# Categories, rings and modules
# A conference in honour of Alberto Facchini

## Abstracts

### 1. Plenary talks

**Francesco de Giovanni** (Università degli Studi di Napoli)
*Abstract properties of linear groups*

The aim of this talk is to describe the behaviour of abstract group theoretical properties, either absolute or of embedding type, within the universe of linear groups over a field.

**Marino Gran** (Université catholique de Louvain)
*Some new examples of pretorsion theories*

The notion of *pretorsion theory* is a natural extension of the classical notion of torsion theory in an abelian category. The idea is to associate, with any pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of a category $\mathcal{C}$, a corresponding notion of $\mathcal{Z}$-trivial morphism, where $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ is the induced subcategory of "trivial objects" in $\mathcal{C}$. When $\mathcal{C}$ is pointed and $\mathcal{Z} = \{0\}$, with 0 the zero object of $\mathcal{C}$, then the notion of pretorsion theory reduces to the usual notion of torsion theory. There are, however, several examples of pretorsion theories that are not torsion theories, for instance in the categories of preordered sets [FF] and of (small) categories [?]. In this talk we shall recall some basic properties of pretorsion theories [FFG], the universal property [BCG] of the *stable category* (in the sense of A. Facchini and C. Finocchiaro [FF]), and some motivating examples. We shall then focus our attention on some new examples of pretorsion theories in the category $\mathsf{PreOrd}(\mathsf{Grp})$ of preordered groups [GM] and $\mathsf{Cat}$ of small categories [BCGT].

This work is in collaboration with Alberto Facchini, Carmelo Finocchiaro, Francis Borceux, Federico Campanini, Aline Michel and Walter Tholen.


**References**

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[FFG] A. Facchini, C. Finocchiaro and M. Gran, *Pretorsion theories in general categories,* J. Pure Appl. Algebra 225 (2021) 106503.

[GM] M. Gran and A. Michel, *Torsion theories and coverings of preordered groups*, Algebra Universalis 82, 22 (2021).

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**Pedro Guil Asensio** (Universidad de Murcia)
*Strongly exchange rings*

Two elements $a, b$ in a ring $R$ form a right coprime pair, written $\langle a, b\rangle$, if $aR + bR = R$. Right coprime pairs have shown to be quite useful in the study of left cotorsion or exchange rings. In this paper, we define the class of strongly right exchange rings in terms of descending chains of them. We show that they are semiregular and that this class of rings contains left injective, left pure-injective, left cotorsion, local and left continuous rings. This allows us to give a unified study of all these classes of rings in terms of the behaviour of descending chains of right coprime pairs. (joint work with M. Izurdiaga)

**George Janelidze** (University of Cape Town)
*Nearly Semi-Abelian Categories*

A category $\mathcal{C}$ with finite limits and finite coproducts is said to be *semi-abelian* [4], if it is

(a) Barr exact;

(b) Bourn protomodular;

(c) pointed.

The purpose of the talk is:

1. To recall conditions (a)-(c) above and their first basic consequences, presenting them as a foundation of what we call semi-abelian categorical algebra.

2. To initiate a development of *nearly semi-abelian* categorical algebra, that is, a similar theory, under a weaker form of condition (c), which only requires the unique morphism $0 \to 1$ to be reqular epimorphism, not necessarily $0 \approx 1$, as it would be in the pointed case; here 0 and 1 denote an initial and a terminal object (in $\mathcal{C}$), respectively.

Every variety of groups with multiple operators [2] is semi-abelian. Every algebra in such a variety has a unique one-element subalgebra, which is not the case for merely nearly semi-abelian varieties. The variety of unital rings should be considered as the 'first' example of a nearly semi-abelian non-semi-abelian variety. And, just as in this example, it is important to have an intrinsic theory of ideals, which, among other things, makes the results of [1] better applicable. Note also, that since every regular epimorphism in a Barr exact category is an effective descent morphism, our weaker form of condition (c) allows us uning Grothendieck descent/monadicity techniques, as in [3].

Our approach should be (this is not done yet!) carefully compared with the universal-algebraic one, proposed in [5.].

**References**


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[2.] P. J. Higgins, *Groups with multiple operators*, Proceedings of the London Mathematical Society (3)6, 1956, 366-416

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[4.] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, Journal of Pure and Applied Algebra 168, 2002, 367-386

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**Manuel Reyes** (University of California, Irvine)
*Categories of hypergroups and hyperstructures*

A hyperoperation on a set M is an operation that associates to each pair of elements a subset of M, thus generalizing a binary operation. Hypergroups and hyperrings are structures defined in terms of hyperoperations. While they were respectively defined in the 1930s and 1950s, they have recently become more prominent through various appearances in number theory, combinatorics, and absolute algebraic geometry. However, to date there has been relatively little attention given to categories of hyperstructures.

We will discuss several categories of hyperstructures, which we view as structures defined in the category of relations but whose morphisms are ordinary set maps. A common theme is that in order for these categories to enjoy good properties (such as (co)completeness), we must allow for the product or sum of two elements to be the empty subset, a condition that is forbidden for hypergroups. Most significantly, we will discuss a category containing canonical hypergroups that has a closed monoidal structure reminiscent of the tensor product of abelian groups. This is joint work with So Nakamura.

**Agata Smoktunowicz** (University of Edinburgh)
*On connections between pre-Lie rings and braces*

In 2014, Wolfgang Rump presented a connection pathway from pre-Lie algebras to braces. This pathway can also be described using the group of flows of a pre-Lie algebra. An advantage of this construction is that the additive group of the pre-Lie algebra and the obtained brace are the same. It is not yet clear if every brace of cardinality $p^n$ for $p > n$ can be obtained from a pre-Lie ring in this way. An affirmative answer to this question would yield an extension of the classical Lazard correspondence between p-adic Lie groups and p-adic Lie rings to the correspondence between braces and pre-Lie rings.

In this talk we will show that if A is a brace of cardinality $p^n$ where $p > n + 1$ then the brace $A/ann(p^4)$ is obtained as the group of flows of some left nilpotent pre-Lie ring. This answers the above question up to elements whose additive order is at most $p^4$. Here $ann(p^4)$ denotes the set of elements whose additive order is $p^i$ for $i4$.

Rump introduced braces in 2007. They are a generalisation of Jacobson radical rings with the two-sided braces being exactly the Jacobson radical rings. One of the main motivations for investigating braces is their connections with set theoretic solutions of the Yang-Baxter equation. Another is the relationship of braces to homological group theory since braces are exactly groups with bijective 1-cocycles. The theory of braces is also connected to algebraic number theory and its generalisations through the concept of Hopf-Galois extensions of abelian type (which was demonstrated by David Bachiller).

Some of this talk relates to work done in collaboration with Aner Shalev.

**References**
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3. A. Smoktunowicz, *On the passage from finite braces to pre-Lie algebras*, Adv. Math. 409 (2022), 108683.
4. A. Smoktunowicz, *From pre-Lie rings back to braces*, arXiv:2208.02535 [math.RA].

**Leandro Vendramin** (Vrije Universiteit Brussel)
*Skew braces, cabling and indecomposable solutions to the Yang-Baxter equation*

We will mention combinatorial results about the structure of finite indecomposable solutions to the Yang-Baxter. These results show an intriguing connection between the cycle structure of a particular permutation constructed from the solution (known as the diagonal map), the Dehornoy class (a positive integer associated with every solution which has its origins in the theory of Garside groups), and the additive structure of a particular finite skew brace associated with the solution. The talk contains concrete examples, open problems and conjectures.

**Efim Zelmanov** (University of California San Diego)
*Basic Algebraic Structures*

This is a leisurely and subjective overview of 200 years of development of Abstract Algebra from Lagrange to our time.

## 2. Contributed Talks

**Federico Campanini** (Université catholique de Louvain)
*Weak forms of the Krull-Schmidt Theorem*

According to the classical Krull-Schmidt Theorem for modules, any module of finite composition length decomposes as a direct sum of indecomposable modules in an essentially unique way, that is, unique up to isomorphism and permutation of the indecomposable summands. In 1975, Warfield posed a problem, essentially asking whether the Krull-Schmidt holds for finite direct sums of uniserial modules. The negative answer to this question was given by A. Facchini in 1996. He showed that even though the Krull-Schmidt

Theorem does not hold for serial modules, it is possible to prove a weak version of it. This phenomenon can be found not only for serial modules, but also for other classes of modules whose endomorphism rings have at most two maximal right ideals. In this talk we shall provide some examples of additive categories in which finite direct-sum decompositions can be described by suitable invariants.

This talk is based on joint works with Alberto Facchini and Sussan F. El-Denken.

**References**

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**Carmelo Finocchiaro** (Università degli Studi di Catania)

*On multiplicative lattices and their spectra*

Prime ideals of commutative rings play a central role both in Commutative Algebra and Algebraic Geometry, being foundation of scheme theory and a key tool for studying ideal-theoretic properties of commutative rings. However a spectrum can be attached to many algebraic structures, such as non commutative rings, monoids, etc. In this talk we will provide a general framework where to study spectra: that of multiplicative lattices. We will discuss some ideas from [1] and [2].

**References**

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**Manuel Cortes Izurdiaga** (Universidad de Malaga)

*Maximal ideals in module categories*

An ideal $\mathcal{I}$ in a preadditive category $\mathbf{C}$ is an additive subfunctor of the Hom bifunctor. As in the case of rings, one can consider maximal and minimal ideals in the category $\mathbf{C}$. We are interested in maximal ideals in the category Mod-$R$ of right modules over a ring $R$. While minimal ideals in $\mathsf{Mod}$-$R$ are well understood, as a consequence of a result, proved by A. Facchini, which establishes a one-to-one correspondence between minimal ideals in $\mathsf{Mod}$-$R$ and simple modules, there is no such description of maximal ideals. The main goal of the talk is to prove that actually there do not exist maximal ideals in $\mathsf{Mod}$-$R$ (in fact, there do not exist in Grothendieck categories).

**Sergio López-Permouth** (Ohio University)

*Graphs and binary operations*

In recent years, the word magma has been used to designate a pair of the form $(S, *)$ where $*$ is a binary operation on the set $S$. Inspired by that terminology, we use the notation and terminology $\mathcal{M}(S)$ (the magma of $S$) to denote the set of all binary operations on the set $S$ (i.e. the set of all magmas with underlying set $S$.)

Our study concerns a monoid structure $(\mathcal{M}(S), \triangleleft)$ on the magma of an arbitrary set $S$. Due to time limitations, we will show only, among the many properties of this *magma monoid* provided in [1], an estimate of the number of its idempotent elements when $|S| = n$.

Various ways in which simple graphs with vertices $V$ can be used to induce binary operations on a set $S$ containing $V$ are considered. In two cases, the corresponding graph-induced operations are subsemigroups of the magma monoid $(\mathcal{M}(S), \triangleleft)$. The fact that those graph-induced operations are closed under $\triangleleft$ induces corresponding liftings $\triangleleft_1$ and $\triangleleft_2$ of $\triangleleft$ to the corresponding inducing families of graphs.

Embeddings of the family $\mathcal{G}(S)$ of simple graphs with vertices $V$ into the magma monoids of $S$ and of $\mathcal{G}(S)$ are also considered as are the properties of said embeddings. We will characterize the idempotent elements in these graph-induced submonoids and will see the impact of these characterizations on the estimates mentioned above.

This is a preliminary report on a project in progress with Asiyeh Rafieipour and Isaac Owusu-Mensah.

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**Mara Pompili** (Karl-Franzens Universität Graz)
*Commutators in the category SKB of skew braces*

We examie the semiablian category SKB of left skew braces. We study the notion of commutator of ideals in a left skew brace. We show that the so-called (Huq=Smith) condition holds for left skew braces. Finally, we give a set of generators for the commutator of two ideals and we apply the theory of multiplicative lattices to the lattice of ideals of skew brace. This is based on a joint work with D. Bourn and A. Facchini.

**Leonid Positselski** (Czech Academy of Sciences, Prague)
*Fp-projective periodicity*

The periodicity phenomenon in homological algebra was discovered by Benson and Goodearl and rediscovered by Neeman. Saroch and Stovicek proved, using complicated set-theoretical techniques, that all fp-projective-periodic modules over a coherent ring are fp-projective. Recently Bazzoni, Hrbek, and the speaker have found a simpler proof of a more general result, namely, that all fp-projective-periodic modules over an arbitrary ring are weakly fp-projective. The elementary proof is based on a strong form of pure pure-projective periodicity (essentially due to Neeman) and the Hill lemma. In this talk, I would present a brief introduction into periodicity theorems and possibly sketch our new proof of fp-projective periodicity.

**Pavel Příhoda** (Charles University Prague)
*Infinite direct sums of finitely generated torsion free modules*

We study one dimensional commutative noetherian domains with module finte normalization having the property that every pure projective torsion free module is a direct sum of finitely presented modules. Over such rings it is often easier to classify non-finitely generated pure projective torsion free modules than the finitely generated ones. In case of Bass domains direct sum decompositions of non-finitely generated pure projective torsion free modules are unique "up to finitely many permutations" so a form of weak Krull-Schmidt theorem holds in this class of modules.

**Lorenzo Stefanello** (Università di Pisa)
*Some explicit constructions of skew braces*

In the last few years a great attention has been devoted to skew braces, objects of group theoretic flavour introduced by Leandro Guarnieri and Leandro Vendramin.

The many interactions with other topics, such as radical rings, the Yang-Baxter equation, and Hopf-Galois structures, motivated many mathematicians to the study of skew braces, carried on from many different perspectives and points of view.

A recent direction of study is to construct new examples of skew braces, possibly satisfying certain properties. These can be used as samples to obtain insights on new results, to test conjectures or, more generally, to better understand the implications of skew braces in other settings.

In this talk, after a brief overview on the main definition and applications, we focus our attention on new constructions of skew braces, involving different concepts of group and ring theory, obtained in recent joint works with Andrea Caranti and with Senne Trappeniers.

**Roger Wiegand** (University of Nebrask-Lincoln)
*Monoids of modules*

Let $R$ be a commutative local Noetherian ring. We consider the monoid of isomorphism classes of finitely generated $R$-modules, with the semigroup operation induced by the direct sum. More specifically, given a finitely generated $R$-modules $M$, we consider the submonoid $\mathsf{add}(M)$ consisting of isomorphism classes of modules that are direct summands of the direct sum of some finite number of copies of $M$. We give a characterization of the monoids that arise in this way, including a ”realization“ theorem for building suitable rings and modules. This approach yields some nice properties that hold for all decompositiong. For example, one *cannot* have indecomposable modules $A$ and $B$ such that $A \oplus A \oplus A \cong B \oplus B$. It also allows one to construct strange examples. For instance, one can have four pairwise non-isomorphic indecomposable $R$-modules $A, B, C, D$ such that $A \oplus B \oplus C \cong D^{(8)}$ (the direct sum of 8 copies of $D$).

**Mohamed Yousif** (The Ohio State University) *Modules with the exchange property*

A right $R$-module $M$ is said to satisfy the (full) exchange property if for any two direct sum decompositions $L = M \oplus N = \oplus_{i \in I} N_i$ there exist submodules $K_i \subseteq N_i$ such that $L = M \oplus (\oplus_{i \in I} K_i)$. If this holds only for $|I| < \infty$, then $M$ is said to satisfy the finite exchange property. The exchange property is of importance because it provides a way to build isomorphic refinements of different direct sum decompositions, which is precisely what is needed to prove the famous Krull-Schmidt-Remak-Azumaya Theorem. It is an open question due to Crawley and Jónsson whether the finite exchange property always implies the full exchange property.

In this talk we present the latest results on this open question and its relationship with clean rings and modules. This is a joint work with Yasser Ibrahim of both Taibah and Cairo Universities.