

Localic Groups in Formal Topology and the Closed Subgroup Theorem

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A localic group is the pointfree analogue of a topological group, although it does not provide a genuine generalization of the latter. This fact is strikingly exemplified by Isbell’s Closed Subgroup Theorem, which states that every subgroup of a localic group is necessarily closed.

A constructive (topos-valid) version of this theorem was given by Peter Johnstone in 1989 [3] and relies on overtiness together with a suitable weakening of the classical notion of closed sublocale: indeed, every overt subgroup of a localic group can be shown to be (overt and) weakly closed.

It turns out that the notion of an overt, weakly closed sublocale essentially coincides—up to certain foundational subtleties (!)—with that of a formal closed subset in the sense of Giovanni Sambin [1]. (This also shows that Johnstone’s notion, although originally introduced to address a specific problem, arises quite naturally within the “dynamic” interpretation of topology developed in [4].)

In this talk, after recalling the basic “dictionary” needed to translate between the languages of locale theory and formal topology [2, 5], I will investigate what group structures on (inductively generated) formal topologies look like and what is needed to prove the Closed Subgroup theorem in this framework. My aim is twofold: first, to show that translating into the language of formal topology can lead to more transparent and concrete definitions and proofs (even when one’s favourite foundational framework is impredicative); second, to provide one of the first genuinely mathematical applications of Sambin’s formal closed subsets (and the related notion of a binary positivity relation).

References

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