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Modules with the Exchange Property

Joint work with Yasser Ibrahim of both Tebah and Cairo Universities

A right R-module M is said to satisfy the (full) exchange property if for any two direct sum decompositions $M \oplus N = \oplus_{i \in I} N_i$, there exist submodules $K_i \subseteq N_i$ such that $M \oplus N = M \oplus (\oplus_{i \in I} K_i)$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite exchange property.

It is an open question due to Crawley and Jónsson whether the finite exchange property always implies the full exchange property. This question was provided a positive answer for the following classes of modules:

1. Modules with indecomposable decompositions by Zimmermann-Zimmermann.

2. Quasi-injective modules by L. Fuchs, where a module M is quasi-injective if it is invariant under endomorphisms of its injective hull.

3. Quasi-continuous modules by Mohamed and Müller & Oshiro and Rizvi, where a module M is quasi-continuous if it is invariant under idempotent-endomorphisms of its injective hull.

4. Auto-invariant Modules by P. Guil Assensio and A. Srivastava, where a module M is called auto-invariant if it is invariant under automorphisms of its injective hull.

5. Square-free modules by P. Nielsen, where M is called square-free if it does not contain a submodule isomorphic to a square $A \oplus A$.

Exchange rings are closely related to another interesting class of rings called clean rings that was first introduced by K. Nicholson, where a ring R is called clean if every element is the sum of an idempotent and a unit. A module M_R is called clean if $End(M_R)$ is a clean ring.

Semiperfect rings, unit-regular rings, strongly-regular rings, and rings of linear transformations of vector spaces are clean.

1. Nicholson proved that every clean ring is an exchange ring, and a ring with central idempotents is clean iff it is an exchange ring.

2. Camillo and Yu showed that an exchange ring with no infinite sets of orthogonal idempotents is clean.

While, we still don't have an answer to the following question:

$$R \text{ is clean iff } R \text{ is exchange} + (*) \text{ ?}$$

It will be nice to unify and extend the above results by Nicholson and by Camillo & Yu, and obtain new classes of modules where the finite exchange implies the full exchange.

This is what we know so far:

1. A module with an indecomposable decomposition is clean iff it has the (finite) exchange property.

2. (Assensio and Srivastava) Auto-invariant Modules are clean;

3. (Camillo, Khurana, Lam, Nicholson and Zhou) Continuous modules are clean;

4. (Nielsen) A square-free module is clean iff it has the finite exchange property, iff it has the full exchange property.

Dualizing Nielsen's Result

Definition 1 *A module M is called dual-square-free (DSF) if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. A ring R is called right DSF-ring, if it is a DSF-module as a right R-module.*

We should note in passing, being right dual-square-free ring is equivalent to saying that every maximal right ideal of R is two-sided. Rings whose maximal right ideals are two-sided are called quasi-duo, and it is an open question whether every right quasi-duo is left quasi-duo.

Theorem 2 *(Ibrahim-Yousif) If M is a DSF-module, then M has the finite exchange property iff M is clean, iff M has the full exchange property.*

Nicholson, Camillo & Yu Results

Next, we combine the result of Nicholson that says that abelian exchange rings are clean and that of Camillo & Yu that says that exchange rings with no infinite sets of orthogonal idempotents are clean.

$C4$-Modules with the restricted ACC on summands

Definition 3 *A module M is called a $C4$-module if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $\ker f \subseteq^{\oplus} A_1$, we have $\mathrm{Im} f \subseteq^{\oplus} A_2$.*

Definition 4 *A module M is called (summand-)square-free if it contains no non-zero isomorphic (direct-summand) submodules A and B with $A \cap B = 0$.*

Definition 5 *A module M is said to satisfy the restricted ACC condition on direct summands if, M has no strictly ascending chains of non-zero direct summands:*

$$A_1 \subsetneq A_2 \subsetneq \cdots$$
$$B_1 \subsetneq B_2 \subsetneq \cdots$$

with $A_i \cong B_i$ and $A_i \cap B_i = 0$ for all $i \geqslant 1$.

Clearly every summand-square-free module and every module with the ACC on direct summands (equivalently DCC on direct summands) satisfies the *restricted* ACC *on direct summands.* In particular, modules with finite Goldie (or finite dual Goldie) dimension are examples of modules with the restricted ACC on direct summands. Thus semilocal rings and rings with no infinite sets of orthogonal idempotents satisfy both the left and right restricted ACC on direct summands.

Theorem 6 *Let M be a $C4$-module with the restricted ACC on direct summands. Then M satisfies the finite exchange property iff M is clean, iff M has the full exchange property.*

Corollary 7 *If M is a summand-square-free module, then M has the finite exchange property iff M is clean, iff M has the full exchange property.*

$D4$-**Modules with the restricted** DCC **on summands**

Definition 8 *A module M is called a $D4$-module if, whenever A and B are submodules of M with $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism with* $\mathrm{Im} f \subseteq^{\oplus} B$, *then* $\ker f \subseteq^{\oplus} A$.

Definition 9 *A module M is called (summand-)dual-square-free if M has no proper (summand) submodules A and B with $M = A + B$ and $M/A \cong M/B$.*

Definition 10 *A module M is said to satisfy the restricted DCC condition on direct summands if, M has no strictly descending chains of non-zero direct summands:*

$$A_1 \supsetneqq A_2 \supsetneqq \cdots$$
$$B_1 \supsetneqq B_2 \supsetneqq \cdots$$

with $(A_i+B_i)/A_i \cong (A_i+B_i)/B_i$ and $A_i+B_i \subseteq^{\oplus} M$ for all $i \geqslant 1$.

Clearly every summand-dual-square-free module and every module with the DCC on direct summands (equivalently ACC on direct summands) satisfies the *restricted DCC on direct summands.* In particular, modules with finite Goldie (or finite dual Goldie) dimension are examples of modules with the restricted DCC on direct summands. Thus semilocal rings and rings with no infinite sets of orthogonal idempotents satisfy both the left and right restricted DCC on direct summands.

Theorem 11 *Let M be a $D4$-module with the restricted DCC on direct summands. Then M satisfies the finite exchange property iff M is clean, iff M has the full exchange property.*

Corollary 12 *If M is a summand-dual-square-free module, then M has the finite exchange property iff M is clean, iff M has the full exchange property.*

Corollary 13 *Exchange rings with the restricted DCC on direct summands are clean.*

Corollary 14 *(Camillo & Yu) Exchange rings with no infinite sets of orthogonal idempotents are clean.*

Since abelian rings are summand-dual-square-free, we have:

Corollary 15 *(Nicholson) Abelian exchange rings are clean.*

Example 16 *If $\mathcal{P}$ is an infinite set of distinct primes $p \in \mathbb{Z}$, and M is the $\mathbb{Z}$-module $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$, then M is an exchange, clean, square-free module (hence a $C4$-module with the restricted ACC on direct summands), and dual-square-free module (hence a $D4$-module with the restricted DCC on direct summands). However, M is an infinite direct sum of indecomposables.*

Thank you