A stochastic view on the deterministic Navier-Stokes equation

GDM Seminar July 2021

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Arnold's characterisation of the Euler flow

Recall the characterisation of the Euler flow as a geodesic on the group of volume preserving diffeomorphisms $G_V(M)$, M a manifold . In this picture one considers, not the Cauchy problem

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & t \geq 0, x \in M \\ \nabla \cdot u = 0, & t \geq 0, x \in M \\ u(0, \cdot) = u_0, & t = 0 \end{cases} \tag{1}$$

but rather

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & 0 \leq t \leq 1 \\ \nabla \cdot u = 0, & 0 \leq t \leq 1 \\ g_1 = h \end{cases} \tag{2}$$

where g_t is the Lagrangian flow $\dot{g}_t = u(g_t),\ g_0 = x \in M$, h a vol. preserving diffeom.

The Lagrangian flow solves the variational problem in G_V

$$\min \frac{1}{2} \int_{[0,1]\times M} |\partial_t g_t(x)|^2 dtdx, \quad g_0 = Id,\ g_1 = h$$

and u is recovered by $u(t,x) = (\partial_t g_t)(g_t^{-1}(x))$.

From the Geometric Mechanics point of view, Euler equation is a particular case of Euler-Poincaré equations, which, in a general (right invariant) Lie group read

$$\partial_t u(t) = -\mathrm{ad}^*_{u(t)} u(t)$$

when applied to the diffeomorphisms group G_V.

Navier-Stokes

For a time dependent vector field $u(t, \cdot)$ such that $\nabla \cdot u(t, \cdot) = 0 \quad \forall t \in [0, T]$ and for a constant $\nu > 0$, let g^u be the solution of the stochastic differential equation (here in the flat case)

$$dg_t^u(x) = \sqrt{2\nu} dW_t + u(t, g_t^u(x))dt$$

with $g_0^u(x) = x$, $t \in [0, 1]$.
We have a diffusion with generator $Lf = \nu \Delta f + (u \cdot \nabla f)$; in particular

$$\frac{d}{dt} E\, f(g_t^u(x)) = E\, Lf(g_t^u(x))$$

Define the action functional

$$A[g^u] = \frac{1}{2} E \int_{[0,1]\times M} |D_t g_t^u(x)|^2 dtdx$$

where

$$D_t g_t = \lim \frac{1}{\varepsilon} E_t[g_{t+\varepsilon} - g_t]$$

E_t the conditional expectation given the σ-algebra generated by the past of t. Consider the (left) variations defined by

$$\partial_v A[g^u] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A[\exp(\varepsilon v) \circ g^u(\cdot)]$$

for smooth divergence free vector fields v, $v(0) = v(1) = 0$.

Theorem. *Let u be a smooth time-dependent divergence free vector field. Then g^u is critical for A iff u satisfies the Navier-Stokes equation*

$$\partial_t u + \nabla_u u = \nu \Delta u - \nabla p$$

and u can be recovered through $u(t,x) = D_t g_t(g_t^{-1}(x))$.

• In Stochastic Geometric Mechanics, this equation is, as in the classical case, an application to G_V of the stochastic Euler-Poincaré reduction method. More generally,

$$\partial_t u(t) = -\mathrm{ad}^*_{u(t)} u(t) + K(u(t))$$

where K is some second order positive operator.

• For a (compact) Riemannian manifold M, the result in $G_V(M)$ gives N.S. equation with the de Rham-Hodge Laplacian $K = \square = dd^* + d^*d$.

• Many other dissipative systems can be studied in this way (e.g. Camassa-Holm).

Proof of the Theorem:

Writing $g_t^u = g_t$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A[\exp(\varepsilon v) \circ g(\cdot)] = E \int_0^1 \left(\int D_t g_t(x) \cdot D_t v(g_t(x)) dx \right) dt$$

By Itô's formula,

$$\int dx\ d\Big(D_t g_t \cdot v(g_t)\Big) = \int dx\ [dD_t g_t \cdot v(g_t) + D_t g_t \cdot dv(g_t) + dD_t g_t \star dv(g_t)]$$

The last (Itô's contraction) term is equal to

$$2\nu(\int (\nabla v \otimes \nabla u)(g_t) dx$$

where $\nabla v \otimes \nabla u = \sum_{i,j=1}^2 \partial_j v^i \partial_j u^i$.

Since $v(0) = v(1) = 0$, the derivative of the action is equal to

$$-E\int_0^1 (\int (D_t D_t g_t(x)dx)dt - 2\nu E\int_0^1 (\int (\nabla v \otimes \nabla u)(g_t(x))dx)dt$$

On the other hand

$$D_t D_t g_t = (\frac{\partial}{\partial t}u + (u.\nabla)u + \nu\Delta u)(g_t)$$

and, using the invariance of the measure dx,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S[\exp_{.}(\epsilon v)\circ g_{.}] = -E\int_0^1 \int \Big(\frac{\partial}{\partial t}u+(u.\nabla)u-\nu\Delta u)\cdot v\Big)(t, g_t(x))dxdt$$

$$= -\int_0^1 \Big(\int [\frac{\partial}{\partial t}u + (u.\nabla)u - \nu\Delta u]\cdot v(t,x))dx\Big)\, dt$$

Extensions: we can deal with advected quantities, covering the case of compressible Navier-Stokes.

We can also consider boundary conditions (with suitable stochastic processes).

Probabilistic methods for **existence** of solution:

A possible approach is via forward-backward sde's (second order equations in stochastic analysis).

Here, for the rest of the talk, we adopt a weaker approach.

Brenier's generalised framework (for Euler)

One minimises a kinetic energy, but now averaged by probability measures Q on the path space $\Omega = C([0,1]; M)$

$$\min \frac{1}{2} E_Q \int_0^1 \|\dot{X}_t\|^2 dt, \quad Q_{01} = \pi,$$

$Q_{01} := (X_0, X_1)_* Q$.
Here $dQ_t = dx \quad \forall t \quad (Q_t = (X_t)_* Q)$ and π is a probability measure on $M \times M$ s.t. its marginals satisfy $d\pi_0 = d\pi_1 = dx$. The solutions P only charge absolutely continuous paths, since the kinetic energy is understood to be ∞ otherwise.

$$\text{Then } \begin{cases} dP_t = dx \;\; \forall t \quad \text{and} \quad P_{01} = \pi \\ \ddot{X}_t + \nabla p(t, X_t) = 0, \; \forall t, \; P - a.e. \end{cases}$$

In this approach one recovers the velocity field by defining a probability measure σ on $[0,1] \times M \times TM$,

$$\int f(t,x,u)\sigma(dt,dx,du) := \int_0^1 \int f(t,X_t,\dot{X}_t)dPdt$$

as DiPerna-Majda solutions;

Introducing viscosity

In the spirit of the stochastic approach before, we consider Brownian-type paths (not abs. continuous). For Q the corresponding law on the path space, kinetic energy is replaced by the forward “mean" velocity:

$$\overset{\rightharpoonup}{u}_t^Q := \lim_{h \to 0^+} \frac{1}{h} E_Q(X_{t+h} - X_t \mid X_{[0,t]})$$

Consider the reference measure R

$$R = \int R^x dx,$$

R^x the law of the Brownian motion starting from x with diffusion constant $\sqrt{2\nu}$.

On the other hand recall the notion of relative entropy of a measure Q with respect to a measure R

$$H(Q|R) := \int \log(\frac{dQ}{dR})\, dQ \in (-\infty, \infty]$$

By Girsanov theorem, to any measure Q on Ω with a finite relative entropy w.r.t. R corresponds a predictable (time dependent) vector field $\vec{u}$ s.t. Q is the law of the process with generator

$$Lf = \nu \Delta f + \vec{u} \cdot \nabla f$$

meaning that for every regular f

$$f(X_t) - f(X_0) - \int_0^t Lf(X_t)dt \qquad \text{is a} \quad Q-\text{martingale}$$

and, in particular $\frac{d}{dt} E_Q f(X_t) = E_Q Lf(X_t)$.
Moreover

$$H(Q|R) = H(Q_0|R_0) + \frac{1}{2} E_Q \int_0^1 |\vec{u}(t, X_t)|^2 \, dt$$

(in our case $dR_0 = dx$).

So we naturally consider the problem

$$\min \frac{1}{2} E_Q \int_0^1 |\overrightarrow{u}(t, X_t)|^2 \, dt$$

with $Q_{01} = \pi$ and $Q_t = \mu_t$ prescribed measures (Lebesgue measure for incompressibility constraint), which is the entropy minimisation problem. We may ask only $Q_t = \mu_t$ for $t \in S \subset [0, 1]$ $(\pi(\cdot \times M) = \mu_0, \pi(M \times \cdot) = \mu_1)$.

We can also define the backward velocity

$$\overleftarrow{u}_t^Q := \lim_{h \to 0^+} \frac{1}{h} E_Q(X_t - X_{t-h} \mid X_{[t,1]})$$

and, since R is reversible, we also have

$$H(Q|R) = \frac{1}{2} E_Q \int_0^1 |\overleftarrow{u}(t, X_t)|^2 \, dt + H(Q_1|R_1)$$

$(R_1 = R_0)$.

The dual problem

The (primal) entropy minimisation problem

$$\inf \{H(Q|R_0) : Q \text{ prob. measure on } \Omega, Q_t = \mu_t \; \forall t \in S, Q_{01} = \pi\}$$

is equivalent to the dual problem

$$\sup_{\{(p,\eta)\in\mathcal{A}}\Big\{ \langle p, \alpha\mu_t\rangle + \langle \eta, \pi\rangle$$

$$- \int_X \log E_{R^x} \exp\left(\int_S p(t, X_t)\, \alpha(dt) + \eta(x, X_1)\right) \mu_0(dx)\}$$

Here $\mathcal{A}$ is a dense set of bounded measurable functions on $(S \times M) \times M^2$, α is a probability measure supported in S and μ_t is a flow of probability measures, weakly continuous in t.

- constraint $P_{0,1} = \pi \quad \longrightarrow \quad$ Lagrange multiplier $\eta(X_0, X_1)$
- constraint $dP_t = dx \quad \longrightarrow \quad$ Lagrange multiplier $\int_0^1 p(t, X_t)dt$

This is a particular case of a **general** convex duality result.

Theorem.

1) If the inf is finite, the primal problem admits a unique solution.

In the case of the torus the primal problem admits a unique solution.

2) If p and η are are bounded measurable functions on M and M^2 resp., if only a finite number of marginals μ_{t_k} is prescribed, both the primal problem and the dual problem are attained respectively at P and (p, η), also the constraint $P_{01} = \pi$ is satisfied. Then P has the form

$$P = \exp\left(\eta(X_0, X_1) + \sum_{s_k} \theta_{s_k}(X_{s_k}) + \int_S p(t, X_t)\, dt\right) R$$

where θ_s are some measurable functions.

In the case where an infinite number of marginal laws is prescribed (Navier-Stokes) we can show that

$$P = \exp\Big(A(X) + \eta(X_0, X_1)\Big) R$$

with A an additive functional, but we could not prove that $A(X) = \int_T p(t, X_t) dt$ for some function p.

More recently, A. Baradat proved the existence of a function p, using pde methods.

The dynamics

P is the law of a process X_t such that

$$dX_t = dM_t + \overrightarrow{u}_t dt, \qquad P - a.s.$$

where M_t is a P-martingale (the Brownian motion for the case where the reference measure is the law of the Brownian motion), $\overrightarrow{u}$ a predictable vector field.
Comparing the expression we have for P with the one issued from Girsanov's theorem, namely

$$\frac{dP}{dR} = \frac{dP_0}{dR_0}(X_0) \exp\Big(\int_0^1 \beta_t^P \cdot dX_t - \frac{1}{2} \int_0^1 |\beta_t^P|^2 \, dt \Big), \quad P - a.s.$$

we have,

$$\overrightarrow{u}_t(X_{[0,t]}) = \overrightarrow{u}_t(X_0, X_t) = \nabla \psi_t^{X_0}(X_t), \quad \forall 0 \le t \le 1, \; P - a.s.,$$

with

$$\psi^x(t,z) := \log E_R\Big[\exp\Big(\eta(x,X_1) + \sum_{s\in S, s>t} \theta_s(X_s) + \int_{T\cap(t,1]} p(r,X_r)\,dr\Big)\,\Big|\,X_t = z\Big], \quad R_t - a.s.$$

Note that for $t = 1$, we have $\psi^x(1,\cdot) = \eta(x,\cdot)$. Furthermore ψ^x is the solution of the Hamilton-Jacobi-Bellman equation

$$\big[(\partial_t + \nu\Delta)\psi + \frac{1}{2}|\nabla\psi|^2 + p\big](t,z) = 0, \qquad 0 \le t < 1,\ t \notin S,\ z \in X,$$

$$\psi(t,\cdot) - \psi(t^-,\cdot) = -\theta(t,\cdot),\ t \in S,$$

$$\psi(1,\cdot) = \eta(x,\cdot), t = 1.$$

The forward velocity satisfies the equation

$$(\partial_t + \overset{\rightharpoonup}{u}^x \cdot \nabla)(\overset{\rightharpoonup}{u}^x) = -\nu\Delta(\overset{\rightharpoonup}{u}^x) - \nabla p, \qquad t < 1,\ t \notin S,$$

$$\overset{\rightharpoonup}{u}^x_t - \overset{\rightharpoonup}{u}^x_{t^-} = -\nabla\theta_t, \quad t \in S$$

$$\overset{\rightharpoonup}{u}^x_1 = \nabla_y \eta(x, \cdot), \quad t = 1$$

(notice the “wrong sign")

The backward velocity of $P(\cdot|X_1 = y)$ solves

$$(\partial_t + \overleftarrow{u}^y \cdot \nabla)(\overleftarrow{u}^y) = \nu\Delta(\overleftarrow{u}^y) - \nabla p, \qquad t > 0,\ t \notin S,$$

$$u_t^{\rightarrow,y} - u_{t^-}^{\leftarrow,y} = \nabla\theta_t, \quad t \in S$$

$$u_0^{\leftarrow,y} = \nabla_y \eta(\cdot, y), \quad t = 0$$

Moreover $\overrightarrow{u}_t^y = \nabla\varphi_t^y(z),\ \ t \notin S$, with

$$(\partial_t - \nu\Delta)\varphi + \frac{1}{2}|\nabla\varphi|^2 + p = 0, \qquad t > 0,\ t \notin S$$

$$\varphi(t,\cdot) - \varphi(t^-,\cdot) = \theta(t,\cdot), \quad t \in S,$$

$$\varphi(0,\cdot) = -\eta(\cdot, y), \quad t = 0.$$

Remarks.

1. The current velocity $u_t = \frac{1}{2}(\overrightarrow{u}_t + \overleftarrow{u}_t)$ solves the continuity equation

$$\partial_t \mu_t + \nabla.(\mu_t u_t) = 0$$

2. $t \to P_{0,t} = (X_0, X_t)_* P$, probability measure on M^2 corresponds to a relaxation of the Lagrangian paths $g_t(\cdot)$.

3. The pressure and the “potentials" θ do not depend on the final position, only on the actual position.

The general solution of our generalization of Brenier's type problem can be described by

$$P = \int_X P(\cdot | X_1 = y)$$

(corresponding to the gradient drift field $u^{\leftarrow, y} = \nabla \varphi^y$) but the average backward velocity

$$\overleftarrow{u}_t(z) = \int \nabla_z \, \varphi_t^y(z) \, P_1^{tz}(dy)$$

$P_1^{tz} := P(X_1 \in dy | X_t = z)$, is not a gradient, due to the nonlinearity of the equations.
This superposition phenomenon is reminiscent of Brenier's multiphase vortex sheets model.

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