What is a space, and why should a logician care?

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Abstract

The logician is familiar with the idea that filters are generalised points. Topos theorists push this idea further by asking: what is a generalised space?

This talk will focus on one such answer: LT-Topologies ("Lawvere-Tierney Topologies"). In the 1980s, Hyland introduced the Effective Topos, and showed that the Turing Degrees embed into its poset of LT-Topologies. Progress was slow until 2013, where Lee-van Oosten showed a meaningful interaction between LT-topologies and combinatorial complexity.

In recent work with T. Kihara (arXiv:2602.08138), we advance this perspective considerably by introducing a game-theoretic version of Katetov's Order. On the set theory side, this yields a new partial order on filters of $\mathbb{N}$, strictly coarser than classical Katetov order on filters yet still supporting a rich internal structure. On the computability theory side, a computable analogue of this order coincides with the usual partial order on LT-topologies.

This connection brings into focus a striking insight: structural questions in computability theory and in the combinatorics of filters on $\mathbb{N}$ can be seen to be governed by the same underlying mechanism. From this viewpoint, the longstanding difficulty of analysing the lattice of subtoposes within Eff is no accident — progress requires advances in combinatorial set theory on the one hand, and advances in computability-theoretic analyses on the other. The talk will conclude by reflecting on what this perspective suggests for category theory, in particular for our understanding of elementary toposes with a natural numbers object.