Commutators in the Category *SKB* of Skew Braces

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Lattices

Definition

A **lattice** is a partially order set $(L, \leq)$ such that every two elements $a, b \in L$ have a least upper bound (denoted by $a \vee b$ and called a **join** for a and b) and a greatest lower bound (denoted by $a \wedge b$ and called a **meet** for a and b).

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A lattice is **complete** if every infinite subset of L has a meet and a join.

Multiplicative Lattices

Definition (A. Facchini, C.A. Finocchiaro, G. Janelidze, 2022 [1])

A **multiplicative lattice** is a complete lattice L equipped with a further binary operation $\cdot : L \times L \to L$ (multiplication) satisfying $x \cdot y \leq x \wedge y$ for all $x, y \in L$.

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The smallest and the largest elements of the complete lattice L will be denoted by 0 and 1, respectively.

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Some Examples

- Let G be a group and $(\mathcal{N}(G), \cap, \cdot)$ the complete lattice of all normal subgroups of G. With the multiplication of any two normal subgroups N and M given by the commutator $[N, M]$, $\mathcal{N}(G)$ is a commutative (not associative) multiplicative lattice.

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- Let R be a ring and $(\mathcal{I}(R), \cap, +)$ the complete lattice of all two-sided ideals of R. As multiplication in $\mathcal{I}(\mathcal{R})$, take the product IJ of any two two-sided ideals I and J of R. Then $\mathcal{I}(R)$ is an associative (not commutative) multiplicative lattice.

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- Let R be a ring and $(\mathcal{I}(R), \cap, +)$ the complete lattice of all two-sided ideals of R. As multiplication in $\mathcal{I}(\mathcal{R})$, take the product IJ of any two two-sided ideals I and J of R. Then $\mathcal{I}(R)$ is an associative (not commutative) multiplicative lattice.
- We can also consider different multiplication in the same lattice. For instance $\mathcal{I}(R)$ is a multiplicative lattice also with respect to the multiplication given by $I \cdot J = [I, J]$ or $I \cdot J = I \cap J$.

Some Definitions

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1. The **lower central series** of x is the descending series

$$x =: x_1 \geq x_2 \geq x_3 \cdots$$

with $x_{n+1} := x_n \cdot x$. If $x_n = 0$ for some $n \geq 1$ then x is **(left) nilpotent.**

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2. The **derived series** of x is the descending series

$$x =: x^{(0)} \geq x^{(1)} \geq x^{(2)} \ldots$$

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If L is commutative, left nilpotency=right nilpotency.
If L is associative, left nilpotency=right nilpotency=solvability.

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4. The **right annihilator** of x is the element $r.\operatorname{ann}_L(x) := \bigvee\{y \in L \mid x \cdot y = 0\}$. Similarly the **left annihilator** of x is the element $l.\operatorname{ann}_L(x) := \bigvee\{y \in L \mid y \cdot x = 0\}$.

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5. The **right center** of x is the element $r.Z(x) := x \wedge r.\operatorname{ann}_L(x)$ and similarly the **left center** of x is the element $l.Z(x) := x \wedge l.\operatorname{ann}_L(x)$.

The Multiplicative Lattice $\mathcal{N}(G)$

For the multiplicative lattice $(\mathcal{N}(G), [-,-])$ of a group G, the element $1 = G$ of $\mathcal{N}(G)$ is nilpotent (resp. solvable, abelian) as an element of the multiplicative lattice if and only if the group G is nilpotent (resp. solvable, abelian).

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The center of the element $1 = G$ in $\mathcal{N}(G)$ is the center $Z(G)$ of the group G.

The Multiplicative Lattice $\mathcal{I}(R)$

For the multiplicative lattice $\mathcal{I}(R)$ of a ring R with operation the product of two ideals, the element $1 = R$ of $\mathcal{I}(R)$ is left nilpotent (=right nilpotent=solvable) as an element of the multiplicative lattice if and only if the ring R is nilpotent, and is an abelian element if and only if the ring R is an additive abelian group with the zero multiplication.

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Notice that the centralizer of an ideal and the center of a ring in the usual sense is not an ideal, but it is just a subring and they do not coincide with our definition of annihilator and center.

Left Skew Braces

Definition

A **(left) skew brace** is a triple $(A, *, \circ)$, where $(A, *)$ and $(A, \circ)$ are groups such that

$$a \circ (b * c) = (a \circ b) * a^{-*} * (a \circ c),$$

for every $a, b, c \in A$.
We indicate with $a^{-*}, a^{-\circ}$ the inverses of a respectively to the $*$ operation and the $\circ$ operation.

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Notice that $1_{(A,\circ)} = 1_{(A,*)}$. We will write 1 for the unique identity element.

Ideals of a Left Skew Brace

A subset I of a left skew brace $(A, *, \circ)$ is called an **ideal** of A if it is a normal subgroup of both the groups $(A, \circ)$ and $(A, *)$ such that $a * I = a \circ I$ for every $a \in A$. We will indicate with $\mathcal{I}(A)$ the set of all the ideals of A.

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Remark

A subset I of A is an ideal of A if and only if $(I, \circ)$ is a normal subgroup of $(A, \circ)$, $a * I = I * a$ and $\lambda_a(I) \subseteq I$ for every $a \in A$, where $\lambda\colon (A, \circ) \to \mathrm{Aut}(A, *)$, is the group homomorphism given by $\lambda\colon a \mapsto \lambda_a$, where $\lambda_a(b) = a^{-*} * (a \circ b)$.

Proposition

For any left skew brace A, there is a one-to-one correspondence between the set $\mathcal{I}(A)$ of all its ideals and the set $\mathcal{C}(A)$ of all its congruences, namely the set of all equivalence relations on A compatible with its two operations.

Remark

Let I, J be two ideals of A. Then

$$I * J = \bigcup_{i \in I} i * J = \bigcup_{i \in I} i \circ J = I \circ J.$$

Hence we will refer to this set as the product IJ of I and J.

Moreover, one can easily prove that IJ is an ideal of A.

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Can we find one a "right" notion of multiplication for the lattice $\mathcal{I}(A)$?

Commutators

Commutators play an important role in the study of algebraic structures.

[2] S.A. Huq, *Commutator, nilpotency and solvability in categories*, Quart. J. Oxford.

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The two most important generalizations of commutator for a generic algebraic structure are due to S. Huq[2] and J. Smith. [3]

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The **Huq commutator** of I and J is the smallest ideal H of A for which the map

$$\mu\colon I \times J \to A/H; \quad (i,j) \mapsto i * j * H$$

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Definition

The **Smith commutator** of I and J is the smallest ideal S of A for which the map

$$p\colon \{(x,y,z) \in A^3 \mid x{*}y^{-*} \in I,\ y{*}z^{-*} \in J\} \to A/S; \quad (x,y,z) \mapsto x{*}y^{-*}{*}z{*}S$$

is a well-defined skew brace morphism.

Commutator of Ideals of a Skew Brace

Theorem (Bourn, Facchini, P., 2022 [4])

Let I, J be two ideals of a left skew brace $(A, *, \circ)$. The Huq Commutator and the Smith Commutator of I and J coincide and it is the ideal generated by the union of the following three sets:

1. *the set* $\{i * j * i^{-*} * j^{-*} \mid i \in I, j \in J\}$;
2. *the set* $\{i \circ j \circ i^{-\circ} \circ j^{-\circ} \mid i \in I, j \in J\}$;
3. *the set* $\{i * j * (i \circ j)^{-*} \mid i \in I, j \in J\}$.

[4] D. Bourn, A. Facchini and M. Pompili, *Aspects of the category SKB of skew braces*, Comm. Algebra (2022), published online 5 Dec 2022.

The Multiplicative Lattice $\mathcal{I}(A)$

Consider the "commutator operation" given by

$$[-,-]\colon \mathcal{I}(A) \times \mathcal{I}(A) \to \mathcal{I}(A).$$

Remark

1. $[I, J] \subseteq I \cap J$;
2. $[I, J] = [J, I]$;
3. $[I, JK] = [I, J][I, K]$;

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$(\mathcal{I}(A), [-,-])$ is a **commutative multiplicative** lattice.

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- An ideal I of A is **abelian** if $[I, I] = 1$.

Center and Centralizer

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We have also the following:

- the **centralizer** (annihilator) $C_{(A,*,\circ)}(I)$ of an ideal I of A is the greatest ideal of A that contains $C_{(A,\circ)}(I) \cap C_{(A,*)}(I) \cap \mathrm{Ker}(\lambda\,|^{I} \colon (A, \circ) \to \mathrm{Aut}(I, *))$;

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- the **center** $Z(A, *, \circ)$ of A is the ideal $Z(A, *) \cap Z(A, \circ) \cap \mathrm{Ker}\,\lambda = \{a \in A \mid a \circ b = b \circ a = a * b = b * a, \text{ for all } b \in A\}$.

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Observe that A is abelian if and only if $Z(A, *, \circ) = A$.

Thank you for your attention!