Turing degrees, constructively

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Synthetic computability is an axiomatization of computability theory that approaches the subject anti-classically and implicitly. That is, we adopt axioms that contradict the law of excluded middle, and we never speak about computations or machines explicitly. For example, the synthetic Rice's theorem states that a decidable predicate on a set is trivial if the set has the fixed-point property. The classic Rice's theorem arises when we interpret the synthetic one in the effective topos.

The topic of the talk is a synthetic approach to Turing degrees. A synthetic partial oracle is a pair of disjoint subsets of the natural numbers, the totality of which forms a directed-complete partial order (dcpo). A Turing reduction is a Scott-continuous endomap on the dcpo of oracles whose graph is enumerable. We study the Kleene-Post theorem, which states that there exist incomparable Turing degrees, as well as the stronger Friedberg-Muchnik theorem about the existence of incomparable computably enumerable Turing degrees. Their synthetic counterparts turn out to be variants of the Baire category theorem.

This is joint work with Andrew Swan.