

# Measure Theory via Locales

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There are two, often unspoken, truths in measure theory:

1. In practice, all useful measures are given by Radon measures.
2. One does not really care so much about the sigma-algebra of measurable sets, but rather about its quotient by the ideal of null sets.

In the case of a given Radon measure, the quotient of measurable sets by null sets is an example of what is called a *measurable locale*. Measurable locales form a full subcategory of the category of locales, which is anti-equivalent to the category of commutative von Neumann algebras via an analogue of Gelfand duality. We argue that the measurable locale obtained from a Radon measure can be constructed directly from a Grothendieck topology on the poset of compact sets. This opens the door to a purely sheaf-theoretic perspective on measure theory that bypasses the traditional approach to measure theory via  $\sigma$ -algebras entirely, and makes it possible to view measure theory entirely as a special case of the theory of locales. As an application, we show that the locale of sublocales of a given Hausdorff space  $X$  equipped with a Radon measure can be equipped with a natural extension of the measure, invariant under measure-preserving homeomorphisms. This result is a strengthening of a theorem obtained independently by Simpson and Leroy. As a consequence, the concept of measurability becomes superfluous when using locales.

This talk is based on the article [1].

## References

- [1] Georg Lehner. Measure theory via locales, 2025. URL: <https://arxiv.org/abs/2510.08826>, arXiv:2510.08826.