# Infinite direct sums of finitely generated torsion free modules

(Joint work with Dolors Herbera and Román Álvarez Arias)

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# Setting and notation

- Let $R$ be a commutative noetherian domain of Krull dimension 1, $\tilde{R}$ its normalization (i.e., the integral closure of $R$ in its field of quotiens). We assume that $\tilde{R}$ is finitely generated $R$-module.
- A finitely generated torsion free module over $R$ is called a *lattice*.
- If $\mathfrak{m} \in \mathrm{Max}(R)$ and $M \in \mathrm{Mod}\text{-}R$ then $M_\mathfrak{m} := M \otimes_R R_\mathfrak{m}$.
- Two $R$-modules $M, N$ are *in the same genus* ($[M] = [N]$) if $M_\mathfrak{m} \simeq N_\mathfrak{m}$ in $\mathrm{Mod}\text{-}R_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$.

# The goal

- A ring $R$ has property (FD) if the class of direct sums of lattices is closed under direct summands.
- For noetherian domains (FD) means that every pure projective torsion free $R$-module is a direct sum of finitely generated modules.
- Describe commutative noetherian domains of Krull dimension 1 with module finite normalization satisfying (FD).

# Summary of the talk at ASTA 2014

**Theorem**

*Let $R$ be a commutative noetherian domain of Krull dimension 1 with module finite normalization $\tilde{R}$. If $R$ has (FD), then*

(a) *For every $\mathfrak{m} \in \mathrm{Max}(R)$ the normalization of $R_\mathfrak{m}$ is a discrete valuation domain.*

(b) *If $\mathfrak{m} \neq \mathfrak{n} \in \mathrm{Max}(R)$, $N$ is an indecomposable $R_\mathfrak{n}$-lattice and $M$ is an indecomposable $R_\mathfrak{m}$-lattice, then the ranks of $M$ and $N$ are coprime.*

*Moreover, $R$ is locally lattice finite then (a) and (b) imply (FD).*

# Wiegand's corollary

## Corollary

*(R. Wiegand) If $R$ is a commutative noetherian domain of Krull dimension 1 with module finite normalization and $R$ satisfies (FD), then there exists at most one $\mathfrak{m} \in \mathrm{Max}(R)$ such that $R_{\mathfrak{m}}$ is not a Bass domain.*

## Definition

A reduced commutative noetherian ring with module finite normalization is *Bass* if each of its ideals can be generated by at most 2 elements.

# Bass domains

For reduced one-dimensional noetherian rings with module finite normalization, H. Bass considered the following conditions

(a) Each indecomposable torsion-free $R$-module is isomorphic to an ideal.

(b) Each ring between $R$ and its normalization is Gorenstein.

(c) Every ideal of $R$ is 2-generated.

Bass showed (b) and (c) are equivalent and imply (a).
If $R$ is also local domain, then (a), (b), and (c) are equivalent.

# Factor categories and infinite direct sums

### Definition

Let $\mathcal{D}$ be a class of $\Lambda$-modules, $M$ an arbitrary $\Lambda$-module, and $I$ an ideal of $\mathrm{End}_\Lambda(M)$. The ideal $\mathcal{I}$ of $\mathcal{D}$ is called *the ideal associated to $I$* if $\mathcal{I}(X, Y) = \{f : X \to Y \mid gfh \in I \; \forall \, g : Y \to M, h : M \to X\}$.

### Example

(Harada, Sai) Let $\mathcal{D}$ be a class all direct sums of modules with local endomorphism ring, $M$ a module with local endomorphism ring $I = J(\mathrm{End}_\Lambda(M))$. Then $\mathcal{D}/\mathcal{I}$ is equivalent to $\mathrm{Mod}\text{-}\mathrm{End}_\Lambda(M)/J(\mathrm{End}_\Lambda(M))$ and the 'dimension' of the image in $\oplus_{i \in I} M_i$ in $\mathcal{D}/\mathcal{I}$ is the cardinality of $\{i \in I \mid M_i \simeq M\}$.

# A result we could apply

**Theorem**

*Let $M_i, i \in I$ be a family of finitely generated $\Lambda$-modules with semilocal endomorphism ring, $X, Y \in \mathrm{Add}(\oplus_{i \in I} M_i)$. Then $X \simeq Y$ if and only if $X$ and $Y$ are isomorphic in any factor category of the form $\mathrm{Add}(\oplus_{i \in I} M_i)/\mathcal{I}$, where $\mathcal{I}$ is the ideal associated a maximal ideal of $\mathrm{End}_R(M_i)$ for some $i \in I$.*

# Associated ideals and localization

- Let $R$ be a commutative ring, $M$ a finitely generated $R$-module, $\mathfrak{n}$ a maximal two-sided ideal of $\mathrm{End}_R(M)$.
- Consider $\varphi\colon R \to \mathrm{End}_R(M)$ and let $\mathfrak{m} := \varphi^{-1}(\mathfrak{n})$. Then $\mathfrak{m} \in \mathrm{Max}(R)$.
- Let $M_i, i \in \mathbb{N}$ be finitely presented $R$-modules (or finitely generated torsion free $R$-modules) and consider $A, A' \leq_{\oplus} \bigoplus_{i \in \mathbb{N}} M_i$.
- If $A_{\mathfrak{m}} \simeq A'_{\mathfrak{m}}$ then $A$ and $A'$ are isomorphic modulo the ideal of $\mathrm{Mod}\text{-}R$ associated to $\mathfrak{n}$.

# Summands of infinite direct sums have trivial genus

## Corollary

*Let $R$ be a commutative ring, $M_1, M_2, \ldots$ finitely presented $R$-modules with semilocal endomorphism rings. If $A, A'$ are direct summands of $\bigoplus_{i \in \mathbb{N}} M_i$ then $A \simeq A'$ if and only if $[A] = [A']$ (i.e., $A_\mathfrak{m} \simeq A_\mathfrak{m}$ for every $m \in \mathrm{Max}(R)$).*

## Remark

*(Facchini, Herbera) Finitely presented modules over semilocal rings have semilocal endomorphism rings.*

# Semilocal case

### Theorem

*Let $R$ be a commutative noetherian semilocal domain of Krull dimension 1 with module finite normalization $\tilde{R}$. Assume that*

(a) *For every $\mathfrak{m} \in \mathrm{Max}(R)$ the normalization of $R_\mathfrak{m}$ is a discrete valuation domain.*

(b) *There is at most one $\mathfrak{m} \in \mathrm{Max}(R)$ such that $R_\mathfrak{m}$ is not a Bass domain*

*Then $R$ satisfies (FD).*

Idea of the proof: If $A$ is a direct summand of $\bigoplus_{i\in\mathbb{N}} M_i$ then $A_\mathfrak{m}$ has to be a direct sum of $R_\mathfrak{m}$-lattices for every $\mathfrak{m} \in \mathrm{Max}(R)$. Use Package deal Theorem od Levy and Odenthal to create $A' = \oplus_{i\in\mathbb{N}} N_i$ where $N_i$ are lattices and $A_\mathfrak{m} \simeq A'_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$.

# Globalization

## Lemma

*Let $R$ be a noetherian domain of Krull dimension one with module finite normalization $\tilde{R}$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the list of maximal ideals of $R$ such that $R_\mathfrak{m}$ is a principal ideal domain any maximal ideal $\mathfrak{m} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$ and let $\Sigma = R \setminus \cup_{i=1}^k \mathfrak{m}_i$. Further let $M \subseteq \tilde{R}^{(\omega)}$ be such that $M_\Sigma$ is a direct sum of finitely generated $R_\Sigma$ modules. Then $M$ is a direct sum of finitely generated modules.*

## Corollary

*The previous theorem holds for an arbitrary commutative noetherian one-dimensional domain with module-finite normalization.*

# Projective modules over Dedekind domains

Let $R$ be a Dedekind domain. Note that all nonzero ideals of $R$ are in the same genus but the ideal class group of $R$ can be non-trivial. On the other hand, any projective $R$-module is either finitely generated or free.

Hence if $I_1, I_2, \ldots$ and $J_1, J_2, \ldots$ are nonzero ideals of $R$ then $\bigoplus_{i\in\mathbb{N}} I_i \simeq \bigoplus_{i\in\mathbb{N}} J_i$.

# Genus of infinite direct sums

**Proposition**

*Let $\Lambda$ be a module-finite algebra over a 1-dimensional noetherian domain $R$ such that $\Lambda_0$ is simple artinian. If $M_i, N_i \in \mathrm{Mod}\text{-}\Lambda$ are nonzero finitely generated torsion free $R$-modules and $M = \bigoplus_{i \in \mathbb{N}} M_i$, $N = \bigoplus_{i \in \mathbb{N}} N_i$ then $M \simeq N \Leftrightarrow [M] = [N]$.*

# Weak Krull-Schmidt theorem

**Theorem**

*(Facchini, 1996) Let $U_1, \dots, U_n$, $V_1, \dots, V_m$ be non-zero uniserial modules over a ring $\Lambda$. Then $\bigoplus_{i=1}^n U_i \simeq \bigoplus_{i=1}^m V_i$ if and only if $n = m$ and there are permutations $\sigma, \tau \in S_n$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for $i = 1, 2, \dots, n$.*

**Theorem**

*Let $R$ be commutative noetherian one dimensional domain with module finite normalization satisfying (FD) and $U_i, V_i, i \in \mathbb{N}$ non-zero ideals of $R$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ the list of maximal ideals of $R$ containing the conductor of $R$. Then $\bigoplus_{i\in\mathbb{N}} U_i \simeq \bigoplus_{i\in\mathbb{N}} V_i$ if and only if there are $\sigma_1, \dots, \sigma_k \in S(\mathbb{N})$ such that*

$$(U_i)_{\mathfrak{m}_j} \simeq (V_{\sigma_j(i)})_{\mathfrak{m}_j}, i \in \mathbb{N}, j = 1, \dots, k\,.$$

# End.

Thank you for your attention.