivalent to:

$$\frac{a\,s\,b \quad b \lhd V}{a \lhd s^{-}V} \qquad\qquad \frac{a\,s\,b \quad a \ltimes s^{*}V}{b \ltimes V}$$

Several other equivalent characterizations are also possible.

Given any basic topology $\mathcal{S}$, one can always define the image of $\mathcal{S}$ along any relation $s : S \to T$, by setting $s\mathcal{S} \equiv \mathcal{T}' \equiv (T, s^{-*}\mathcal{A}s^{-}, s\mathcal{J}s^{*})$. This is a basic topology in which formal open subsets are just the universal images of formal open subsets of $\mathcal{S}$, and dually formal closed subsets are just the existential images of formal closed subsets of $\mathcal{S}$. It is the coarsest basic topology which makes $s$ a continuous relation.

Following this definition of image, it can happen that $\mathcal{S}$ satisfies $\downarrow$-Right, while its image $\mathcal{T}'$ does not. So the notion of continuous relation is not the right notion of morphism between formal topologies. As the notion of formal topology was obtained by describing axiomatically the formal side of a concrete space, that is a basic pair satisfying B1 and B2, now the correct definition of morphism between formal topologies is obtained by describing axiomatically the right component of a relation-pair which preserves the validity of B1 and B2.

So assume that $\mathcal{S}$ is the formal topology which is presented by a concrete space $\mathcal{X} = (X, \Vdash, S)$. One can see that the image of $\mathcal{S}$ along a relation $s : S \to T$ is the same thing as the basic topology presented by the basic pair $(X, s \circ \Vdash, T)$. Since $(s \circ \Vdash)^{-} = \mathsf{ext}\, s^{-}$, this satisfies B1 and B2 if $\mathsf{ext}\, s^{-}T = X$ and $\mathsf{ext}\, s^{-}b \cap \mathsf{ext}\, s^{-}d = \mathsf{ext}\, s^{-}(b \downarrow d)$, for any $b, d \in T$. But then, since $\mathcal{X}$ satisfies B1 and B2 (that is $\mathsf{ext}\, S = X$ and $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\,(U \downarrow V)$), these two equations are equivalent to $\mathsf{ext}\, s^{-}T = \mathsf{ext}\, S$ and $\mathsf{ext}\,(s^{-}b \downarrow s^{-}d) = \mathsf{ext}\, s^{-}(b \downarrow d)$, and hence finally, by the isomorphism

$Sat(\mathcal{A}) \cong Red(\mathsf{int})$, also to $\mathcal{A}s^- T = \mathcal{A}S$ and $\mathcal{A}(s^- b \downarrow s^- d) = \mathcal{A}s^-(b \downarrow d)$, for any $b, d \in T$. In the notation with $\lhd$, these are equivalent to

$$\text{totality} \quad S \lhd s^- T \qquad\qquad \text{convergence} \quad s^- b \downarrow s^- d \lhd s^-(b \downarrow d)$$

So a morphism between formal topologies is defined to be a continuous relation which satisfies totality and convergence; it is called a *formal map*. Now one can easily check that the image $\mathcal{T}' = s\mathcal{S}$ of a formal topology $\mathcal{S}$ along a formal map is a formal topology too.

The notion of morphism between formal topologies presented in [21] is easily seen to be a special case of formal map. It is important to observe that the conditions of totality and of convergence are automatically satisfied by a relation $s$ when it is the right component of a relation pair $(f, s)$, where $f : X \to Y$ is a function and $(X, \Vdash, S)$, $(Y, \Vdash, T)$ are concrete spaces (the proof is left to readers). This shows that, when only functions are considered, the notion of continuity includes that of convergence.

The reason motivating the name of formal maps is that they induce functions between the formal spaces determined by the formal topologies; actually, they are the predicative way to present such maps. In fact, it is routine to check that whenever $s : \mathcal{S} \to \mathcal{T}$ is a formal map between formal topologies and $\alpha$ is a formal point of $\mathcal{S}$, then the image $s\alpha$ of the subset $\alpha$ along $s$ is a formal point of $\mathcal{T}$. Hence $s$ induces a map between $Pt(\mathcal{S})$ and $Pt(\mathcal{T})$.

Actually also the converse holds (confirm the correctness of definition of formal maps): one can see that under suitable conditions (that is, when $\mathcal{S}$ and $\mathcal{T}$ are spatial), the mapping $\alpha \mapsto s\alpha$ is a function from $Pt(\mathcal{S})$ into $Pt(\mathcal{T})$ if and only if the relation $s : S \to T$ is a formal map.

## 2.7 Generating positivity by coinduction

It has been shown in [7] that the most general way to generate a formal cover on a set $S$ is to start from a family of sets $I(a)$ *set* $(a \in S)$ and a family of subsets $C(a, i) \subseteq S\,(a \in S, i \in I(a))$. The intuition is that $I(a)$ is a set of indexes for the covers of $a$, and that $C(a, i)$ is the cover of $a$ with index $i$, taken as an axiom. Then a cover $\lhd$ (I mean, $\lhd$ reflexive and transitive) is generated inductively simply by the rules (see [7])

$$\text{(3)} \qquad \frac{a \,\epsilon\, U}{a \lhd U} \qquad \frac{i \in I(a) \quad C(a, i) \lhd U}{a \lhd U}\,.$$

The new idea now is to generate the largest predicate $\ltimes$ compatible with $\lhd$ by coinduction, that is by forcing compatibility to hold by successively taking away elements which do not satisfy it. Given that $\lhd$ is generated from axioms $a \lhd C(a, i)$, to force compatibility it is enough to consider this case. And of course one must also force reflexivity to hold. So the rules are

$$\frac{a \ltimes U}{a \,\epsilon\, U} \qquad \frac{i \in I(a) \quad a \ltimes U}{C(a, i) \ltimes U}$$

The relation $\lhd$ is the minimal relation satisfying the rules written in (3). This means that for every subset $U$, the subset $\mathcal{A}U \equiv \{a \in S : a \lhd U\}$ is the least among the subsets $P$ satisfying $U \subseteq P$ and $C(b, i) \subseteq P \to b \,\epsilon\, P$ for any $b \in S$ and $i \in I(b)$. In other terms, the following principle of induction holds:

$$\frac{a \lhd U \quad U \subseteq P \quad \begin{array}{c}[i \in I(b), C(b, i) \subseteq P] \\ | \\ b \,\epsilon\, P\end{array}}{a \,\epsilon\, P}$$

Dually, for every $U$ the subset $\mathcal{J}U \equiv \{a \in S : a \ltimes U\}$ is the largest among subsets $Q$ such that $Q \subseteq U$ and $b \,\epsilon\, Q \to C(b, i) \between Q$ for any $b \in S$ and $i \in I(b)$. So the following principle of coinduction holds:

$$\frac{a \,\epsilon\, Q \quad Q \subseteq U \quad \begin{array}{c}[b \,\epsilon\, Q, i \in I(b)] \\ | \\ C(b, i) \between Q\end{array}}{a \ltimes U}$$

Using these two principles, it is possible to prove that $(S, \lhd, \ltimes)$ is indeed a basic topology. By combining this with the treatment of $\downarrow$-Right in [7], one can also easily generate convergent basic topologies. This shows at least that there is a wealth of examples for the new definitions.

Moreover, there is a wealth of examples also of continuous relations. In fact, assume that $\mathcal{S}$ is generated as above by $I$, $C$ and that $\mathcal{T}$ is similarly generated by $J(b)$ *set* $(b \in T)$ and $D(b, j) \subseteq T\,(b \in T, j \in J(b))$. Assume that $s : S \to T$ is any relation respecting the axioms, that is satisfying $s^- b \lhd_{\mathcal{S}} s^- D(b, j)$ for any $b \in T$ and $j \in J(b)$. Then one can prove by induction that $s^{-*}$ is formal open and by coinduction that $s$ is formal closed.

The idea of a coinductive generation of $\ltimes$ first came to Martin-Löf, in July 1996 soon after several conversations by the author on the basic picture and in particular on the $\forall\exists$-$\exists\forall$ duality between open and closed subsets. The joint paper [17] includes also a game theoretic interpretation of $\lhd$ and $\ltimes$. Valentini [32] has later shown that one can force $\ltimes$ to satisfy any given axioms, fully independently of the axioms for $\lhd$. This shows that there is a wealth of examples in which formal closed subsets are by no means determined by formal open ones.

## 2.8 Overlap algebras and the algebraization of topology

The algebraic structure traditionally associated with a topological space is that of its open subsets, which form a complete lattice satisfying infinite distributivity (of arbitrary joins over binary meets). This is called frame or locale, according to the direction of arrows (see [12]). Due to the topological interpretation of intuitionistic logic (propositions as open subsets), the structure of locale is also the intuitionistic algebraic counterpart of the structure of the power of a set; in this context, it is often called a complete Heyting algebra, to stress the presence of implication (which is anyway impredicatively definable in any locale).

The discovery and development of the basic picture has shown that closed subsets should be defined and treated independently of open subsets, and not just as their complement. To be able to reflect into an algebraic definition also the presence of a primitive notion of closed subsets, the notion of locale has to be modified. Then there are apparently two main possible choices to express topological notions algebraically: either we assume, as it is usually done, that the right structure associated to the intuitionistic notion of the power of a set is that of locale, or we prepare ourselves to change that too.

In the first case, the algebraic version of the definition of basic topology is that of a frame, or better of a locale because of the direction of arrows, equipped both with a saturation and with a reduction operator, corresponding to $\mathcal{A}$ and to $\mathcal{J}$, respectively, in a basic topology. The problem with this notion is that one cannot express the link between $\mathcal{A}$ and $\mathcal{J}$ which is assumed to hold in a basic topology, namely compatibility, since locales lack any notion corresponding to overlap $\between$. In a locale, one can express only the equational, or negative, version of compatibility, which is equivalent to proper compatibility only classically.

More interesting is the second alternative. To express compatibility in full form, using only algebraic notions, one has to add an algebraic counterpart of overlap $\between$ between subsets. This brings to a new algebraic description of the structure of the power of a set $\mathcal{P}S$, which is obtained from the notion of locale by adding a new primitive $\gtrless$, corresponding to $\between$ (just like $\leq$ corresponds to $\subseteq$; the shape of the new sign $\gtrless$ should recall this). The problem then is to find the right axioms to be assumed for $\gtrless$. My proposal is to define an algebra of subsets with overlap, briefly overlap algebra, or o-algebra, to be a locale $(\mathcal{P}, \leq, \wedge, \vee, 0, 1)$ equipped with an extra relation $\gtrless$ defined on every element $p, q : \mathcal{P}$ which satisfies:

symmetry  if $p \gtrless q$ then $q \gtrless p$,  for every $p, q$ in $\mathcal{P}$;

and which is connected with the other components by the following conditions:

preservation of meet: if $p \gtrless q$ then $p \gtrless p \wedge q$

splitting of join: $p \gtrless \bigvee_{i \in I} q_i$  iff  there exists $i \in I$ such that $p \gtrless q_i$;

$$\text{density}\qquad \frac{\begin{array}{c} q \ggreater r \\ | \\ p \ggreater r \end{array}}{p \leq q} \text{ (where } r \text{ must be an arbitrary element of } \mathcal{P}\text{).}$$

It is also interesting to study structures in which some conditions, like density or distributivity (of the underlying locale) are not assumed.

The definition of overlap algebra is not vacuous because clearly for any set $X$ the structure $(\mathcal{P}X, \subseteq, \between, \cup, \cap, \emptyset, X)$ is an example, and actually the motivating example. It can be shown shown that atomic o-algebras are exactly the same as powers of sets. Still, the notion of o-algebra is more general than that of $\mathcal{P}S$ and there are very natural examples of nonatomic overlap algebras. In particular, global H-subsets of a set form a non-atomic overlap algebra.

The presence of the algebraic overlap $\ggreater$ allows to express compatibility in full. It is natural to define an o-basic topology, that is a basic topology over an arbitrary o-algebra, to be an overlap algebra $\mathcal{P}$ with a saturation operator $\mathsf{a}$ and a reduction operator $\mathsf{j}$ linked by compatibility:

$$\mathsf{a}\, p \ggreater \mathsf{j}\, q \text{ if and only if } p \ggreater \mathsf{j}\, q.$$

A basic topology then becomes exactly the same as an o-basic topology over the specific o-algebra $\mathcal{P}S$.

Also all other notions of the basic picture can be similarly generalized. A systematic way to do this is to repeat step by step the development of the basic picture, starting from a generalization of basic pairs in which arbitrary o-algebras $\mathcal{P}$ and $\mathcal{Q}$ replace the power of sets $\mathcal{P}X$ and $\mathcal{P}S$, respectively. The first obstacle (and essentially the only one) is to define the algebraic counterpart of the forcing relation $\Vdash$ between $X$ and $S$. In other terms, the problem is to find an algebraic characterization of relations. This problem is solved by exploiting the presence of $\ggreater$.

We have seen that the operators on subsets $\mathsf{ext}$, $\mathsf{rest}$, $\Diamond, \Box$ form two pairs of adjoint functors, $\mathsf{ext} \dashv \Box$ and $\Diamond \dashv \mathsf{rest}$. But if we consider two arbitrary pairs of adjoint functors $F \dashv G$ and $F' \dashv G'$, with $F, G' : \mathcal{P}X \to \mathcal{P}S$ and $F', G : \mathcal{P}X \leftarrow \mathcal{P}S$, nothing is said about the fact that they are induced by the *same* relation, or better by a relation and by its converse.

It is well known that any adjunction $F \dashv G$ between $\mathcal{P}X$ and $\mathcal{P}S$ is induced by a relation between $X$ and $S$. In fact, the left adjoint functor $F$ respects unions, and hence one can define $r : X \to S$ by putting $x\, r\, a \equiv a \;\epsilon\; F\{x\}$ and then obtain that $F(D) = \bigcup_{x \epsilon D} F\{x\} \equiv \bigcup_{x \epsilon D} r\, x \equiv r(D)$. So $F$ is the same as the existential image along $r$. Since $r \dashv r^*$, by the uniqueness of adjoints $G = r^*$ holds. Assuming a second pair $F' \dashv G'$ of adjoint functors in the opposite direction tells us that there is a second relation $r' : X \leftarrow S$ such that $F' = r'$ and $G' = r'^*$. So now the problem is: which condition should one add to characterize abstractly the fact that $r' = r^-$? The answer is, a posteriori, incredibly simple: $r' = r^-$ holds if and only if the two existential functors $F$ and $F'$ are linked by

(4) $\qquad F(D) \between U$ if and only if $D \between F'(U)$, for any $D \subseteq X$, $U \subseteq S$.

In fact, if $r' = r^-$ then (4) holds, because for an arbitrary $r$ we have

(5) $\qquad r(D) \between U$ if and only if $D \between r^-(U)$, for any $D \subseteq X$, $U \subseteq S$.

(this is easily checked by intuitionistic logic). Conversely, if $F = r$, $F' = r'$ and (4) holds, then for every $x \in X$ and $a \in S$ we have $x \;\epsilon\; r'a$ iff $\{x\} \between r'a$ iff $r\, x \between \{a\}$ iff $\{x\} \between r^- a$ iff $x \;\epsilon\; r^- a$, that is $r' = r^-$. I call (5) the *fundamental symmetry*, and say that two functors $F$, $F'$ satisfying (4) are *symmetric*, written $F {\cdot}|{\cdot} F'$.

It is thus proved that four functors $F, G' : \mathcal{P}X \to \mathcal{P}S$ and $F', G : \mathcal{P}X \leftarrow \mathcal{P}S$ are those induced by a relation $r : X \to S$, that is $F = r$, $G = r^*$, $F' = r^-$, $G' = r^{-*}$, if and only if they form two adjoint pairs $F \dashv G$, $F' \dashv G'$ and, moreover, $F$ is symmetric to $F'$, $F {\cdot}|{\cdot} F'$. This I suggest to call a *symmetric pair of adjunctions*.

The algebraic counterpart of relations is therefore the expression in the language of o-algebras of the characterization of a relation as a symmetric pair of adjunctions. Thus we say that an *o-relation* from the o-algebra $\mathcal{P}$ into the o-algebra $\mathcal{Q}$ is a quadruple of functions $F = \langle f, f^-, f^*, f^{-*} \rangle$, where $f, f^{-*} : \mathcal{P} \to \mathcal{Q}$ and $f^-, f^* : \mathcal{Q} \to \mathcal{P}$, such that:

$f$ is left adjoint to $f^*$, written $f \dashv f^*$, that is $fp \leq q$ iff $p \leq f^*q$;

$f^-$ is left adjoint to $f^{-*}$, written$f^- \dashv f^{-*}$, that is $f^-q \leq p$ iff $q \leq f^{-*}p$;

$f$ is symmetric to $f^-$, written $f{\cdot}|{\cdot}f^-$, that is $fp \between q$ iff $p \between f^-q$.

Then an *o-basic pair* is given by two o-algebras $\mathcal{P}$ and $\mathcal{Q}$ and an o-relation $F : \mathcal{P} \to \mathcal{Q}$. between them.

One can now see that the definition o-basic topology is just the result of transferring the structure of an o-algebra $\mathcal{P}$ into a second o-algebra $\mathcal{Q}$ through an o-relation $F : \mathcal{P} \to \mathcal{Q}$. In fact, putting $\mathsf{a} \equiv f^{-*}f^-$ and $\mathsf{j} \equiv ff^*$ one can easily prove (as with any adjunction) that $\mathsf{a}$ is a saturation operator and $\mathsf{j}$ is a reduction operator. The novelty is compatibility, and here is its proof:

$$\mathsf{a}\,p \between \mathsf{j}\,q \equiv f^{-*}f^-p \between ff^*q \quad \text{by definition of } \mathsf{a} \text{ and } \mathsf{j}\,;$$

$$\text{iff } f^-f^{-*}f^-p \between f^*q \quad \text{because } f{\cdot}|{\cdot}f^-;$$

$$\text{iff } f^-p \between f^*q \quad \text{because } f^-f^{-*}f^- = f^-;$$

$$\text{iff } p \between ff^*q \quad \text{because } f{\cdot}|{\cdot}f^-;$$

$$\equiv p \between \mathsf{j}\,q \quad \text{by definition of } \mathsf{j}\,.$$

Following these ideas, one can show that most of the definitions of the basic picture and formal topology can be expressed in a purely algebraic language and that the axioms of overlap algebras are sufficient to prove the main results about them. This is a straightforward but quite instructive exercise. For instance, the operation $\downarrow$ is defined in a basic topology by means of singletons. So, unless the underlying o-algebra is atomic, to express convergence in an o-basic topology (by assuming $\downarrow$-right) one needs an extra primitive. In other words, the notion of o-formal topology needs a further primitive for convergence.

Thus one reaches an "algebraization" (and generalization) of the basic picture, which apparently is not possible using locales. The source of the richer expressive power is the presence of the new primitive $\between$, corresponding to the existential statement of overlap between two subsets.

This fact has a clear foundational interest, since now the notion of overlap algebra can help and give a mathematical (rather than ideological) answer to a common question about predicativism. A typical question of somebody acquainting with the matter is: what am I allowed to use (of the common classical or impredicative methods) in developing predicative mathematics, topology in particular? Now a partial answer can be: all what can be transcribed into the theory of overlap algebras.

## 2.9 From completeness to invariance

I have already noticed several times that the method to obtain the definition of a formal notion is that of taking as formal axioms all the relevant properties which hold in the presentable case. It is now time to analyse this more carefully. The main problems are: what does it mean to take *all* properties? how can one be sure that there are no other? And in any case, which are the right axioms for $\lhd$ and $\ltimes$?

It should be clear that the answers depend both on the choice of the language (that is, of the primitives) *and* on the choice of the foundational theory. We now see how the different choices give different results, in particular on three specific questions: should closed subsets be uniquely determined by open subsets? should the cover be always assumed to be inductively generated? should one assume the condition of positivity $(\mathsf{Pos}(a) \to a \lhd U) \to a \lhd U$? I will argue that in the most basic approach the answer must be no to each question.

Assuming classical logic, as we have seen in section 2.1, in any basic pair the equation $\mathsf{cl} = -\,\mathsf{int}\,-$ holds. By the same reason, on the formal side $\mathcal{J} = -\mathcal{A}-$. Moreover, classical logic guarantees compatibility of $-\mathcal{A}-$ with $\mathcal{A}$ to hold: $\mathcal{A}U \between -\mathcal{A}-V \leftrightarrow U \between -\mathcal{A}-V$ is classically equivalent to $\mathcal{A}U \subseteq \mathcal{A}-V \leftrightarrow U \subseteq \mathcal{A}-V$, which is the characteristic property of closure operators. So classically the definition of basic topology boils down to that of a set $S$ with a closure operator $\mathcal{A}$. In this sense, it is not visible in a classical foundation. Note, however, that reasoning classically *after* having assumed our definition of basic topology is not enough to make it trivial. That is, adding the law of double negation on subsets $--U = U$ is not enough to force $\mathcal{J} = -\mathcal{A}-$ to hold: in fact, when $\mathcal{A}$ is the identity, any interior operator $\mathcal{J}$ is trivially compatible with it. This seems to mean that the structure of basic topologies has after all some stability which goes beyond foundations.

If definitions through quantification over subsets are allowed, like in topos theory, then on any set $X$ with an interior operator $\mathsf{int}$ one can define closure as usual by quantifying over all open subsets. That is, the definition of $\mathsf{cl}$ in a basic pair, namely $x \epsilon \mathsf{cl}\, D \equiv \forall a(x \epsilon \mathsf{ext}\, a \rightarrow \mathsf{ext}\, a \between D)$, is turned into

$$x \epsilon \mathsf{CL}\,(D) \equiv \forall E(x \epsilon \mathsf{int}\, E \rightarrow \mathsf{int}\, E \between D).$$

One can check directly that such $\mathsf{CL}$ is indeed a closure operator compatible with $\mathsf{int}$, and that actually it is the greatest of such operators. However, it is more instructive to note that impredicatively the collection of open subsets is actually a set, defined by $\{D \subseteq X : D = \mathsf{int}\, D\}$. Then the above definition of $\mathsf{CL}$ coincides with the definition of $\mathsf{cl} \equiv \mathsf{rest}\,\Diamond$ in the basic pair formed by $X$, the set of open subsets and $x \Vdash D \equiv x \epsilon D$.

On the formal side, by symmetry one can always define a cover $\lhd_{\ltimes}$ impredicatively in terms of a positivity predicate $\ltimes$:

$$a \lhd_{\ltimes} U \equiv \forall W(a \ltimes W \rightarrow U \ltimes W).$$

To get an intuitive grasp of this definition, one should compare it with the pointwise definition of cover $\forall x(x \Vdash a \rightarrow x \Vdash U)$, and recall that $a \ltimes W$ expresses formally the existence of a point in $\mathsf{ext}\, a \cap \mathsf{rest}\, W$.

So also in the impredicative case the full structure of basic topology would not be visible, since one can always choose the cover, and hence formal open subsets, to be uniquely determined by formal closed subsets. Moreover, when $\lhd_{\ltimes}$ is defined as above, the formal topology $(S, \lhd_{\ltimes}, \ltimes)$ is actually presentable (with $X$ the set of formal closed subsets and $U \Vdash a \equiv a \epsilon U$). So the reasons for introducing basic topologies in this context are not so compelling. It is still unknown to me whether, conversely, one can find a similar impredicative definition of the positivity relation in terms of a given cover $\lhd$. In the special case of a positivity predicate $\mathsf{Pos}$ which moreover satisfies positivity, this is well known, and the definition is $\mathsf{Pos}(a) \equiv \forall U(a \lhd U \rightarrow U \between S)$.

Also in a foundation based on the notion of computation, such as Martin-Löf's type theory, there are some good reasons for a less general notion of basic topology than that given here. In fact, because of the validity of the axiom of choice, the cover defined on the formal side of a basic pair is always inductively generated (this was remarked by Martin-Löf, and Milly Maietti has shown that the proof in [7] can be extended to the case in which sets are assumed to have an extensional equality). So it is natural to require inductive generation of the cover $\lhd$ as part of the definition. In this case, a positivity relation is uniquely determined, and it is the greatest positivity relation that is compatible with $\lhd$, which is defined by coinduction (see section 2.7 above). In the same spirit, one can prove that for a relation between two such basic topologies to be a continuous relation it is enough that it respects the axioms (a proof is in [17]). In this sense the two conditions defining continuous relations are no longer independent. It is to be recalled, however, that here too the fact that $\ltimes$ is uniquely determined by $\lhd$ cannot be proved starting from our definition of basic topology, even assuming full Martin-Löf type theory.

The foundation we have been working with so far is essentially just intuitionistic many sorted logic with comprehension restricted to elementary formulae. No assumption is made on the nature of sets; in particular, no axiom of choice and no powerset axiom are assumed to be valid. So there is no principle which allows to reconstruct that half of topology, dealing with existential statements and with closed subsets, in terms of the other half, namely universal statements and open subsets. This is why one can never forget either of them. The main conceptual advantage is that the resulting mathematics respects both the intuition of computation, which underlies the justification of the axiom of choice, and the intuition of some kind of continuity, by which one can sometimes be in the position of knowing a statement of the form $\forall x \exists y$ to hold also without having a function giving $y$ in terms of $x$.

Once a language $\mathcal{L}$ is fixed, the "problem of completeness" is made rigorous in the following way. For every basic topology, one defines the theory in $\mathcal{L}$ of a basic topology $\mathcal{S}$, that is put

$$Th(\mathcal{S}) \equiv \text{ all statements which can be expressed in the language } \mathcal{L} \text{ and which hold in } \mathcal{S}$$

Then one extends this to any class $K$ of basic topologies, putting:

$$Th(K) \equiv \text{ all statements which are in } Th(\mathcal{S}), \text{ for every } \mathcal{S} \text{ in } K$$

Then completeness would be the following fact: if $BT$ is the class of all basic topologies, and $RBT$ the class of all representable basic topologies, then

$$Th(BT) = Th(RBT) \tag{6}$$

I don't know whether this holds, for a reasonable choice of $\mathcal{L}$. Would the definition of basic topology be better if it were given in such a way that completeness as formulated above holds?

Perhaps yes; but in my opinion, not at any cost. For instance, imagine we could reach completeness by adding a condition $C$ to the definition, where however $C$ is very complicated, and the proof that the completeness holds also is very complicated, and perhaps uses nonconstructive methods. Would that make the definition better? My opinion is that it would not be worthwhile to add $C$ to the basic definition (which does not mean, of course, that I am against studying what happens when $C$ is assumed!).

However, we analyse better what we should do to be able to obtain a completeness result. This will be helpful to organize what I want to say. As I said above, we should first of all specify the language in which $Th(\mathcal{S})$ is expressed.

After the specification of the language, the problem of completeness is in perfectly rigorous terms. In fact, now the equation (6) has a proper sense. It is enough to specify that the theory $Th(\mathcal{S})$ (and hence $Th(K)$ for a class $K$) is formed by all properties of $\mathcal{S}$ which can be written in the language of overlap algebras, with functors between them.

How can we check that a certain property of $\mathcal{P}S$ is written in the language of overlap algebras? It is just easier to formalize the notion of $\mathcal{P}S$ directly in that language, which is the same as to replace $\mathcal{P}S$ (and $\mathcal{P}X$) directly with overlap algebras. An isomorphic copy of $\mathcal{P}S$ is just an atomic overlap algebras. And then basic topologies are replaced with o-basic topologies. An o-basic topology $(\mathcal{P}, \mathsf{a}, \mathsf{j})$ is said to be representable if it is the image under a morphism $\langle f, f^-, f^*, f^{-*}\rangle$ of a discrete 0-basic topology $(\mathcal{P}, id, id)$. Then the question becomes: is the theory of o-basic topologies over an atomic overlap algebra the same as the theory of images under an o-continuos relation of a discrete o-basic topology over an atomic overlap algebra? I purposely put it in words, so that one can see that one restriction is totally artificial from an algebraic point of view: there is no reason to restrict to atomic overlap algebras. In fact, they still have a logical flavour, because of the presence of atoms (by which one can define elements by comprehension: every subset of the atoms determines an element of the o-algebra). If we also suppress the restriction to discrete o-basic topologies, we get the following question: is the theory of o-basic topologies the same as the theory of images of o-basic topologies under an o-continuos relation?

In this way the original question has become more mathematical, since now it is equivalent to saying that the definition of basic topology must be invariant under transfer along continuous relations. It is also more general, since not all overlap algebras are atomic.

This seems to me a more solid criterion for the correctness (and completeness) of the definition of basic topology. I conjecture that the definition of basic topology given above is the best one can do, i.e., it is the strongest definition which is invariant under continuous relations. Work on this is still going on.

## 2.10 The dark side of the moon

The treatment of existential statements, or of statements of the form $\exists\forall$ like that in the definition of interior, is the dark side of the mathematical planet. They have usually been reduced either to a negation (as in classical logic, where $\exists$ is the same as $\neg\forall\neg$ and hence closed is the same as complement of open) or to an impredicative quantification (closure defined in terms of all open subsets). The main aim of the basic picture, and of formal topology developed on it, is the beginning of a more direct, positive exploration of that kind of information which is usually treated as negative.

Specifically, the aim is to develop a mathematics which keeps on the scene as primitive also the notions which are connected with existential quantifications. The introduction of the notation $\between$ has this purpose; in fact, it allows to transform logical argumentations involving the existential quantifier into mathematical arguments involving $\between$, which are based on a spatial intuition. The first step is then to treat closed subsets as independent of open subsets. On the formal side, this brings to the introduction of the positivity relation $\ltimes$, which means a direct treatment of the notion of closed in pointfree terms. Another interesting example is the generation of the positivity relation $\ltimes$ by coinduction.

Only time and further work will tell whether the mathematics which is beginning to come out is interesting and with interesting new applications. My expectation is that it should find applications in those sciences in which a careful management of information seems important, like computer science, theoretical biology and perhaps quantum theory.

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