Spread representation of the n-simplex and Brouwer's fixed-point theorem

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Brouwer's quest for continuum as infinitely proceeding sequences led him to introduce the notion of spread. A spread is a certain subset of Baire space $\mathbb{N}^{\mathbb{N}}$ from which intended mathematical objects are obtained as its continuous image. In particular, the real numbers can be obtained as a continuous image of the ternary spread [2, Section 3.3]. In [3], we showed that the notion of real numbers by the spread representation is geometric, so that they can be naturally understood as the points of a certain point-free topology. This observation gave rise to a notion of point-free real numbers based on the ternary spread. In particular, the spread representation of the unit interval [4, Chapters 6, Section 3] determines the subspace of the point-free real numbers that corresponds to the formal unit interval $[0, 1]$.

In this talk, we extend the above representation to the n-simplex Δ, which can be naturally seen as an n-dimensional unit interval. The basic opens of Δ consist of all vertices of successive barycentric subdivisions of the finite order $[n] = \{0, 1, 2, \ldots, n\}$ $(0 < 1 < \cdots < n-1 < n)$, which are elements of successive finite powers of $[n]$. These basic opens are then equipped with the order opposite of the element-hood, forming a finite branching tree with $(n+1)$-roots $0, 1, \ldots, n$. This process gives rise to a point-free representation of Δ (in contrast to the geometric realisation of Δ usually considered in algebraic topology).

With respect to this point-free notion, the Heine–Borel theorem holds and Lebesgue's numbers can be computed, the latter of which is used to show the existence of a simplicial approximation of a point-free continuous map. Still in the point-free setting, we formulate an approximate version of Brouwer's fixed-point theorem in any dimension, whose point-free proof can be naturally reduced to the combinatorial lemma by Sperner [1, Chapter 1].

References

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