# Toposes with enough points as categories of étale spaces

Umberto Tarantino[1]
joint work with Sam van Gool[2] and Jérémie Marquès[3]

[1] IRIF, Université Paris Cité, Paris, France
[2] LMF, ENS Paris Saclay, Paris, France
[3] Università degli Studi di Milano, Milan, Italy

**Background.** In the landmark paper [1], Barr extended Manes' characterization of compact Hausdorff spaces as the algebras for the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$ to arbitrary topological spaces, which he recovers as the titular *relational algebras* for $\beta$. Concretely, this means that a topology on a set $X$ can be equivalently described as a suitable relation between elements of $X$ and ultrafilters on $X$ — i.e. a map $X \times \beta X \to \mathbf{2}$ — which we can think of as specifying a notion of *convergence*: an ultrafilter $\nu \in \beta X$ converges to a point $x \in X$ if every neighborhood of $x$ lies in $\nu$. In other words, a spatial locale can be recovered from its set of points, once the latter is endowed with appropriate extra structure defined in terms of ultrafilters.

**Contribution.** The aim of this talk, based on [5], is to present a reconstruction theorem for *toposes with enough points* which categorifies Barr's result. Playing the role of the ultrafilter monad, we now have the *ultracompletion* pseudomonad $\mathbb{\beta}\colon \mathbf{CAT} \to \mathbf{CAT}$, originally introduced by Rosolini [3]: for a category $C$, objects of $\mathbb{\beta}C$ are formal ultraproducts (or *ultrafamilies*) in $C$, i.e. tuples $(c_i)_{i\in I}$ of objects of $C$ together with an ultrafilter $\nu$ on the index set $I$, which an algebra functor $\mathbb{\beta}C \to C$ maps to 'actual' ultraproducts in $C$.

The role of relations, between categories, is played by *profunctors*: following Barr's blueprint, we therefore introduce *ultraconvergence spaces* [4, 5] as a profunctorial analogue of convergence relations. Intuitively, ultraconvergence spaces replace a convergence relation $X \times \beta X \to \mathbf{2}$ with a profunctor $C^{\mathrm{op}} \times \mathbb{\beta}C \to \mathbf{Set}$ which associates, to an object $c$ and an ultrafamily $(d_i)^\nu_{i\in I}$ in $C$, a set of *ultra-arrows* $c \prec (d_i)^\nu_{i\in I}$ acting as 'witnesses' of the convergence of $(d_i)^\nu_{i\in I}$ to $c$. In the same spirit we introduce *continuous maps* of ultraconvergence spaces, categorifying topological continuity, and suitable 2-cells between them, obtaining a 2-category **UltSp**. We also introduce the category $\mathbf{Et}(X)$ of *étale maps* over an ultraconvergence space $X$, extending the topological notion of the same name.

For a topos $\mathcal{E}$, its category of points $\mathrm{pt}(\mathcal{E})$ carries a natural ultraconvergence structure, where an ultra-arrow $p \prec (q_i)^\nu_{i\in I}$ is a natural transformation $p^* \Rightarrow \prod_{i:\nu} q_i^*$. Here, $\prod_{i:\nu} q_i^*\colon \mathcal{E} \to \mathbf{Set}$ is the *ultraproduct* of (the inverse images of) the ultrafamily of points $(q_i)^\nu_{i\in I}$, i.e. the functor

$$\mathcal{E} \xrightarrow{\langle q_i^* \rangle_{i\in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

where the functor $\prod_{i:\nu}(-)\colon \mathbf{Set}^I \to \mathbf{Set}$ computes usual ultraproducts of sets. With this definition, our theorem reads as follows; in particular, we recover the famous equivalence $\mathrm{Sh}(\mathcal{O}(X)) \simeq \mathbf{Et}(X)$ in the localic case.

**Theorem.** *Let $\mathcal{E}$ be a topos with enough points. Then, $\mathcal{E} \simeq \mathbf{Et}(\mathrm{pt}(\mathcal{E}))$.*

Time permitting, we will explain how this corresponds to a *(strong) conceptual completeness theorem* for geometric logic, extending Makkai's analogous result for coherent logic via ultracategories [2], and we will comment on joint work in progress of the presenting author with Quentin Aristote, aimed to recover ultraconvergence spaces algebraically by extending the pseudomonad $\mathbb{\beta}$ to profunctors.

## References

[1] Michael Barr. Relational algebras. In *Reports of the Midwest Category Seminar IV*, pages 39–55, Berlin, Heidelberg, 1970. Springer.

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[3] Giuseppe Rosolini. Ultracompletions, 2024. Talk at CT2024.

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[5] Sam van Gool, Jérémie Marquès, and Umberto Tarantino. Toposes with enough points as categories of étale spaces, 2025. arXiv:2508.09604.