Weak forms of the Krull-Schmidt Theorem

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Classical Krull-Schmidt Theorem for modules [Krull 1925, Schmidt 1929]

Let R be a ring and M be an R-module* of finite composition length. Then there exists a decomposition

$$M = M_1 \oplus \cdots \oplus M_r$$

into indecomposable submodules, which is essentially unique: if $M = N_1 \oplus \cdots \oplus N_t$ is another decomposition of M into indecomposable submodules, then $r = t$ and there exists a permutation σ of $\{1, 2, \ldots, r\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \ldots, r$.

***Convention:**

- ring = associative ring with 1
- R-module = unital right R-modules

Can we drop the assumption "finite composition length"?

(Recall: Finite length = Artinian + Noetherian)

- Krull knew that Noetherian modules decompose as a direct-sum of indecomposable modules, but that their direct-sum decompositions are not essentially unique (1932).

- Krull asked if the Krull-Schmidt Theorem holds for Artinian modules (1932).

... 63 years later...

- In 1995, Facchini, Herbera, Levy and Vámos showed that the Krull-Schmidt Theorem fails also for Artinian modules.

Theorem [Facchini, Herbera, Levy and Vámos 1995]:

For any integer $n \geq 2$ there exists an artinian module M_R over a suitable ring R such that M_R can be written as a direct sum of k indecomposable modules for $k = 2, \ldots, n$.

For modules of finite composition length:

"being indecomposable" = "having a local endomorphism ring"

Krull-Schmidt-Remak-Azumaya Theorem 1950:

Let R be a ring and let $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ be two families of modules with local endomorphism rings. Then

$$\bigoplus_{i \in I} M_i \cong \bigoplus_{j \in J} N_j$$

if and only if there exists a bijection $\sigma : I \to J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.

We want to deal with finite direct-sum decompositions.

(Our version of the) Krull-Schmidt Theorem

Let $M_1, \dots, M_r, N_1, \dots, N_t$ be $r+t$ R-modules with local endomorphism rings. Then

$$\bigoplus_{i=1}^{r} M_i \cong \bigoplus_{j=1}^{t} N_j$$

if and only if $r = t$ and there exists a permutation $\sigma \in S_r$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, \dots, r$.

What happens if we drop the assumption "having a local endomorphism ring"?

Weak form of the Krull-Schmidt Theorem:

Take indecomposable objects $M_1, \dots, M_r, N_1, \dots, N_s$ into a suitable category such that

$$\bigoplus_{i=1}^{r} M_i \cong \bigoplus_{j=1}^{s} N_j \iff \text{"uniqueness up to a (fixed) number of permutations/invariants"}$$

A weak form of the Krull-Schmidt Theorem

An R-module U is said to be uniserial if the lattice of its submodules is linearly ordered under inclusion, that is, for every $V, W \leq U$, either $V \subseteq W$ or $W \subseteq V$.

Proposition [Facchini, 1996]

The endomorphism ring $\operatorname{End}(U_R)$ of a non-zero uniserial module U_R has at most two maximal right ideals: the completely prime two-sided ideals

$$I_{U,m} := \{\, f \in \operatorname{End}(U_R) \mid f \text{ is not a monomorphism} \,\}$$

and

$$I_{U,e} := \{\, f \in \operatorname{End}(U_R) \mid f \text{ is not an epimorphism} \,\},$$

or only one of them.

Two R-modules M and N are said to have:

- the same monogeny class, denoted by $M \sim_m N$, if there exist two monomorphisms:

$$M \rightarrowtail N \qquad \text{and} \qquad N \rightarrowtail M$$

- the same epigeny class, denoted by $M \sim_e N$, if there exist two epimorphisms:

$$M \twoheadrightarrow N \qquad \text{and} \qquad N \twoheadrightarrow M$$

Finite direct sums of uniserial modules are classified via their monogeny class and their epigeny class.

A weak form of the Krull-Schmidt Theorem [Facchini, 1996]

Let $U_1, \dots, U_r, V_1, \dots, V_t$ be non-zero uniserial R-modules. Then

$$U_1 \oplus \cdots \oplus U_r \cong V_1 \oplus \cdots \oplus V_t$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \dots, r\}$ such that $U_i \sim_m V_{\sigma(i)}$ and $U_i \sim_e V_{\tau(i)}$ for every $i = 1, 2, \dots, r$.

In the same paper, Facchini constructed an example of n^2 ($n \geq 2$) pairwise non-isomorphic uniserial modules $U_{j,k}$ ($j, k = 1, 2, \ldots, n$) over a suitable ring R in order to show that a module that is a direct sum of n uniserial modules can have $n!$ pairwise non-isomorphic direct-sum decompositions.

Example for $n = 2$:

$$R := \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & 0 & 0 \\ p\mathbb{Z}_p & \mathbb{Z}_p & 0 & 0 \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_q & \mathbb{Z}_q \\ \mathbb{Q} & \mathbb{Q} & q\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} \subseteq \mathbf{M}_4(\mathbb{Q}).$$

- Consider the 4 uniserial right R-modules:

$$U_{1,1} = \left(\frac{\mathbb{Q}}{\mathbb{Z}_p},\ \frac{\mathbb{Q}}{\mathbb{Z}_p},\ \mathbb{Z}_q,\ \mathbb{Z}_q\right) \qquad U_{1,2} = \left(\frac{\mathbb{Q}}{p\mathbb{Z}_p},\ \frac{\mathbb{Q}}{\mathbb{Z}_p},\ \mathbb{Z}_q,\ \mathbb{Z}_q\right)$$

$$U_{2,1} = \left(\frac{\mathbb{Q}}{\mathbb{Z}_p},\ \frac{\mathbb{Q}}{\mathbb{Z}_p},\ q\mathbb{Z}_q,\ \mathbb{Z}_q\right) \qquad U_{2,2} = \left(\frac{\mathbb{Q}}{p\mathbb{Z}_p},\ \frac{\mathbb{Q}}{\mathbb{Z}_p},\ q\mathbb{Z}_q,\ \mathbb{Z}_q\right)$$

- $U_{1,1} \sim_m U_{1,2}$ & $U_{2,2} \sim_m U_{2,1}$ and $U_{1,1} \sim_e U_{2,1}$ & $U_{2,2} \sim_e U_{1,2}$
- $U_{1,1} \oplus U_{2,2} \cong U_{\sigma(1),\tau(1)} \oplus U_{\sigma(2),\tau(2)}$ for every pair of permutations σ and τ

"Weak forms of the Krull-Schmidt Theorem in the case two"

There are other examples (e.g. kernel of morphisms between indecomposable injective modules, couniformly presented modules, ...)

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Beyond the case two...

The category of morphisms between modules

Let Morph(Mod-R) denote the morphism category, defined as follows:

- **objects:** R-module morphisms between right R-modules.

  We will denote by M and N generic objects $M\colon M_0 \to M_1$ and $N\colon N_0 \to N_1$.

- **morphisms:** a morphism $u\colon M \to N$ in the category Morph(Mod-R) is a pair of R-module morphisms (u_0, u_1) making the following diagram commute:

$$\begin{array}{ccc} M_0 & \xrightarrow{M} & M_1 \\ {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u_1} \\ N_0 & \xrightarrow{N} & N_1 \end{array}$$

Theorem [Fossum, Griffith, Reiten 1975, Green 1982]

$$\text{Morph}(\text{Mod-}R) \quad \cong \quad \text{Mod-}\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$$

- Let $\mathcal{U}$ be the class of all non-zero uniserial right R-modules.
- Let Morph($\mathcal{U}$) be the full subcategory of Morph(Mod-R) whose objects are all morphisms between two modules of $\mathcal{U}$.

Proposition:

Let M be an object of Morph($\mathcal{U}$). Then End(M) has at most four maximal right ideals, which are among the completely prime two-sided ideals

$$I_{M,0,m} := \{ (u_0, u_1) \in \mathrm{End}(M) \mid u_0 \text{ is not a monomorphism } \},$$

$$I_{M,1,m} := \{ (u_0, u_1) \in \mathrm{End}(M) \mid u_1 \text{ is not a monomorphism } \},$$

$$I_{M,0,e} := \{ (u_0, u_1) \in \mathrm{End}(M) \mid u_0 \text{ is not an epimorphism } \},$$

$$I_{M,1,e} := \{ (u_0, u_1) \in \mathrm{End}(M) \mid u_1 \text{ is not an epimorphism } \}.$$

For every pair of objects $M, N \in \mathrm{Morph}(\text{Mod-}R)$ we will write:

- $M \sim_{0,m} N$ if there exist two morphisms $(u_0, u_1) \in \mathrm{Hom}(M, N)$ and $(v_0, v_1) \in \mathrm{Hom}(N, M)$ such that both u_0 and v_0 are injective right R-modules morphisms;

- $M \sim_{1,m} N$ if there exist two morphisms $(u_0, u_1) \in \mathrm{Hom}(M, N)$ and $(v_0, v_1) \in \mathrm{Hom}(N, M)$ such that both u_1 and v_1 are injective right R-modules morphisms;

- $M \sim_{0,e} N$ if there exist two morphisms $(u_0, u_1) \in \mathrm{Hom}(M, N)$ and $(v_0, v_1) \in \mathrm{Hom}(N, M)$ such that both u_0 and v_0 are surjective right R-modules morphisms;

- $M \sim_{1,e} N$ if there exist two morphisms $(u_0, u_1) \in \mathrm{Hom}(M, N)$ and $(v_0, v_1) \in \mathrm{Hom}(N, M)$ such that both u_1 and v_1 are surjective right R-modules morphisms.

Theorem [— , El-Deken and Facchini, 2018]:

Let $M_1, \dots M_r, N_1, \dots, N_t$ be $r+t$ objects in Morph($\mathcal{U}$). Then

$$\bigoplus_{j=1}^{r} M_j \cong \bigoplus_{k=1}^{t} N_k$$

if and only if $r = t$ and there exist four permutations $\sigma_{0,m}, \sigma_{1,m}, \sigma_{0,e}, \sigma_{1,e}$ of $\{1, 2, \dots, r\}$ such that $M_j \sim_{i,a} N_{\sigma_{i,a}(j)}$ for every $j = 1, \dots, r$, $i = 0, 1$ and $a = m, e$.

"Weak forms of the Krull-Schmidt Theorem in the case n"

We have:

- An additive category $\mathcal{C}$ and a fixed positive integer n.
  ($n =$ the number of "invariants").

- A class $\mathcal{U}$ of (indecomposable) objects of $\mathcal{C}$ such that for every $M \in \mathcal{U}$, the endomorphism ring $\mathrm{End}_{\mathcal{C}}(M)$ has at most n maximal ideals which are among n completely prime two-sided ideals $I_{M,1}, \dots, I_{M,n}$.

- n equivalence relations $\sim_i$ ($i = 1, \dots, n$) on $\mathrm{Ob}(\mathcal{C})$ related to the maximal ideals of the objects.

Weak form of the Krull-Schmidt Theorem

Let $M_1, M_2 \dots, M_r, N_1, N_2, \dots, N_s$ be $r + s$ objects of $\mathcal{U}$. Then

$$\bigoplus_{k=1}^{r} M_k \cong \bigoplus_{\ell=1}^{s} N_\ell$$

in the category $\mathcal{C}$ if and only if $r = s$ and there exist n permutations σ_i of $\{1, 2, \dots, r\}$, where $i = 1, \dots, n$, such that $M_k \sim_i N_{\sigma_i(k)}$ for every $k = 1, \dots, r$.

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A key notion: ideals and factor categories

- An ideal $\mathcal{I}$ of an additive category $\mathcal{C}$ assigns to every pair of objects A, B of $\mathcal{C}$ a subgroup $\mathcal{I}(A, B) \trianglelefteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that, for every objects C, D in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(B, D)\mathcal{I}(A, B)\mathrm{Hom}_{\mathcal{C}}(C, A) \subseteq \mathcal{I}(C, D)$$

- Given any ideal of $\mathcal{C}$, we can consider the factor category $\mathcal{C}/\mathcal{I}$:
  - *Objects*: $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}/\mathcal{I})$;
  - *Morphisms*: $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\mathcal{I}(A, B)$.

For any ideal $\mathcal{I}$ of $\mathcal{C}$, we have a canonical additive functor $F\colon \mathcal{C} \to \mathcal{C}/\mathcal{I}$.

- Let A be an object of $\mathcal{C}$ and $I \trianglelefteq \mathrm{End}_{\mathcal{C}}(A)$. The ideal of $\mathcal{C}$ associated to I is the ideal $\mathcal{I}_A$ of $\mathcal{C}$ defined by:

$$f \in \mathcal{I}_A(X, Y) \iff \left(A \xrightarrow{\forall\alpha} X \xrightarrow{f} Y \xrightarrow{\forall\beta} A\right) \in I \qquad \forall X, Y \in \mathcal{C}$$

Equivalence relations in terms of the factor categories

Ingredients:

▷ $\mathcal{U} \subseteq \mathcal{C}$ ▷ $M \in \mathrm{Ob}(\mathcal{U}) \rightsquigarrow \mathrm{Max}(\mathrm{End}(M)) \subseteq \{I_{M,1}, \dots I_{M,n}\}$ ▷ $\sim_1, \dots, \sim_n$

Lemma:

Take $M \in \mathrm{Ob}(\mathcal{U})$, $I_{M,j} \in \mathrm{Max}(\mathrm{End}(M))$ and $\mathcal{I}_{M,j}$ its associated ideal in $\mathcal{C}$.
For any object $N \in \mathrm{Ob}(\mathcal{U})$ we have:

$$M \sim_j N \iff M \cong N \text{ in } \mathcal{C}/\mathcal{I}_{M,j}$$

and

$$M \nsim_j N \iff N \cong 0 \text{ in } \mathcal{C}/\mathcal{I}_{M,j}$$

Moreover, in the first case $I_{N,j} = \mathcal{I}_{M,j}(N, N)$ is a maximal right ideal of $\mathrm{End}(N)$ and $\mathcal{I}_{M,j} = \mathcal{I}_{N,j}$.

For any object M of $\mathcal{U}$, define:

$V(M) := \{\mathcal{I}_{M,i} \mid I_{M,i} \text{ is a maximal ideal of } \mathrm{End}(M)\}$.

Proposition [— , Facchini 2018]

Let $M_1, \dots, M_r$ be objects of $\mathcal{U}$. Then there is a one-to-one correspondence between the maximal two-sided ideals of $\mathrm{End}_{\mathcal{C}}(\bigoplus_{k=1}^r M_k)$ and $\bigcup_{k=1}^r V(M_k)$, given by

$$\Psi : \bigcup_{k=1}^r V(M_k) \longrightarrow \mathrm{Max}\left(\mathrm{End}_{\mathcal{C}}(\bigoplus_{k=1}^r M_k)\right)$$

$$\mathcal{I}_{M_k,i} \quad \mapsto \quad \mathcal{I}_{M_k,i}(\oplus_{k=1}^r M_k, \oplus_{k=1}^r M_k)$$

$$\mathcal{J} \quad \leftarrow\!\shortmid \quad J$$

Theorem [— , Facchini 2018]

Let $M_1, M_2 \dots, M_r, N_1, N_2, \dots, N_s$ be $r+s$ objects of $\mathcal{U}$. For every $j = 1, \dots, n$ define

$$X_j := \{k \mid k = 1, \dots, r,\ I_{M_k,j} \text{ is a maximal ideal of } \operatorname{End}(M_k)\}$$

$$Y_j := \{\ell \mid \ell = 1, \dots, s,\ I_{N_\ell,j} \text{ is a maximal ideal of } \operatorname{End}(N_\ell)\}.$$

Then the following conditions are equivalent:

(1) $\bigoplus_{k=1}^{r} M_k \cong \bigoplus_{\ell=1}^{s} N_\ell$ in the category $\mathcal{C}$;

(2) $r = s$ and there exist n permutations σ_j of $\{1, 2, \dots, r\}$, where $j = 1, \dots, n$, such that $M_k \sim_j N_{\sigma_j(k)}$ for every $k = 1, \dots, r$.

(3) $r = s$ and there exist n bijections $\tau_j : X_j \to Y_j$, where $j = 1, \dots, n$, such that $M_k \sim_j N_{\tau_j(k)}$ for every $k \in X_j$;

Thank you!