Point vortices on non-orientable surfaces

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joint work with Nataliya Balabanova

$\Gamma > 0$

speed $\sim 1/r$

$\Gamma < 0$

Kirchhoff $\to$ Hamiltonian structure

N point vortices

In $\mathbb{R}^2$

$$H = -\frac{1}{2\pi} \sum_{i<j} \Gamma_i \Gamma_j \ln |\underline{x}_i - x_j|$$

Sympl form

$$\Omega = \bigoplus \Gamma_j \sigma_j$$

$\sigma_j = \sigma$ on j^{th} copy of $\mathbb{R}^2$

Phase space is

$$\underbrace{\mathbb{R}^2 \times \dots \times \mathbb{R}^2}_{N} \setminus \Delta$$

some $\underline{x}_i = \underline{x}_j$

Motion of x_i given by Hamilton's equation.

For more general surfaces

$$H = -\sum_{i<j} \Gamma_i \Gamma_j \, G(x_i, x_j)$$

$$\Delta_x G(x,y) = \delta_y(x).$$

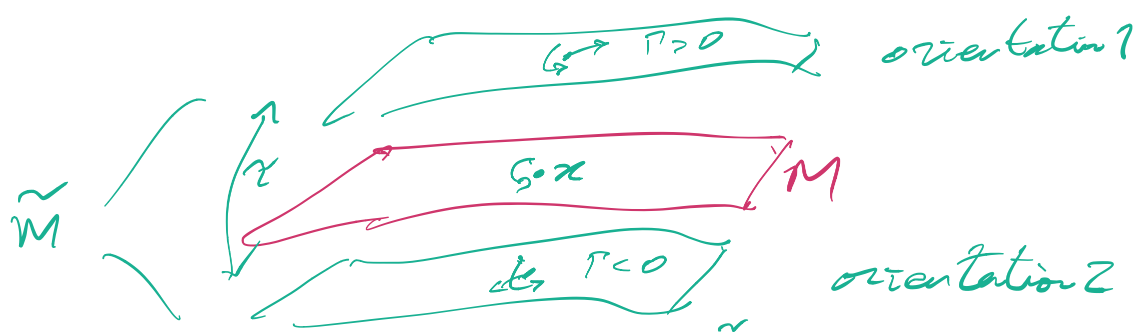
<u>Vorticity</u>:

$\underline{v}$ = velocity field

$\underline{\lambda} = \operatorname{curl} \underline{v}$ is vorticity

twisted / pseudo scalar in 2D

$\underline{\lambda}$ depends on choice of orientation



$$\begin{array}{ccc} \tilde{M} & = & \{ \overset{\tilde{x}}{(x, or_x)} \mid x \in M \} \\ \downarrow & & \downarrow \\ M & & x \end{array}$$

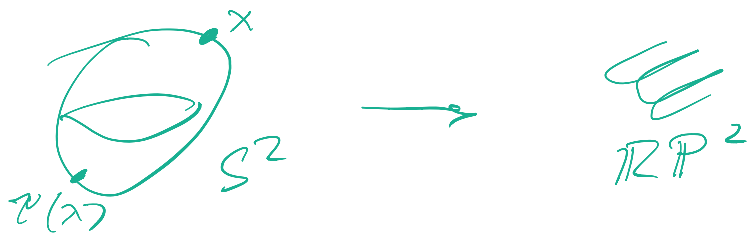
$$\tau(x, or_x) = (x, -or_x)$$

An object ξ is twisted if

$$\tau(\xi(\tilde{x})) = -\xi(\tau(\tilde{x}))$$

$\tilde{M}$ has a tautological orientation

τ is orientation reversing



Marsden–Weinstein (1983)

Flow is via volume preserving diffeomorphisms of M

$SDiff(M)_0$ isotopic to identity.

Vorticity $\sim$ $Lie(SDiff_0)^*$

initial condition $\lambda \to$ vorticity as a 2-form flow / vector field $\underline{v}$

$\underline{v}^\flat$ – 1-form Riem. structure

$\underline{\lambda} = d(\underline{v}^\flat)$ – 2-form

(scalar vorticity λ = Hodge star of $\underline{\lambda}$)

Phase space is

$\underline{\lambda} \in \Omega^2$

$$\mathcal{O}_{\underline{\lambda}} = \{ \varphi^* \underline{\lambda} \mid \varphi \in SDiff_0(M) \}$$

$$\Omega_{\underline{\lambda}} (L_{\underline{u}} \underline{\lambda}, L_{\underline{v}} \underline{\lambda}) = \int_M \underline{\lambda} (\underline{u}, \underline{v}) \, dA$$

$$H(\underline{\lambda}) = \frac{1}{2} \int \rho |\underline{v}|^2 \, dA = \frac{1}{2} \int \langle \underline{\lambda}, \underline{\psi} \rangle \, dA$$

where $\Delta \underline{\psi} = \underline{\lambda}$.

$(\underline{\psi} = \Delta^{-1} \underline{\lambda} .)$

For point vortices:

(scalar) $\lambda = \sum_{j=1}^{N} \Gamma_j \, \delta_{x_j}$ $\qquad$ Γ_j constant

$x_j \in M$

$\mathcal{O}_\lambda$ Parametrized by

$$M \times \dots \times M \setminus \Delta \dashrightarrow \mathcal{O}_\lambda$$

$$(x_1, \dots, x_N) \longmapsto \sum \Gamma_j \, \delta_{x_j}$$

If M is non-orientable this parametrization doesn't work.

Instead parametrize by

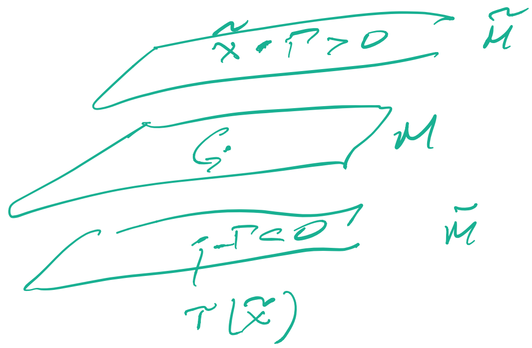
$$\tilde{M} \times \dots \times \tilde{M} \setminus \tilde{\Delta} \longrightarrow \mathcal{O}_\lambda$$

$$(\tilde{x}_1, \dots, \tilde{x}_N) \longmapsto \hat{\lambda} = \sum \Gamma_j \left(\delta_{\tilde{x}_j} - \delta_{\tau(\tilde{x}_j)} \right)$$

2N point vortices in total

$\downarrow$

use N of these



1 vortex on M

$\to$ 2 lifts on $\tilde{M}$

Hamiltonian is

$$H(\tilde{x}) = - G(\tilde{x}, \tau(\tilde{x}))$$

For $S^2/\mathbb{RP}^2$ this is constant

Example of a Robin function.

Eg for Möbius band (flat metric)

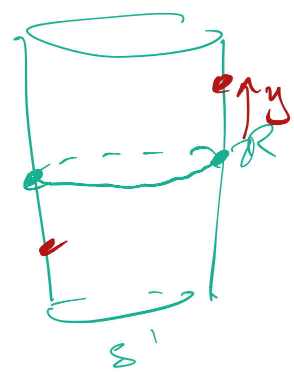
$$R(x,y) = \frac{1}{2\pi} \ln \cosh^2 y .$$

Möbius band:

double cover is cylinder $S^1 \times \mathbb{R}$
$x \quad , \quad y$

identify

$(x, y) \sim (x+\pi, -y)$



More general Hamiltonian is

$$H(x_1,\dots,x_N) = -\sum_{i<j} \Gamma_i \Gamma_j \, G(x_i, x_j) - \frac{1}{2} \sum_j \Gamma_j^2 \, R(x_j)$$

Robin function

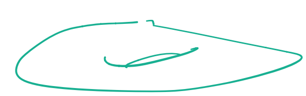
$$R(x) = \lim_{y \to x} \left(G(x,y) - \frac{1}{2\pi} \ln d(x,y) \right)$$

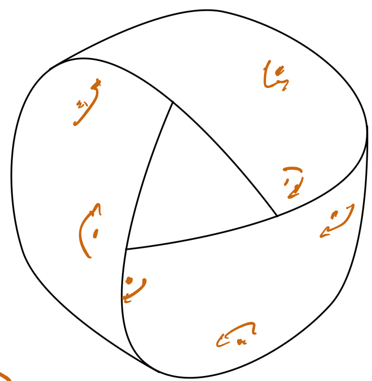
→ regular function

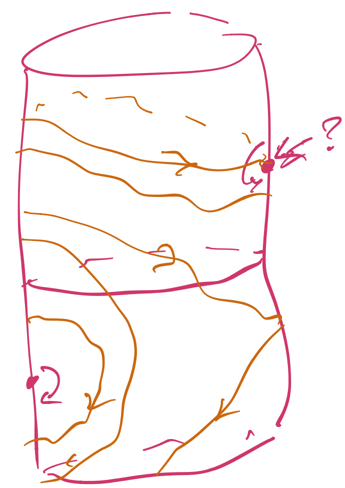
governs 'self-interaction' of a point vortex.

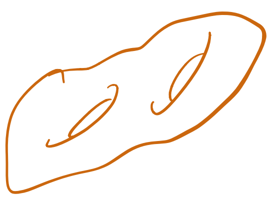
if $\phi : M \circlearrowleft$ is isometry

then $R \circ \phi = R$.

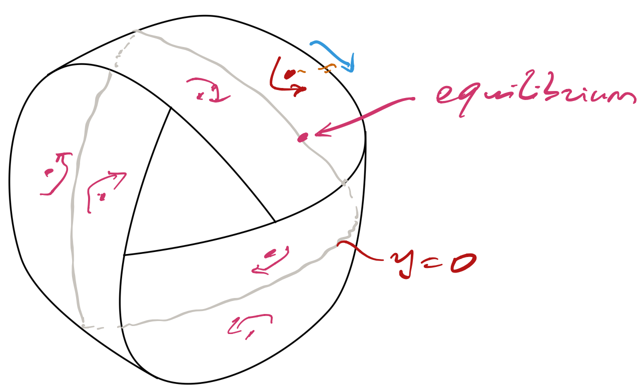








$\dot{x} = \tanh y$



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