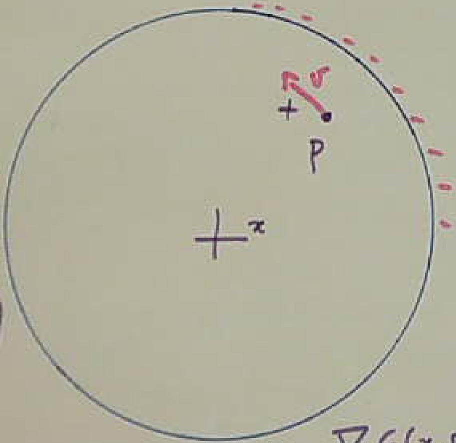
Vortex on a compact surface, Steady vortex metric, and generalizations to higher dimensions.



electric potential

$$\parallel$$

$$0 = G(x,p)$$

$$x \in \partial B_1$$

$$G(x,p) = \underbrace{\log|x-p|}_{\substack{\text{potential}\\ \text{due to}\\ \text{the point}\\ \text{charge}}} + \underbrace{R_B(x,p)}_{\substack{\text{potent.}\\ \text{due}\\ \text{to the}\\ \text{charges}\\ \text{at } \partial B}}$$

$$\nabla_x G(x,p) = \frac{x-p}{|x-p|^2} + \nabla_x R(x,p)$$

- Force upon a charge $F = \nabla_x R_B(x,p)$

- On a domain with a Riemannian metric

$$\Delta_q G(q,p) = \delta_p(q)$$

$$G(q,p) = 0 \qquad q \in \partial\Omega$$

$$R(p) = \lim_{q \to p} G(q,p) - \log \underbrace{d(q,p)}_{\text{Riemannian distance}}$$

Robin function

Compact surfaces

- $\Delta_q G(q,p) = -\delta_p(q) + \frac{1}{V}$ uniform distribution of negative charges (Gauss law)

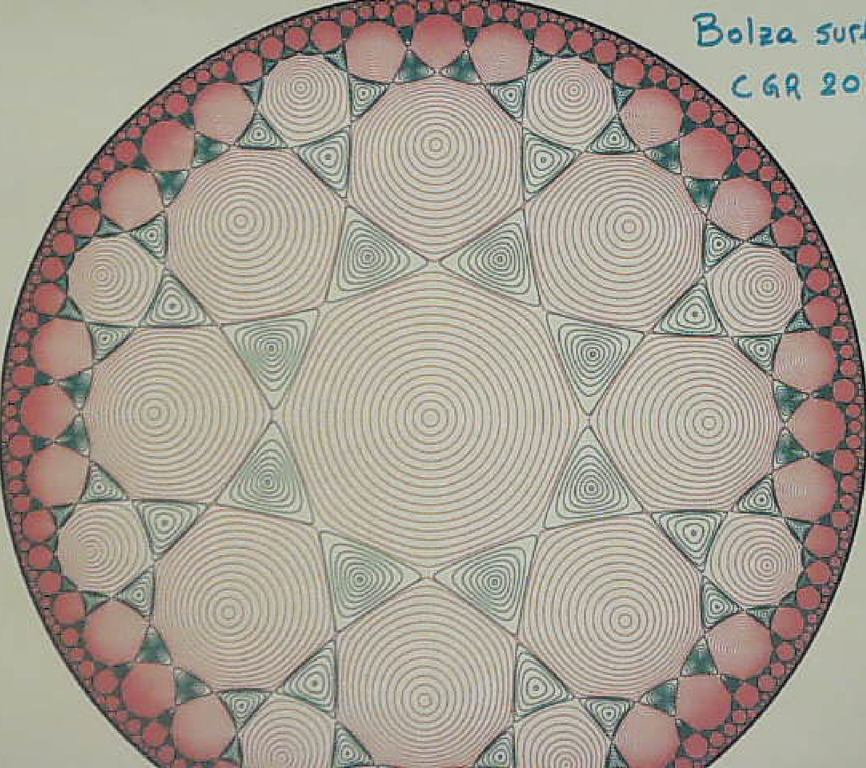
- $R(p) = \lim_{q \to p} G(q,p) - \log d(q,p)$

- The motion of a single vortex is proportional to $\nabla^{\perp} R(p)$

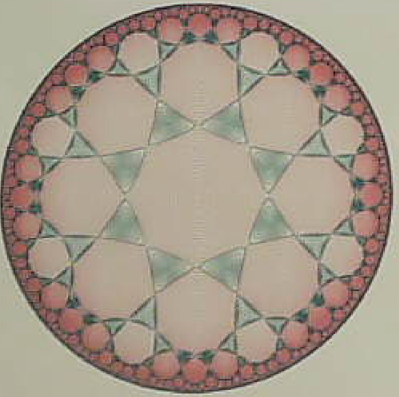
- "Steady Vortex Metric" $\Longleftrightarrow$ $R(p) = cte$

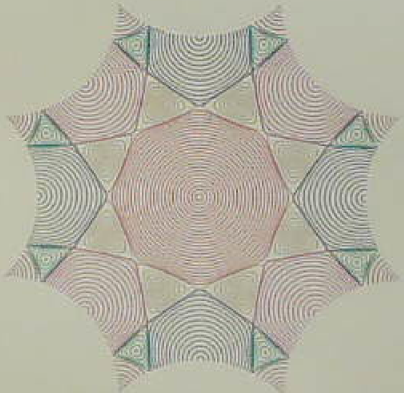
Some references:

- Boatto & Koiller: Vortices on closed surfaces (2008), (2013)
- CGR & Viglioni: Hydrodynamic vortex on surfaces (2017) (surface with ends and force upon a vortex)
- Gustafsson: "On the motion of a vortex..." Tech. Report (1979) (SVM)
- CGR: "The motion of a vortex on a closed..." (2017) (constant neg. curvature)



Bolza surface
CGR 2017

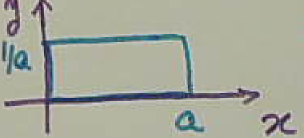




- SVM exists and is unique in surfaces with ends

CGR-Viglioni (2017)

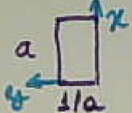
- SVM exists on compact S. (Okikiolu 2009)
- SVM may not be unique (Okikiolu)
- SVM in $\mathbb{T}^2$:



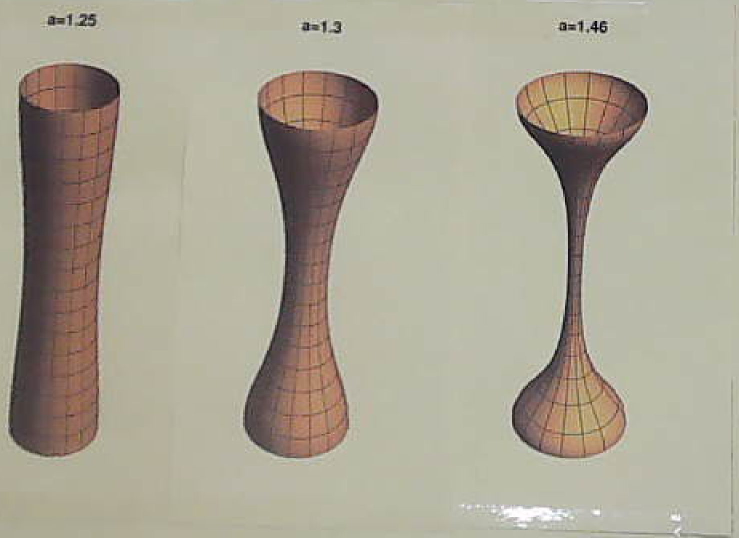
$a \geq 1$

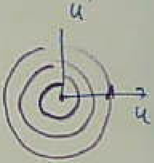
Flat metric is SVM $\begin{cases} \text{unique if } a < \frac{2}{\sqrt{\pi}} \approx 1.1283 \\ \text{nonunique if } a > \sqrt{\frac{\pi}{2}} \approx 1.253 \end{cases}$

Periodic cylinders (tori)



$a > \sqrt{\frac{\pi}{2}} \approx 1.253$





Profile equation $u''(x) = 8\pi(1-e^{u(x)}),\ u(x+a) = u(x)$

Higher dimensions $n \geq 3$ (unfinished)

- $R(p) = \lim\limits_{q \to p} \left[G(q,p) - a_n \, d(q,p)^{n-2} \right]$ does not work

- Heat kernel $\left(\frac{\partial}{\partial t} - \Delta \right) K(q,p,t) = 0$, $K(q,p,0) = \delta_p($

$\Rightarrow$ $K(q,p,t) \xrightarrow[t \to \infty]{} \frac{1}{V}$ conservation of energy

- $G(q,p) = \int_0^\infty \left\{ K(q,p,t) - \frac{1}{V} \right\} dt$

- Robin function $n \geq 3$ odd

$$R(p) = \lim_{q \to p} \left\{ G(q,p) - \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p) \left(\frac{4}{d^2(q,p)} \right)^{\frac{n}{2}-k-1} \Gamma\left(\frac{n}{2}-k-1\right) \right\}$$

$u_k(q,p)$ given by the short-time asymtotics of the Heat Kernel (Minakshisundaram - Plejel)

- Zeta function of M.P: $\zeta(p,q,s) = \sum_k \frac{\phi_k(q)\phi_k(p)}{\lambda_k^s}$

$\Delta \phi_k + \lambda_k \phi_k = 0$ $\qquad$ $\mathrm{Re}\, s > \frac{n}{2}$

- n - odd $\quad \zeta(p,p,1) = R(p)$

- n - even $\quad \lim_{s \to 1} \left\{ \zeta(p,p,s) - \frac{1}{4\pi} \frac{1}{s-1} \right\} = R(p) + cte$

Probabilistic interpretation of $R(p)$, $n=2$

NET = narrow escape time

escape window $B_\varepsilon(p)$

NET = Expected time of escape $= -\log \varepsilon + R(p) + \cdots$