Symbolic dynamics and oscillatory motions in the 3 body problem

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The planar three body problem

- Three co-planar bodies of masses $m_0, m_1, m_2 > 0$ under Newtonian gravitational force:
  \[
  \frac{d^2 q_i}{dt^2} = \sum_{j=0, j\neq i}^{2} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_0, q_1, q_2 \in \mathbb{R}^2
  \]

- Long term dynamics?

- Two strongly related problems:
  - Chaotic motions
  - Final motions
Chaotic motions

(One of) the paradigmatic chaotic dynamics is the shift – the Smale horseshoe.

Take either $X = \{1, \ldots, M\}$ or $X = \mathbb{N}$ and $X^\mathbb{Z}$.

Shift $\sigma : X^\mathbb{Z} \rightarrow X^\mathbb{Z}$ defined as

$$(\sigma \omega)_k = \omega_{k+1}$$

Properties:

- Sensitive dependence on initial conditions.
- Topologically transitive: dense orbits in $X^\mathbb{Z}$.
- Periodic orbits are dense in $X^\mathbb{Z}$.
- Positive topological entropy

Can we embed this dynamics in the 3BP?
Chazy (1922): Final motions for the 3 body problem

Call $r_i$ the mutual distances between bodies.

Final motions: Possible behaviors of the 3BP when $t \to \pm \infty$.

- **Hyperbolic ($H^\pm$):** $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$.
- **Hyperbolic-Parabolic (HP_k^\pm):** $|r_i| \to \infty \forall i$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$.
- **Hyperbolic-Elliptic (HE_k^\pm):** $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \pm \infty$.
- **Parabolic-Elliptic (PE_k^\pm):** $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \infty$.
- **Parabolic ($P^\pm$):** $|r_i| \to \infty$, $|\dot{r}_i| \to 0$.
- **Bounded ($B^\pm$):** $\sup_{t \geq t_0} |r_i| < \infty$.
- **Oscillatory (OS^\pm):** $\limsup_{t \to \pm \infty} \sup_i |r_i| = \infty$, $\liminf_{t \to \pm \infty} \sup_i |r_i| < \infty$. 
Final motions for the 3BP

- In the limit $m_1, m_2 \to 0$: Two uncoupled two 2BP.
- Only $H, P, HP_k, HE_k, PE_k, B$ (All except oscillatory motions!).
  - Past and future final motions must coincide,
- Questions by Chazy (1922) for the 3BP:
  - Do oscillatory motions exist?
  - Can one combine different past and future final motions?
- Long literature on oscillatory motions (and also on chaotic motions) for the 3BP. But:
  - Most for the Restricted 3BP: $m_2 = 0$ so $q_0, q_1$ perform a 2BP.
  - Quite strict assumptions on the masses of the bodies.
First result:

- **Sitnikov (1960)** considered the Restricted spatial elliptic 3BP.
- Existence of oscillatory motions when
  - $m_1 = m_2 = 1/2$ and $q_1, q_2$ move on ellipses of small eccentricity.
  - $q_3$ ($m_3 = 0$) moves on the (invariant) vertical axis.
- Free combination of past and future final motions.
Oscillatory motions: Past results

- **Moser** (1973): New proof of Sitnikov results via chaotic motions.

- **Restr. Planar Circular 3BP**: $q_0, q_1$ (masses $\mu$ and $1 - \mu$, $\mu \in (0, \frac{1}{2}]$) perform circular motion and $q_2$ is coplanar:
  - Simó and Llibre (1980): Oscillatory motions for $0 < \mu \ll 1$.
  - G.-Martín-Seara (2016): Oscillatory motions for all masses: $\mu \in (0, \frac{1}{2}]$.

- Other approaches, other results for the Restricted 3BP: Kaloshin-Galante, G.-Martín-Sabbagh-Seara, Xia, Seara-Zhang...

- Oscillatory motions for the **full 3BP** ($m_2 > 0$):
  - Alexeev (1969) for a Sitnikov-like model: $m_0 = m_1 \gg m_2$.
  - Moeckel (2007): For a “large” (non-generic) choice of masses. The proof relies on the passage close to triple collision.
Abundance of the Final motions

- Measure of each $X^- \cap Y^+$?
- It is known for each $X^- \cap Y^+$ whether it has positive or zero measure except for $OS^- \cap OS^+$.
- (Wide open) Conjecture (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.
- According to Arnold this is the central problem of Celestial Mechanics.
- Kaloshin–Gorodetski (2011): The Hausdorff dimension of oscillatory motions is maximal for both the Sitnikov problem and the RPC3BP.
- Recall that most existence results deal with Restricted models/narrow ranges of parameters!
Main result: Final motions

**Theorem (G.–Martín–Seara)**

Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then,

$$X^{-} \cap Y^{+} \neq \emptyset$$

with $X, Y = OS, B, PE_2, HE_2$.

In particular, $OS^{-} \cap OS^{+} \neq \emptyset$.

- The bodies of masses $m_0$ and $m_1$ perform (approximately) circular motions. That is, $|q_0 - q_1|$ is approximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.
- We can combine all possible negative energy final motions.
Main result: Chaotic motions

Theorem (G.–Martín–Seara)

Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$ and denote by $\Phi_t$ its flow. Then, there exists a section $\Pi$ transverse to $\Phi_t$ such that the induced Poincaré map

$$\mathcal{P} : \mathcal{U} = \mathcal{U} \subset \Pi \to \Pi$$

has an invariant set $\mathcal{X}$ which is homeomorphic to $\mathbb{N}^\mathbb{Z}$ such that $\mathcal{P}|_{\mathcal{X}}$ is topologically conjugated to the shift.

- The set $\mathcal{X}$ is a hyperbolic set once the 3BP is reduced by its classical first integrals.
- Previous results also in other regions of phase space (but for quite strict mass choices): Bolotin-McKay, Bolotin, Marco-Niederman, Arioli, Wilczak-Zgliczynski, Capinski,...
Main result: Chaotic motions

- $q_0$ and $q_1$ evolve at a bounded distance from their center of mass.
- $q_2$ makes excursions close to a parabolic motion.

Let $N_k$ be the number of complete revolutions of $q_0$, $q_1$ between two consecutives passages of $q_2$ through $\Pi$.

There exist integers $L \gg 1$ such that for any $\omega \in \mathbb{N}^\mathbb{Z}$, there exists a solution of the 3BP such that $N_k = L + \omega_k$. 
The proof of the two theorems

- We follow the Moser approach for the Sitnikov Problem
- Its implementation for the 3BP presents several difficulties
- Plan for the rest of the talk:
  - Moser’s proof for the Sitnikov model
  - Differences between the Sitnikov model and the planar 3BP
  - Proof for the planar 3BP.
Moser’s approach for the Sitnikov problem

- Sitnikov model: \( H(p, q, t) = \frac{p^2}{2} - \frac{1}{\sqrt{q^2 + R(t)}} \)

- \( R(t) \) is the radius of the ellipses: \( R(t) = \frac{1}{2} + \frac{\varepsilon}{2} \cos t + O(\varepsilon^2) \).

- \( P_{\pm} = (q, p, t) = (\pm \infty, 0, t), \ t \in \mathbb{T} \) are periodic orbits at infinity

- Compactification (McGehee) change of coordinates \( q = f(x) \) so that both become \( P = (0, 0, t) \).

- **Step 1:** Check that \( P \) has stable and unstable invariant manifolds
  Not obvious! Its linearization is not hyperbolic but parabolic, i.e. degenerate linearization (McGehee 1972).
Moser’s approach for the Sitnikov problem

- Take the stroboscopic Poincaré map \( f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \),

- **Step 2:** Check that the invariant manifolds split for \( 0 < \varepsilon \ll 1 \).

- Melnikov Method: \( \alpha(\varepsilon) = M\varepsilon + O(\varepsilon) \) with \( M \neq 0 \).

- If \( P \) were hyperbolic: Smale Theorem would lead to a Smale horseshoe which contains \( P \) on its closure.
Moser approach for the Sitnikov problem

- \( P \) is a topological saddle:
  
  Locally
  
  $$
  \begin{align*}
  \dot{x} &= -(x + y)^3 x (1 + \text{h.o.t}) \\
  \dot{y} &= (x + y)^3 y (1 + \text{h.o.t}) \\
  \dot{t} &= 1
  \end{align*}
  $$

- **Step 3:** Parabolic Lambda lemma for the local dynamics.

- **Step 4:** Construct an isolating block (plus cone fields) to build the Smale horseshoe.

- **Key point:** The model can be reduced to a 2D area preserving map.

- Same happens for the RPC3BP and the Alexeev model.
How can one implement Moser approach for the 3BP?

- After reducing by the classical first integrals: 3 degrees of freedom (dimension 6).

Analog of infinity in a fixed energy level is: $\mathbb{D} \times \mathbb{T} \subset \mathbb{R}^2 \times \mathbb{T}$.

2 parameter family of periodic orbits at infinity.

There are “center” directions: much harder to build hyperbolic sets.

- We want results for all values of the masses $m_0, m_1, m_2 > 0$ ($m_0 \neq m_1$).

Transversality of the invariant manifolds of infinity?

We need a close to integrable regime.
A good system of coordinates

- Jacobi coordinates to reduce conservation of linear momentum:
  
  \[ Q_1 = q_1 - q_0 \]
  
  \[ Q_2 = q_2 - c_{01}, \quad c_{01} = \frac{m_0}{m_0 + m_1} q_0 + \frac{m_1}{m_0 + m_1} q_1 \]

- \( Q_1 \) performs approximate ellipses \(\rightarrow\) Delaunay coordinates \((\ell, g, L, \Gamma)\) (Action-angle coordinates for the 2BP with negative energy)
  
  - \( \ell \) mean anomaly
  - \( L \) square of the semi-major axis
  - \( g \) argument of the perihelion
  - \( \Gamma \) angular momentum

- \( Q_2 \) performs approximate parabolas \(\rightarrow\) Polar coordinates \((r, \theta, y, G)\)
  
  - \( y \) radial momentum and \( G \) angular momentum.
A good system of coordinates

- Variables: \((\ell, g, L, \Gamma)\) and \((r, \theta, y, G)\)
- Invariance by rotation \(\Theta = \Gamma + G\) is a first integral
- The Hamiltonian depends on \(g\) and \(\theta\) through \(\phi = g - \theta\).
- Reducing by the rotations: 3 d.o.f Hamiltonian depending on the parameter \(\Theta\).
  \[ H = H(\ell, \phi, r, L, \Gamma, y; \Theta) \]
- Delaunay coordinates are not well suited for close to circular motion.
- Poincaré variables: \((\lambda, \alpha, r, L, \beta, y)\) with
  \[ \lambda = \ell + \phi, \quad \alpha = \sqrt{2(L - \Gamma)}e^{i\phi}, \quad \bar{\alpha} = \sqrt{2(L - \Gamma)}e^{-i\phi}. \]
- New Hamiltonian \(H = H(\lambda, \alpha, r, L, \beta, y; \Theta)\).
Parabolic infinity and its manifolds

\[ H = H(\lambda, \alpha, r, L, \beta, y; \Theta) \]

- Reaching infinity with zero velocity: \( r = \infty \) and \( y = 0 \).
- At \((r, y) = (\infty, 0)\): Hamiltonian is \( H = -\frac{1}{2L^2} \).
- Fixing energy \( \equiv \) Fixing the semimajor axis of the primaries.
- Dynamics at an energy level: \( \lambda \in \mathbb{T} \) and \( \alpha = \bar{\beta} \in \mathbb{D} \subset \mathbb{C} \) and

\[
\begin{align*}
\dot{\lambda} &= \frac{1}{L^3}, \\
\dot{\alpha} &= \dot{\beta} = 0.
\end{align*}
\]

- It is foliated by periodic orbits \( \mathbb{T}_{\alpha_0, \beta_0} = \{\alpha = \alpha_0, \beta = \beta_0\} \).
- Baldomá-Fontich-Martín: \( \mathbb{T}_{\alpha_0, \beta_0} \) have invariant manifolds.
Moser Approach: Scheme

Transversality of the infinity invariant manifolds
two homoclinic channels

Two Scattering maps

Two global maps

Parabolic Lambda lemma

Local map

Isolating block for a suitable iterate of the return map

Oscillatory motions

Symbolic dynamics
A different nearly integrable regime

- To prove transversality between the invariant manifolds of infinity we must be in a perturbative regime.
- Take $|Q_2| \gg |Q_1| \iff r \gg L$.
- Equivalently: Fix $L \sim 1$ (energy level) and take $\Theta \gg 1$.
- From the third body, the first two are “very close to each other”.
- At first order, the third body only “sees” one body at the center of mass of the other two.
- Conclusion:

$$H = \text{Two uncoupled Kepler problems} + \text{Small perturbation}.$$ 

- Two time scales: Motion on the far away parabola is much slower than the motion on the ellipses.
Transversality of the invariant manifolds

- Fix energy $H = -1/2$. Parabolic infinity:

$$\Lambda_\infty = \{ L = 1, r = +\infty, y = 0, \lambda \in \mathbb{T}, \alpha = \overline{\beta} \in \mathbb{D} \subset \mathbb{C} \}$$

- The parameter $\Theta \gg 1$ measures the distance between the third body and the other two.

**Theorem**

Take any $m_0, m_1, m_2 > 0$, $m_0 \neq m_1$. There exist $z_0^0, z_0^1 \in \mathbb{D}$ such that:

- $\mathbb{T}_{z_0^0}$ and $\mathbb{T}_{z_0^1}$ have homoclinic connections.
- The invariant manifolds of $\Lambda_\infty$ intersect transversally along them.
- The points $z_0^0 = (\alpha_0^0, \beta_0^0)$, $z_0^1 = (\alpha_0^1, \beta_0^1)$ satisfy

$$\alpha_0^1 - \alpha_0^0 = -\frac{6i}{\sqrt{\pi}} \Theta^{9/2} e^{-\frac{\Theta^3}{3}} \left(1 + \mathcal{O}\left(\Theta^{-1/2}\right)\right).$$
Transversality between the invariant manifolds

- Hidden in the theorem: The transversality between the invariant manifolds is exponentially small (in some directions)

\[ \text{angle} \sim \Theta^{-1/2} e^{-\Theta^3/3}, \quad \Theta \gg 1 \]

(Exponentially small splitting of separatrices)

- Why? Fast rotation coupled with slow motion on the invariant manifolds.

- Classical perturbative methods (Melnikov Theory) cannot be applied to construct these transverse homoclinic orbits.

- Deep analysis of the analytic extension of the parameterizations of the perturbed invariant manifolds in complex domains.
How to construct an isolating block

- Take $\lambda$-Poincaré map and reduce by the energy: 4 dimensional symplectic map $\mathcal{P}$.

- We need a 4 dimensional isolating block for $\mathcal{P}$.

- In the $(r, y)$–plane: Proceed as in the Sitnikov problem (Moser construction following the Smale Theorem).

- How do we construct an isolating block for the “center” $(\alpha, \beta)$ variables?

- Main tool: The scattering map.
The scattering map (after Delshams – de la Llave – Seara)

- Normally hyperbolic (parabolic) invariant manifold $\Lambda$
- Its stable and unstable manifolds intersect along a homoclinic channel $\Gamma$.
- Scattering map associated to the homoclinic channel $\Gamma$

$$S : \Lambda \to \Lambda, \quad x_+ = S(x_-)$$

when

$$\emptyset \neq W^s(x_+) \cap W^u(x_-) \in \Gamma$$

- Delshams– de la Llave – Seara: Scattering map is symplectic.
Two scattering maps

In $\mathbb{D}$ we can define two scattering maps given by two homoclinic channels

$$S_i : \mathbb{D}_i \subset \mathbb{D} \to \mathbb{C}.$$  

The homoclinic points $z_0^0$ and $z_1^0$ satisfy

$$z_i^i = S_i(z_i^0)$$

Close to them, $S_i$ are close to integrable twist maps.

They are elliptic fixed points by the scattering maps.
Isolating block for (a suitable iterate of) the scattering map

- There exists $M > 0$ such that $S_1^M \circ S_0$ has an isolating block.
- Tools: Birkhoff normal form, KAM Theorem...
- The Scattering maps encodes heteroclinic orbits to infinity.
- We want an isolating block for the “true” Poincaré map!
- Return map from a neighborhood of infinity to itself:
  - $(r, y)$ plane: it behaves as the one for the Sitnikov problem.
  - $(\alpha, \beta)$ plane: it can be “approximated” by the scattering maps.
- Extra difficulty: One has to control the dynamics close to infinity by a Parabolic Lambda Lemma.
An isolating block for (a suitable iterate of) the return map

- Applying the return map a large number of times one can construct
  the isolating block using:

  - The techniques by Moser in the \((r, y)\) plane.
  - The isolating block of the scattering map in the \((\alpha, \beta)\) plane.
Thank you for your attention