Apartness in domain theory

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Summary I will present a constructive study of the Scott topology on domains illustrated by various concrete examples. I hope to exhibit how a constructive development highlights, and is informed by, the computational intuitions underlying domain theory.

The talk will be based on my *MFPS'21* paper [2] and its journal version [3]. Additionally, I will draw some connections to recent work with Escardó on injective types [4] and discuss how Escardó [5], Tosun [10], and Van der Weide & Frumin [11] have built on my work since.

Abstract for the papers underlying this talk Our starting point will be what we call the *intrinsic apartness* relation: two points of a domain are said to be apart if they can be separated by a Scott open. Under modest decidability conditions, we note that the Bridges–Vîţă apartness topology [1] induced by the intrinsic apartness coincides with the Scott topology, so that their framework applies to domain theory.

We then identify the *sharp* elements of a domain as a class of elements where the intrinsic apartness is well behaved (i.e. where it is tight and cotransitive). In many examples of interest, the compact elements of an algebraic domain are all sharp. We relate sharp elements to located subsets as introduced by Spitters [9] and used by Kawai in a study of the patch and Lawson formal topologies [7].

Another important class of sharp elements is given by the *strongly maximal* elements of a domain, as studied by Smyth and Heckmann [6, 8] in a classical context. We revisit the subspace of strongly maximal elements in a constructive context where it useful even when, classically, the notion of strong maximality collapses to (ordinary) maximality.

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