Hamiltonian $\mathbb{S}^1$-actions and integrable systems

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September 14, 2021

Geometry, Dynamics and Mechanics Seminar

Notation and conventions I

Definition: (M, ω) is a symplectic manifold if

- M smooth manifold,
- ω nondegenerate, closed 2-form on M.

$\rightsquigarrow$ $\dim M$ *must be even*.

(Standard) example: $(M, \omega) = (\mathbb{R}^{2n}, \sum_{k=1}^{n} dp_k \wedge dq_k)$
where $(q_1, \ldots, q_n, p_1, \ldots, p_n) =: (q, p)$ coordinates of $\mathbb{R}^{2n}$.

Darboux's theorem:
Any symplectic manifold looks locally like $(\mathbb{R}^{2n}, \sum_{k=1}^{n} dp_k \wedge dq_k)$.

$\rightsquigarrow$ *locally all symplectic manifolds 'look the same',* i.e., no 'local symplectic invariants' like curvature in Riemannian geometry

Notation and conventions II

Definition: Let (M, ω) be symplectic and $H : M \to \mathbb{R}$ smooth.

- $\omega(X^H, \cdot) = dH$ Hamiltonian vector field
- $z' = X^H(z)$ Hamiltonian equation

$\rightsquigarrow$ In local coordinates $(\mathbb{R}^{2n}, dp \wedge dq)$, we get

$$X^H(q,p) = \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} \qquad \text{Hamiltonian vector field}$$

and

$$\begin{cases} q' = \partial_p H(q,p) \\ p' = -\partial_q H(q,p) \end{cases} \qquad \text{Hamiltonian equations}$$

Special case: Hamiltonian $\mathbb{S}^1$-spaces:

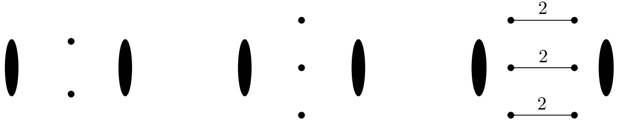
Considering Hamiltonians with periodic flow with minimal period:

Classification by Karshon (1999) in dimension 4:

Let (M, ω) be a 4-dim compact sympl. manifold and let $L : M \to \mathbb{R}$ be the momentum map of an effective Hamiltonian $\mathbb{S}^1$-action. Two such spaces are equivariantly symplectomorphic if and only if their associated labeled, directed graphs (next slide) are equal.

Notation: In what follows, Hamiltonians denoted by L often end up inducing an effective Hamiltonian $\mathbb{S}^1$-action.

Graph of $\mathbb{S}^1$-spaces:

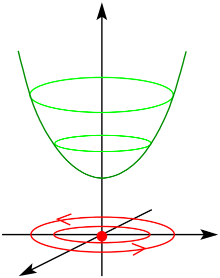


(a) (b) (c)

Graph:

- Vertex set: Fixed point = vertex, Fixed surface = fat vertex
- Edge set: Directed edges between vertices stand for $\mathbb{Z}_k$-sphere, $k \geq 2$, which are connected components of $\{x \in M \mid Stab(x) = \mathbb{Z}/k\mathbb{Z}\}$.
- Labels: value of moment map; fixed surface: volume & genus

Towards Liouville integrability: energy conservation



Energy conservation: H constant along Hamiltonian solution $\gamma' = X^H(\gamma)$

$\rightsquigarrow$ Ham. solutions stay within *one* level set (= 'energy level').

Conclusions:

- Ham. sol. stay in $\leq \dim M - 1$ dimensional subsets.
- in dim $= 2$: regular Ham. sol. known up to parametrization.

Question: Can we confine solutions to even lower dim. subsets?

Liouville integrability:

Idea:

Given $H =: H_1$, look for more functions H_2, H_3, H_4 ... H_k such that all Ham. sol. stay within each others' level sets!

$\rightsquigarrow$ Then: each solution stays in intersection of level sets

Note:

$$\dim\left(H_1^{-1}(r_1) \cap \cdots \cap H_k^{-1}(r_k)\right) \leq \dim M - k$$

Let γ_{H_i} be Hamiltonian solution of H_i and calculate:

$$\left(H_i \circ \gamma_{H_j}\right)' = DH_i.\gamma'_{H_j} = DH_i.X^{H_j} = \omega(X^{H_i}, X^{H_j}) =: \{H_i, H_j\}$$

Criterium:

$$\gamma_{H_j} \text{ stays in level sets of } H_i \quad \Leftrightarrow \quad \{H_i, H_j\} = 0$$

Setting from now on:

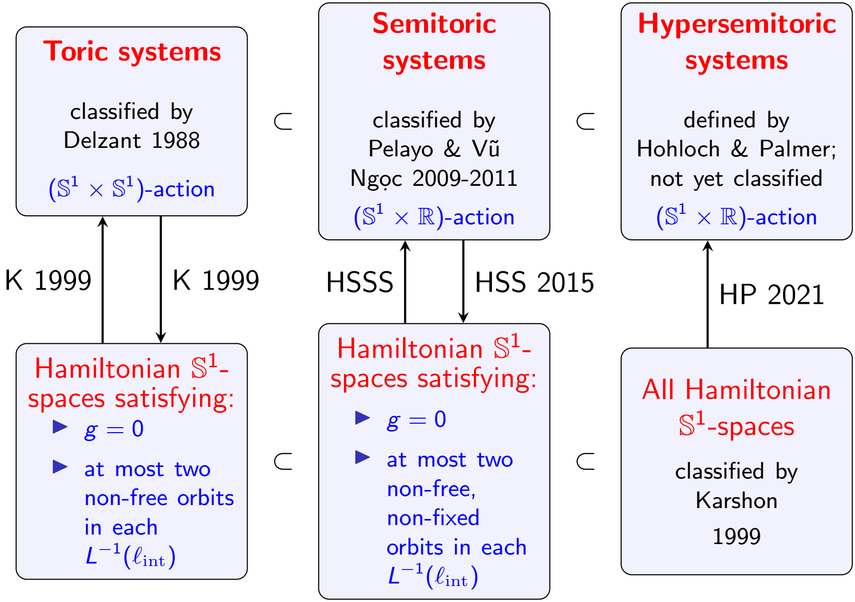
(M, ω) 4-dimensional connected symplectic manifold.

Definitions and Notations:

1) $\Phi = (L, H) : M \to \mathbb{R}^2$ is (the momentum map of) a completely integrable Hamiltonian system if
   - Ham. vector fields X^L, X^H almost everywhere lin. independent
   - L, H Poisson-commute: $0 = \{L, H\} := \omega(X^L, X^H)$.

2) In particular, Hamiltonian flows commute: $\varphi_s^L \circ \varphi_t^H = \varphi_t^H \circ \varphi_s^L$
   $\Rightarrow$ $\mathbb{R}^2$-action: $\quad \mathbb{R}^2 \times M \to M, \quad ((s,t), x) \mapsto \varphi_s^L \circ \varphi_t^H(x)$.

Overview/Aim:



Essential ingredients: Singular points

(M, ω) 4-dimensional connected symplectic manifold.

Definitions and Notations:

$\Phi = (L, H) : M \to \mathbb{R}^2$ completely integrable Hamiltonian system.

1) z singular point if rank $D\Phi|_z < 2$.
2) z nondegenerate fixed point if
   - Hessians $D^2L|_z$ and $D^2H|_z$ linearly independent,
   - $\exists$ linear combination of $\omega^{-1}D^2L|_z$ and $\omega^{-1}D^2H|_z$ having 4 distinct eigenvalues.

Local normal form for nondegenerate singularities:

Theorem (Eliasson, Miranda & Zung. . .):

Sympl. coordinates $(x, \xi) = (x_1, x_2, \xi_1, \xi_2)$ and functions f_1, f_2 with $\{L, f_k\} = 0 = \{H, f_k\}$ near nondegenerate singular point with:

1) Hyperbolic component: $f_k(x, \xi) = x_k \xi_k$.
2) Elliptic component: $f_k(x, \xi) = \frac{1}{2}(x_k^2 + \xi_k^2)$.
3) Focus-Focus component (coming in pairs):
$$\begin{cases} f_{k-1}(x, \xi) = x_{k-1}\xi_k - x_k \xi_{k-1}, \\ f_k(x, \xi) = x_{k-1}\xi_{k-1} + x_k \xi_k. \end{cases}$$
4) Regular component: $f_k(x, \xi) = \xi_k$.

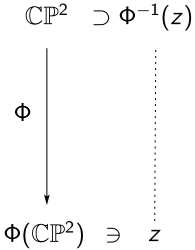
Easiest integrable systems: Toric systems

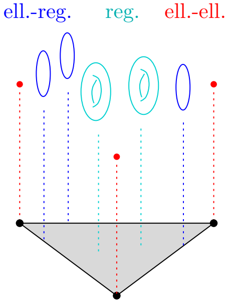
Completely integrable systems $\Phi = (L, H) : M \to \mathbb{R}^2$ where the flows φ^L and φ^H are periodic of same period.

- $\mathbb{R}^2$-action becomes $\mathbb{S}^1 \times \mathbb{S}^1 =: \mathbb{T}^2$-action

Example:

$$\Phi : \mathbb{CP}^2 \to \mathbb{R}^2, \quad \Phi([z_0, z_1, z_2]) := \frac{1}{2}\left(\frac{|z_1|^2}{\sum_{k=0}^2 |z_k|^2}, \frac{|z_2|^2}{\sum_{k=0}^2 |z_k|^2}\right)$$





Singular points:

- *elliptic-elliptic* (rank = 0)
- *elliptic-regular* (rank = 1)

Classification of toric systems: Delzant's Theorem

Delzant 1988: Constructive classification

$$\left\{\begin{matrix}\text{Compact connected } 2n\text{-dim} \\ \text{symplectic manifolds } (M,\omega) \\ \text{with effective Ham. } \mathbb{T}^n\text{-action}\end{matrix}\right\} \Big/ _{\text{equiv. sympl.}} \xrightarrow{1:1} \left\{\begin{matrix}\text{so-called} \\ \text{Delzant} \\ \text{polytopes}\end{matrix}\right\}$$

$$(M, \omega, \mathbb{T}^n, \Phi) \quad \mapsto \quad \Phi(M).$$

Conclusion:

- Toric manifolds determined by **finite** set of data.
- Toric manifolds = combinatorics
- Toric manifolds are a *very* special case of integrable systems.

Overview: Toric extension

Toric systems

classified by
Delzant 1988

$(\mathbb{S}^1 \times \mathbb{S}^1)$-action

Karshon 1999

Karshon 1999

Hamiltonian $\mathbb{S}^1$-spaces satisfying:

- $g = 0$
- at most two non-free orbits in each $L^{-1}(\ell_{\text{int}})$

Semitoric systems:

(M, ω) 4-dimensional connected symplectic manifold.

Definition (Pelayo & Vũ Ngọc)

A semitoric system is a completely integrable system $\Phi = (L, H) : (M, \omega) \to \mathbb{R}^2$ such that

- L is proper,
- L induces an effective Hamiltonian $\mathbb{S}^1$-action,
- Φ admits only nondegenerate singularities,
- no hyperbolic components.

Conclusion: $\mathbb{S}^1 \times \mathbb{R}$-action; possible singularities:

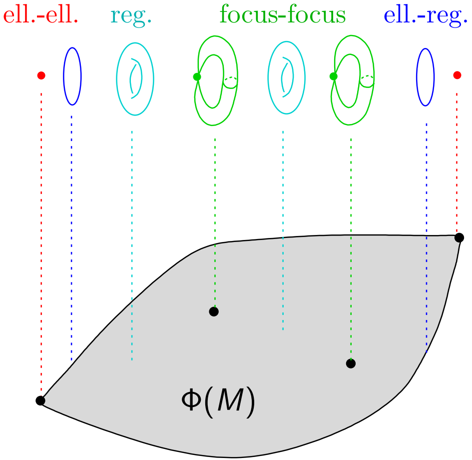
- *focus-focus* (rank $= 0$)
- *elliptic-elliptic* (rank $= 0$)
- *elliptic-regular* (rank $= 1$)

Nondegenerate singularities (no hyp. components):

$$(M, \omega) \supset \Phi^{-1}(z)$$

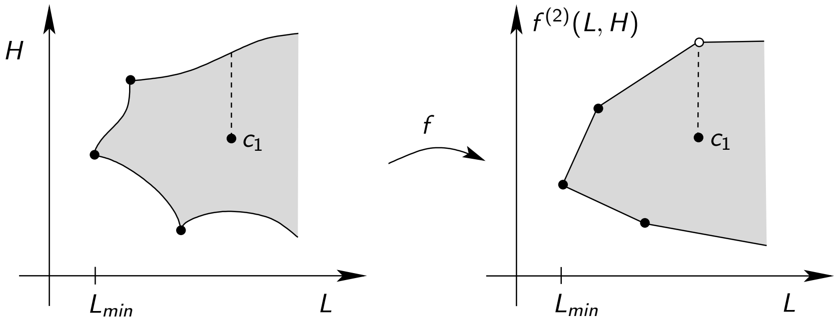
$$\Phi = (L, H) \downarrow$$

$$\mathbb{R}^2 \supset \Phi(M) \ni z$$

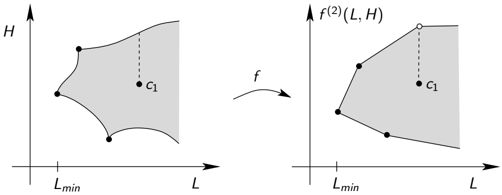


Semitoric systems:

The image of a (semitoric) momentum map $\Phi(M)$ is a 'curved polygon' which can be 'straightened' by some homeomorphism $f(x, y) = (x, f^{(2)}(x, y))$ into polygon(s) with cuts:



Semitoric systems: Classification



Classification invariants (Pelayo & Vũ Ngọc):

(1) m_{FF}, the number of focus-focus singularities.

(2) Taylor series invariant: Taylor series expansion of generating function at focus-focus points.

(3) An equivalence class of generalized polygons.

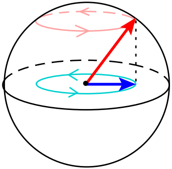
(4) Height invariant: Height of focus-focus value in polygon.

(5) Twisting-index invariant: m_{FF} numbers measuring 'relative twistedness' near focus-focus singularities.

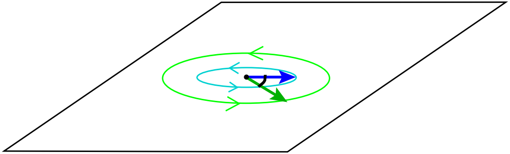
1st example: Coupled spin oscillators

(or Jaynes-Cummings, Gaudin model)

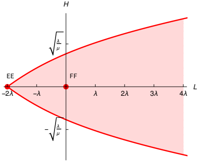
$(\mathbb{S}^2 \times \mathbb{R}^2, \lambda\omega_{\mathbb{S}^2} \oplus \mu\omega_{\mathbb{R}^2})$ with $\lambda, \mu > 0$:



$\times$



$(L, H) : \mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$



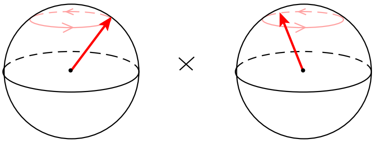
$$L(x, y, z, u, v) = \lambda(z - 1) + \mu\frac{u^2 + v^2}{2}$$

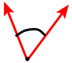
$$H(x, y, z, u, v) = \frac{1}{2}(xu + yv)$$

2nd example: Coupled angular momenta

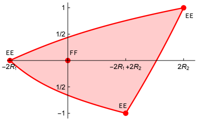
$\mathbb{S}^2 \times \mathbb{S}^2$ with $0 < R_1 < R_2$ and $\omega = -(R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$,

$t \in \,]t^-, t^+[\, \subset \, [0, 1]$.





$(L, H) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^2$

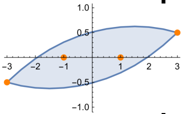


(figure for $t = \frac{1}{2}$)

$$L(x_1, y_1, z_1, x_2, y_2, z_2) = R_1(z_1 - 1) + R_2(z_2 + 1),$$
$$H(x_1, y_1, z_1, x_2, y_2, z_2) = (1 - t)z_1 + t(x_1 x_2 + y_1 y_2 + z_1 z_2) + 2t - 1$$

3rd example: Family with 2 focus-focus singularities

Theorem (Hohloch & Palmer, 2018)
$\mathbb{S}^2 \times \mathbb{S}^2$ with $R := (R_1, R_2)$, $0 < R_1 < R_2$ and $\omega_R = R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2}$ and $t := (t_1, t_2, t_3, t_4) \in \mathbb{R}^4$. Then



$$\begin{cases} L_R := R_1 z_1 + R_2 z_2, \\ H_{\vec{t}} := t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2. \end{cases}$$

is semitoric with 2 focus-focus for certain parameters choices.

Theorem (Hohloch & Palmer, 2018)
This system is semitoric for $s_1, s_2 \in [0, 1]$, $R_1 = 1, R_2 = 2:$

$t_1 = s_1(1 - s_2)$, $\quad t_2 = s_2(1 - s_1)$,

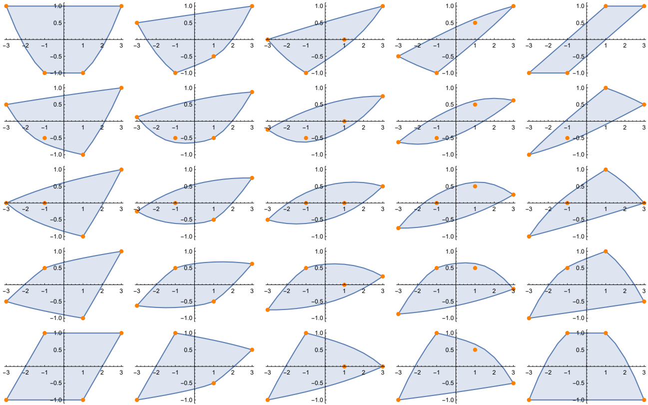
$t_3 = (1 - s_1)(1 - s_2) + s_1 s_2$, $\quad t_4 = (1 - s_1)(1 - s_2) - s_1 s_2$

Momentum map image of 2-parameter family:

Special parameter choices with $s_1, s_2 \in [0,1]$, $R_1 = 1, R_2 = 2:$

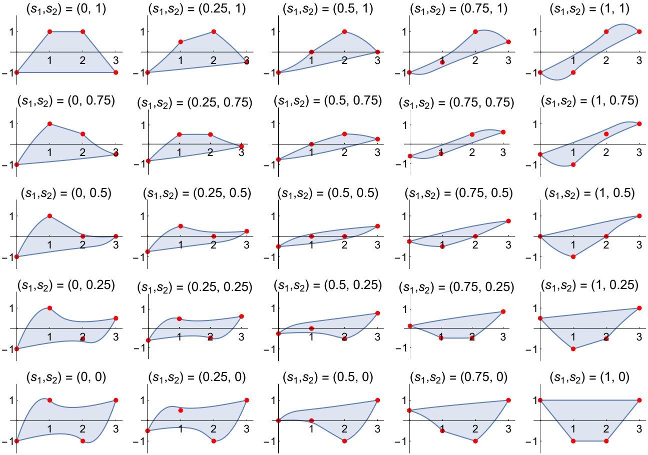
$t_1 = s_1(1 - s_2)$, $\quad t_2 = s_2(1 - s_1)$,

$t_3 = (1 - s_1)(1 - s_2) + s_1 s_2$, $\quad t_4 = (1 - s_1)(1 - s_2) - s_1 s_2$



4th example: Le Floch & Palmer (2018) on $W_2(\alpha, \beta)$:

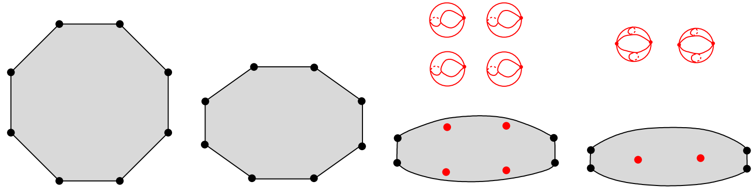
$$\Phi_{s_1,s_2} = (L, H_{s_1,s_2}) : W_2(\alpha, \beta) \to \mathbb{R}^2$$



(here: $\alpha = 1$ and $\beta = 1$)

5th example: De Meulenaere & Hohloch (2021):

1-parameter family with simultaneous moving of singular points:



8 EE	8 EE	4 EE + 4 FF	4 EE + 4 FF
		single pinched	double pinched
$t = 0$	$0 < t < t^-$	$t^- < t < \frac{1}{2}$	$t = \frac{1}{2}$

Overview classification progress I

Some of the symplectic invariants had been calculated for certain parameter values of

- spherical pendulum
- coupled spin oscillator (or Jaynes-Cummings, Gaudin model)
- coupled angular momenta

in various works by

Dullin, Pelayo, Vu Ngoc, Le Floch, Babelon and others...

Open problem: Twisting index; higher order terms of Taylor series invariant; continuous parameters (for all invariants)...

Overview classification progress II:

Alonso & **Dullin** & **Hohloch** (2017, 2018):

- Taylor series and twisting index for coupled spin oscillator.
  $\rightsquigarrow$ Pelayo & Vũ Ngọc's classification completed!
- Taylor series, height invariant, twisting index, (and polygon invariant) for coupled angular momenta.
  $\rightsquigarrow$ Pelayo & Vũ Ngọc's classification completed!
- Also calculated for more general parameter values of coupled angular momenta than the here formulated system...
- Limit behaviour of Taylor series of coupled angular momenta for $t \to t^{\pm}$: it blows up!

Overview classification progress III:

Le Floch & Palmer's Hirzebruch systems:

- number of FF and polygon invariant for all parameter values (Le Floch & Palmer 2018)
- height invariant for selected parameter values (Le Floch & Palmer 2018)

$\rightsquigarrow$ still unknown:

- height invariant for all parameter values
- Taylor series invariant
- twisting index

Overview classification progress IV:

Hohloch & Palmer's 2-FF-system:

- number of FF and polygon invariant
  (Hohloch & Palmer 2018)
- height invariant for selected parameter values
  (Le Floch & Palmer 2018/19)
- height invariant for 1-parameter subfamily
  (Alonso & Hohloch 2021)
- Taylor series invariant and Twisting index
  (Alonso & Hohloch & Palmer, ongoing)

De Meulenaere & Hohloch's octagon-system:

- number of FF
  (De Meulenaere & Hohloch 2019)
- ... ?

Overview: toric and semitoric extensions

Toric systems

classified by Delzant 1988

$(\mathbb{S}^1 \times \mathbb{S}^1)$-action

$\subset$

Semitoric systems

classified by Pelayo & Vũ Ngọc 2009-2011

$(\mathbb{S}^1 \times \mathbb{R})$-action

HSS 2015 = Hohloch & S. Sabatini & D. Sepe

HSSS (ongoing) = Hohloch & S. Sabatini & D. Sepe & M. Symington

K 1999 ↑ K 1999 ↓

HSSS ↑ HSS 2015 ↓

Hamiltonian $\mathbb{S}^1$-spaces satisfying:

- $g = 0$
- at most two non-free orbits in each $L^{-1}(\ell_{\text{int}})$

$\subset$

Hamiltonian $\mathbb{S}^1$-spaces satisfying:

- $g = 0$
- at most two non-free, non-fixed orbits in each $L^{-1}(\ell_{\text{int}})$

Hypersemitoric systems I:

Definition (Hohloch & Palmer 2021)

(M, ω) 4-dim, compact, connected symplectic manifold.
A completely integrable system $\Phi = (L, H) : M \to \mathbb{R}^2$ is **hypersemitoric** if

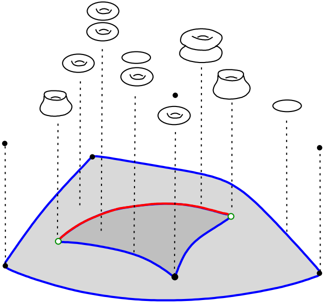
- L induces an effective $\mathbb{S}^1$-action,
- all singularities of Φ are nondegenerate except for maybe finitely many parabolic degenerate points.

Possible nondegenerate singularities:

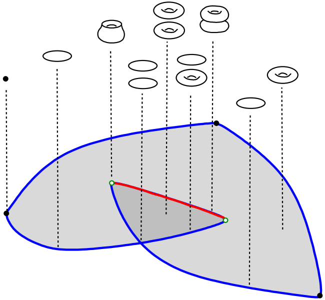
Rank 0: elliptic-elliptic, elliptic-hyperbolic, focus-focus.
Rank 1: elliptic-regular, hyperbolic-regular.

Note: The periodic flow of L prevents the existence of hyperbolic-hyperbolic fixed points.

Hyperbolic-regular and parabolic degenerate points:



Flap



Swallowtail

Definition of parabolic degenerate points

(M, ω, Φ) integrable system, $p \in M$ a singular with $\mathrm{d}f_1(p) \neq 0$ where $(f_1, f_2) = g \circ \Phi$ for some local diffeormorphism g of $\mathbb{R}^2$ in neighborhood of $\Phi(p)$. Define

$$\tilde{f}_2 := \tilde{f}_{2,p} := (f_2)|_{f_1^{-1}(f_1(p))} \colon f_1^{-1}(f_1(p)) \to \mathbb{R}.$$

p is a parabolic degenerate singular point if:

- p is a critical point of $\tilde{f}_2$,
- $\mathrm{rank}(\mathrm{d}^2 \tilde{f}_2(p)) = 1$,
- there exists $v \in \ker(\mathrm{d}^2 \tilde{f}_2(p))$ such that

  $$v^3(\tilde{f}_2) := \frac{\mathrm{d}^3}{\mathrm{d}t^3} \tilde{f}_2(\gamma(t))|_{t=0}$$

  is nonzero, where $\gamma \colon \;]-\varepsilon, \varepsilon[\to f_1^{-1}(f_1(p))$ is a curve satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.
- $\mathrm{rank}(\mathrm{d}^2(f_2 - kf_1)(p)) = 3$, where $k \in \mathbb{R}$ is determined by $\mathrm{d}f_2(p) = k\mathrm{d}f_1(p)$.

Hypersemitoric systems II:

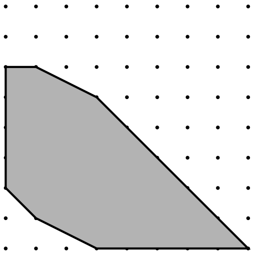
Theorem (Hohloch & Palmer 2021)

Let (M, ω) be a 4-dim, compact, connected symplectic manifold. Then **for any** Hamiltonian L that induces an effective $\mathbb{S}^1$-action there exists a smooth $H : M \to \mathbb{R}$ such that $\Phi := (L, H) : M \to \mathbb{R}^2$ is a **hypersemitoric system**.

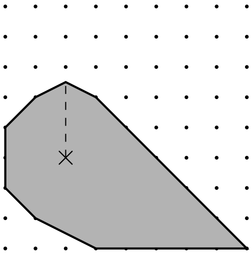
Remark

- Some L force the existence of degenerate points in any extension. But one can choose what type of degenerate points one wants to admit (we opted for parabolic points).
- Hypersemitoric systems are the 'nicest and easiest' class of integrable systems to which an arbitrary L that induces an effective $\mathbb{S}^1$-action can be extended.

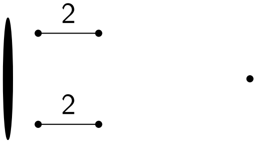
Creating/deleting fixed points (ideas of HSSS):



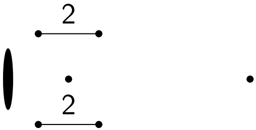
(a) The Delzant polygon of $\mathrm{Bl}^4(\mathbb{CP}^2)$.



(b) A semitoric polygon of $\mathrm{Bl}^5(\mathbb{CP}^2)$.

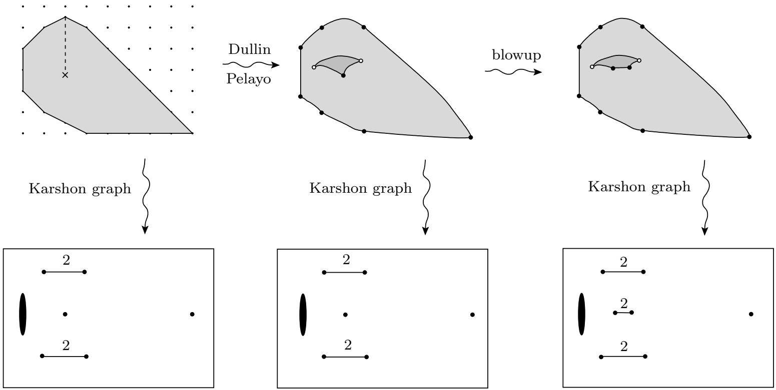


(c) The Karshon graph of the Hamiltonian $\mathbb{S}^1$-space induced by the toric system.

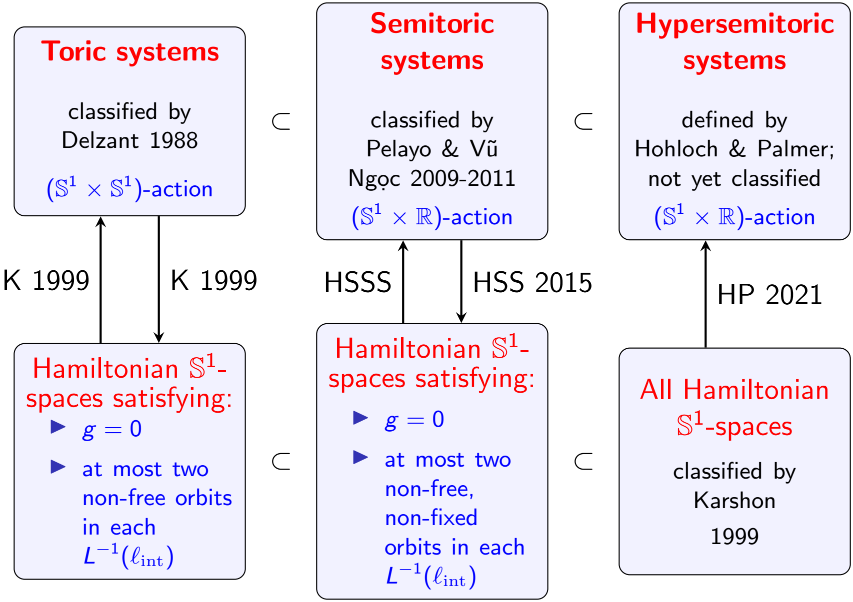


(d) The Karshon graph of the Hamiltonian $\mathbb{S}^1$-space induced by the semitoric system.

Creating/deleting inner edges (HP):



Overview of Results:



Thank you for your attention!