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MAXIMAL IDEALS IN MODULE CATEGORIES

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Dedicated to Alberto Facchini

Based on

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Cortés-Izurdiaga, M., Facchini, A.
Maximal ideals in module categories and applications
Applied categorical structures.
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1. IDEALS IN PREADDITIVE CATEGORIES

- $\mathbf{C} =$ preadditive category.

Ideals in preadditive categories

An **ideal** $\mathcal{I}$ in $\mathbf{C}$ is an additive subfunctor of the Hom functor:

- $\mathcal{I}(A, B) \leq_{\mathbb{Z}} \operatorname{Hom}(A, B)$
- For any diagram

$$X \xrightarrow{f} A \xrightarrow{i} B \xrightarrow{g} Y$$

with $i \in \mathcal{I}(A, B)$, $gif \in \mathcal{I}(X, Y)$.

Inclusion of ideals

$$\mathcal{I} \leq \mathcal{J} \Leftrightarrow \mathcal{I}(A, B) \leq \mathcal{J}(A, B), \forall A, B \in \mathbf{C}$$

Maximal and minimal ideals

The ideal $\mathcal{I}$ is

- **minimal** if it is not zero and does not contain any other non-zero ideal.
- **maximal** if it is not Hom and is not contained in any other proper ideal.

- R not necessarily commutative ring with identity.
- Mod-R is the category of right modules.

Minimal ideals in Mod-R

Facchini, 2009. There is a one-to-one correspondence

$$\{\text{Minimal ideals in Mod-}\} \Leftrightarrow \{\text{Iso-classes of simples}\}$$

Maximal ideals in Mod-R

There is no such description

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OBJECTIVE: Study maximal indeals in module categories.
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Constructing ideals in $\mathbf{C}$

Take $A \in \mathbf{C}$ and $I \trianglelefteq \mathrm{End}_{\mathbf{C}}(A)$ an ideal. We define the **ideal associated to** I, $\mathcal{A}_I$, as

$$\mathcal{A}_I(X,Y) = \{f : X \to Y \mid A \xrightarrow{\alpha} X \xrightarrow{f} Y \xrightarrow{\beta} A \in I \text{ for all } \alpha, \beta\}$$

- If $\mathcal{A}_I$ need not be maximal, even if I is.
  - $\mathcal{A}_I(B,B) = I'$ might not be maximal.
  - Then $\mathcal{A}_I < \mathcal{A}_{I'}$.

Caracterization of maximal ideals

Facchini, Perone, Prihoda, 2011. *An ideal* $\mathcal{M}$ *of* $\mathbf{C}$ *is maximal if and only if* $\mathcal{M} = \mathcal{A}_I$ *and satisfies that*

- $\mathcal{M}(A,A) = \mathrm{End}_{\mathbf{C}}(A)$ *or*
- $\mathcal{M}(A,A)$ *is a maximal ideal.*

Corollary

If $\mathbf{C}$ is additive with split idempotents and $A \in \mathbf{C}$, then the maximal ideals in add(A) are the ideals associated to maximal ideals of End$_{\mathbf{C}}(A)$.

$$\text{add}(A) = \{\, \text{direct summands of } A^n \}$$

What about Mod-R?

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OBJECTIVE: Study maximal indeals in module categories.
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2. A PARTIAL ORDER BETWEEN OBJECTS OF $\mathbf{C}$

- $\mathbf{C}$ $=$ preadditive category.
- $A, B \in \mathbf{C}$.

Strict order

$A \prec B$ if exists $E \subseteq Hom(B, A) \times Hom(A, B)$ with

1. E infinite.
2. $(f, g) \in E \Rightarrow fg = 1_A$.
3. For each $\varphi : A \to B$, $|\{(f, g) \in E :| f\varphi \neq 0\}| < |E|$.

Partial order

$$A \preceq B \Leftrightarrow A \prec B \text{ or } A = B$$

Main example in Mod-R

Let κ be an infinite regular cardinal:

- If A is a non-zero and $< \kappa$-generated, then $A \prec A^{(\kappa)}$.
  - We can embedd A in $A^{(\kappa)}$ in "κ distinct ways".
  - Any morphisms $\varphi : A \to A^{(\kappa)}$ "only touches less than κ of these direct summands".

Corollary

For each non-zero A in Mod-R, there exists B with $A \prec B$.

Vector spaces

If U and V are non-zero vector spaces, then $U \prec V$ if and only if V is infinite dimensional and dim(U) <dim(V).

3. MAXIMAL IDEALS

Theorem

If $A \prec B$ and $I \lhd \mathrm{End}(A)$ is proper, then $\mathcal{A}_I(B, B)$ is proper and not maximal.

Proof.

Set J the ideal of End(B) consisting of all morphisms that factors through A. Then

- $\mathcal{A}_I(B, B) \lhd \mathcal{A}_I(B, B) + J$
  - For any $(f, g) \in E$, $gf \in J$ and not in $\mathcal{A}_I(B, B)$, since $fgfg = 1_A \notin I$.

- $\mathcal{A}_I(B,B)+J \triangleleft \operatorname{End}(B)$
  - If $\varphi = h + \sum g_i f_i$ and $(f,g) \in E$, then

$$f\varphi g = fhg + \sum f g_i f_i g$$

$$A_\varphi = \{(f,g) \in E \mid f\varphi g \notin I\} \subseteq \bigcup_{i=1}^{n} \{(f,g) \in E \mid f g_i f_i g \neq 0\} \Rightarrow A_\varphi \neq E$$

  - However

$$A_{1_B} = \{(f,g) \in E \mid f 1_B g\} = E$$

  - This means that $1_B \notin \mathcal{A}_I(B,B)+J$.

Theorem

If $A \prec B$ and $I \lhd \mathrm{End}(A)$ is proper, then $\mathcal{A}_I(B, B)$ is proper and not maximal.

Corollary 1

If there do not exist maximal objects with respect to $\preceq$, then $\mathbf{C}$ does not have maximal ideals.

Proof.

If $\mathcal{M}$ is proper
$\Rightarrow \mathcal{M}(A, A) \neq A$
$\Rightarrow$ Since A is not maximal, $\exists A \prec B$
$\Rightarrow \mathcal{M}(B, B)$ is not maximal in End(B) by the Theorem.
$\Rightarrow \mathcal{M}$ is not maximal. $\square$

Corollary 2

Mod-R does not have maximal ideals (actually, there are no maximal ideals in any Grothendieck category).

Proof.

If M is a module $\Rightarrow M$ is $< \kappa$-generated $\Rightarrow M \prec M^{(\kappa)}$. $\square$

Corollary 3

If $\mathcal{M}$ is maximal

- $\mathcal{M}(A, A) \neq \mathrm{End}(A)$, then A is maximal with respect to $\preceq$.
- In other words, if A is not maximal with respect $\preceq$, $\mathcal{M}(A, A) = \mathrm{End}(A)$.

4. A STRATEGY FOR COMPUTING MAXIMAL IDEALS

Corollary 3

If $\mathcal{M}$ is maximal

- $\mathcal{M}(A, A) \neq \mathrm{End}(A)$, then A is maximal with respect to $\preceq$.
- In other words, if A is not maximal with respect $\preceq$, $\mathcal{M}(A, A) = \mathrm{End}(A)$.

Computing maximal ideals

- Take $\mathbf{M}(\mathbf{C}) =$ Maximal objects with respect to $\preceq$.
- Take a maximal ideal $\mathcal{M}$ of $\mathbf{M}(\mathbf{C})$.
- $I = \mathcal{M}(A, A) \neq \mathrm{End}(A)$ for some $A \in \mathbf{M}(\mathbf{C})$.
- Set $\mathcal{M}^e = \mathcal{A}_I$ the associated ideal in $\mathbf{C}$.

Computing maximal ideals

- Take $\mathbf{M}(\mathbf{C})$ = Maximal objects with respect to $\preceq$.
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Is $\mathcal{M}^e$ maximal in C?

$\Leftrightarrow \mathcal{M}^e(B, B) = \mathrm{End}(B)$ for each $B \notin \mathbf{M}(\mathbf{C})$

- $\mathbf{C} - \mathbf{M}(\mathbf{C}) = \mathbf{S}(\mathbf{C}) \cup \mathbf{T}(\mathbf{C})$, where
  - $\mathbf{S}(\mathbf{C})$ = non maximal B for which exists $C \in \mathbf{M}(\mathbf{C})$ with $B \prec C$.
  - $\mathbf{T}(\mathbf{C})$ = the rest of them.

Proposition

- $\mathcal{M}^e(B) = \mathrm{End}(B)$ *for each* $B \in \mathbf{S}(\mathbf{C})$.
- *Not always* $\mathcal{M}^e(B) = \mathrm{End}(B)$ *for each* $B \in \mathbf{T}(\mathbf{C})$.

Theorem

Given a preadditive category $\mathbf{C}$*, there is a bijective correspondence between:*

- *Maximal ideals of* $\mathbf{C}$*.*
- *Maximal ideals* $\mathcal{M}$ *of* $\mathbf{T}(\mathbf{C})$ *satisfying* $\mathcal{M}(A, A) = \mathrm{End}(A)$ *for every* $A \in \mathbf{T}(\mathbf{C})$*.*

THANK YOU VERY MUCH!