A Predicative Constructive Effective Model of Formal Topology via the Minimalist Foundation.

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As advocated in Sambin's book Positive Topology [7], the entire field of Formal Topology can be developed within a minimalist constructive and predicative foundation capable of formalizing positive topologies which are inductively and coinductively generated. As shown in [5, 6], a system formally implementing these ideas can be taken as the extensional level of the Minimalist Foundation designed in [2], extended with constructors for generating inductive and coinductive predicates. For such a system, we provide an effective model in a fibred predicative topos designed in [1] as a predicative version of Hyland's effective topos and based on [3, 4]. Such an interpretation extends the Kleene realizability interpretation of arithmetic, and as the latter can serve to extract programs from proofs in Formal Topology. In particular, it validates countable choice and the (formal) Church's Thesis—an axiom internally stating that all numeric functions are computable by a Turing Machine. To facilitate comparison with other effective models, we provide an algebraic presentation of this interpretation as an instance of a free categorical construction using fibered category theory.

References

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