The view from ultraminimalism

Steve Vickers
s.j.vickers.1@bham.ac.uk
https://sjvickers.github.io

March 13, 2026

Abstract

The "minimalist" logic in Sambin's *Positive Topology* [Sam25] makes as few assumptions as possible, and so is intended to be relevant to all schools of constructivism. Nonetheless, there is an *ultra*minimalist account that rejects negative logical connectives, principally $\to$ and $\forall$ – or, type theoretically, $\to$ and Π. Its main known examples are geometric logic and, increasingly, the logic of arithmetic universes (AUs).

I shall set out an ultraminimalist perspective on some of the arguments in [Sam25]. Here are some of the topics I shall cover.

Topology Topology and continuity are the natural state of ultraminimalist mathematics: it is only the negative connectives that give rise to discontinuities. Moreover, the topology is point-free, for a better fit with constructive maths, and generalized (in the sense of Grothendieck), yet the working is pointwise. Technically, ultraminimalist working is an excellent tool for dealing with locales and Grothendieck toposes [Vic99].

Philosophically, this suggests that we should be cautious about how we introduce the negative connectives. Once they are admitted, and the natural topology is poisoned, they are very hard to weed out. To reconstruct the topology is a considerable bureaucratic effort.

Sets, classes, collections ... Such distinctions must be reexamined in the presence of topology.

Observations vs. constructions Building on ideas of Abramsky and Smyth, I have argued [Vic89, Vic10] that the ultraminimalist logic is one of *observations,* and that the axioms in a theory (which do involve $\forall$ and $\to$) are not themselves observable but should be understood as enabling Popperian falsifiability. Observation is not the same as construction. However, they do interact, as in "I have found a proof", or "My program has terminated".

Denotational semantics In assigning mathematical meaning to pieces of program code, this captures a deep connection between construction (by the programmer) and observation (by the user): the topology on a semantic domain captures *what the user can observe by watching the program run.*

Application Program Interfaces (APIs) The construction-observation, program-user, interface can be iterated. The program side might be a

library, *used* in constructing another *program.* This is an important principle in software engineering, since it excuses the application programmer from having to know the implementation details in the library and allows the library to be debugged, upgraded, or even completely reimplemented, without affecting the user: it just has to respect the axiomatics (the specification) of the API.

Ideal vs. concrete mathematics Hard-line constructivism says if you can't construct it it isn't maths: it is unsafe to axiomatize "ideal" objects and then assert their existence by logical fiat. My reading of the two levels of logic in [Sam25] is that you still need the idealizations, and I propose that they play the same role as the user view in an API, with topology barring you from observing at the level of construction details.

Internalization of AUs An AU may have internal AUs. Even though the internal mathematics of the internal AU is ultraminimal, it can be discussed in a more intuitionistic way in the outer AU. Moreover, the internalization can be nested. I conjecture this can be related to the interfaces mentioned above.

References

[Des25] Samuel Desrochers, *Properties of arithmetic universes and locoi*, Ph.D. thesis, University of Ottawa, 2025.

[Mai10] Maria Emilia Maietti, *Joyal's arithmetic universe as list-arithmetic pretopos*, Theory and Applications of Categories **24** (2010), no. 3, 39–83.

[MV12] Maria Emilia Maietti and Steven Vickers, *An induction principle for consequence in arithmetic universes*, Journal of Pure and Applied Algebra **216** (2012), no. 8–9, 2049–2067.

[Sam25] Giovanni Sambin, *Positive topology. A new practice in constructive mathematics*, Oxford Logic Guides, no. 57, Oxford Science Publications, 2025.

[Vic89] Steven Vickers, *Topology via logic*, Cambridge University Press, 1989.

[Vic99] ______, *Topical categories of domains*, Mathematical Structures in Computer Science **9** (1999), 569–616.

[Vic10] ______, *Issues of logic, algebra and topology in ontology*, Theory and Applications of Ontology: Computer Applications (Roberto Poli, Michael Healy, and Achilles Kameas, eds.), Theory and Applications of Ontology, vol. 2, Springer, 2010.

[Vic19] ______, *Sketches for arithmetic universes*, Journal of Logic and Analysis **11** (2019), no. FT4, 1–56.