Abstract Properties of Linear Groups

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Spectra of Groups

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Abstract

The aim of this paper is to investigate the behaviour of prime and semiprime subgroups of groups, and their relation with the existence of abelian normal subgroups. In particular, we study the set Spec(G) of all prime subgroups of a group G endowed with the Zariski topology and, among other examples, we construct an infinite group whose proper normal subgroups are prime and form a descending chain of type $\omega + 1$.

Keywords Prime subgroup · Semiprime subgroup · Zariski spectrum

Mathematics Subject Classification (2020) 20E15 · 20F19

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Examples of linear groups

All finite groups are linear (A. Cayley 1854, C. Jordan, 1870)
Free groups are linear (V.N. Nisnevič, 1940)

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Moreover, the knowledge of linear groups over locally finite fields is crucial in the theory of locally finite groups

William Burnside
1852–1927

In 1902 **Burnside** asked if every finitely generated periodic group is finite, or equivalently if all periodic groups are *locally finite*. Only in 1964 Burnside's question was answered negatively by Golod.

Burnside's theorem: *Let G be a group of finite exponent. If G is linear over a field of characteristic zero, then G is finite*

Issai Schur
1875–1941

In 1911, **Issai Schur** proved that, for any field $\mathfrak{K}$ and any positive integer n, every finitely generated periodic subgroup of $GL(n, \mathfrak{K})$ is finite. Thus:

Every periodic linear group is locally finite, i.e. all its finitely generated subgroups are finite

Jacques Tits
1930–2021

Tits' Alternative

Let G be a finitely generated linear group. Then either G is soluble-by-finite or it contains a free non-abelian subgroup of rank 2

Anatoliĭ I. Mal'cev
1909–1967

Mal'cev's theorem (see also Lie-Kolchin)

Every soluble linear group G is nilpotent-by-abelian-by-finite, i.e. G contains normal subgroups $N \leqslant A$ such that N is nilpotent, A/N is abelian and G/A is finite

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• Any locally nilpotent linear group is hypercentral (M.S. Garaščuk)

• Any locally nilpotent linear group contains a nilpotent subgroup of finite index (K.W. Gruenberg)

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- Linear groups whose proper subgroups are close to being nilpotent, *Comm. Algebra* 49 (2021)
- Groups whose proper subgroups are linear, *J. Algebra* 592 (2022)
- The upper and lower central series in linear groups, *Quarterly J. Math.* 73 (2022)
- Linear groups with restricted conjugacy classes, *Ricerche Mat.* 71 (2022)
- Subnormality in linear groups, *J. Pure Appl. Algebra* 227 (2023)

Topological methods in linear groups

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A topological space $(S, \mathcal{T})$ is called a *Z-space* if all finite subsets of S are closed and S satisfies the minimal condition on closed subsets in this case $\mathcal{T}$ is a *Z-topology*

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Any Z-space is the union in a unique way, of finitely many non-empty connected subsets which are open and closed
(the *connecetd components* of the Z-space)

A group G is called a *CZ-group* if its underlying set carries a Z-topology such that for each $g \in G$ the mappings

$$x \mapsto x^{-1} \qquad x \mapsto gx \qquad x \mapsto xg \qquad x \mapsto x^{-1}gx$$

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Any linear group $G \leqslant GL(n, \mathfrak{K})$ can be made into a CZ-group in a natural way: its Z-topology is induced by the *Zariski topology* of a $\mathfrak{K}$-vector space of dimension n^2

Let G be a linear group and let G^0 be
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Moreover, G^0 is contained in any closed subgroup of finite index of G and centralizes every element of G admitting only finitely many conjugates

It is also important to notice that if G is a linear group over a field $\mathfrak{K}$ and N is a closed normal subgroup of G, then the factor group G/N is linear over $\mathfrak{K}$

The theorems of Schur, Baer and Hall

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The consideration of any infinite extraspecial p-group shows that converses of Schur's and Baer's theorem do not hold in general

The theorem of Merzljakov

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Juriĭ I. Merzljakov
1940–1995

Let G be a linear group. Then $G/\zeta_k(G)$ is finite for some positive integer k if and only if $\gamma_{k+1}(G)$ is finite
(Ju.I. Merzljakov, 1967)

Let $\mathfrak{X}$ be a class of groups. We say that

- a theorem of *Schur type* holds for $\mathfrak{X}$ if,

for any group G such that $G/\zeta(G) \in \mathfrak{X}$, also G' belongs to $\mathfrak{X}$

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- a theorem of *Hall type* holds for $\mathfrak{X}$ if, for any group G such that $\gamma_{k+1}(G) \in \mathfrak{X}$ for some positive integer k, also $G/\zeta_{2k}(G)$ belongs to $\mathfrak{X}$

- a theorem of *Merzljakov type* holds for $\mathfrak{X}$ if, for any linear group G and any positive integer k, the factor group $G/\zeta_k(G)$ belongs to $\mathfrak{X}$ for some positive integer k if and only if $\gamma_{k+1}(G) \in \mathfrak{X}$

It is known that theorems of Schur, Baer and Hall type hold for the class of polycyclic groups

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Sergei N. Černikov
1912–1987

A theorem of Schur type holds for the class of Černikov groups
(S.N. Černikov, 1959)

A group is called a *Černikov group* if it satisfies the minimal condition on subgroups and contains an abelian subgroup of finite index

A theorem of Hall type does not hold
for the class of Černikov groups:
the holomorph of a group of type p^∞ (with $p > 2$)
has Černikov commtator subgroup and trivial centre

	Baer	*Hall*	*Merzljakov*
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic	✓	✓	✓
Soluble minimax	✓	✗	✓
Soluble of finite Rank	✓	✗	$char > 0$: ✓ $char = 0$: ✗

Subnormality in linear groups

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Let X be a subgroup of a group G

- X is *serial* in G if there exists a series between X and G
- X is *ascendant* (*descendant*) in G if there exists an ascending (descending) series between X and G
- X is *subnormal* in G if there exists a finite series between X and G

Let X be a serial subgroup of a periodic linear group $G \leqslant GL(n, \mathfrak{K})$. Then X is ascendant in G of length at most $\omega + \kappa(n)$, where κ is an integer-valued function of n only

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where κ is an integer-valued function of n only

Let X be a descendant subgroup of a periodic linear group G.
Then X is subnormal in G

Wielandt's subnormality criterion

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Helmut Wielandt
1910–2001

A subgroup X of a finite group G is subnormal if and only if X is subnormal in $\langle X, X^g \rangle$ for each $g \in G$
(H. Wielandt, 1974)

This important subnormality criterion cannot be extended to the infinite case, even if the attention is restricted to periodic linear groups

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A subgroup X of a periodic linear group G is ascendant if and only if X is ascendant in $\langle X, X^g \rangle$ for each $g \in G$

Hall's nilpotency criterion

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Philip Hall
1904–1982

Let G be a group containing a nilpotent normal subgroup N such that G/N' is nilpotent. Then G is nilpotent
(P. Hall, 1958)

Let $\mathfrak{X}$ be a class of groups.
$\mathfrak{X}$ is called a *Hall class* if it contains every group G admitting a nilpotent normal subgroup N such that G/N' is in $\mathfrak{X}$

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Most of the relevant group classes have this property

However we have constructed, for any prime q, a linear group G over a field of characteristic q which is not finite-by-(locally nilpotent), although it contains a nilpotent normal subgroup N such that G/N' is finite-by-abelian

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This choice is necessary, as the following result shows

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Let G be a linear group over an algebraically closed field $\mathfrak{K}$ and let N be a nilpotent normal subgroup of G such that G/N' is finite-by-nilpotent. Then G is finite-by-nilpotent, provided that one of the following conditions holds:

- *$\mathfrak{K}$ has characteristic 0;*
- *N or G is connected;*
- *N_u is connected or abelian*

where N_u is the unipotent component of N in its Jordan decomposition