Recent developments of computer-assisted proofs in the Navier-Stokes equations

Jean-Philippe Lessard

McGill UNIVERSITY

Geometry, Dynamics and Mechanics Seminar

Università degli Studi di Padova, Italy

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The Navier-Stokes equations for a fluid of constant density ρ can be expressed as

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f \\ \nabla \cdot u = 0, \end{cases}$$

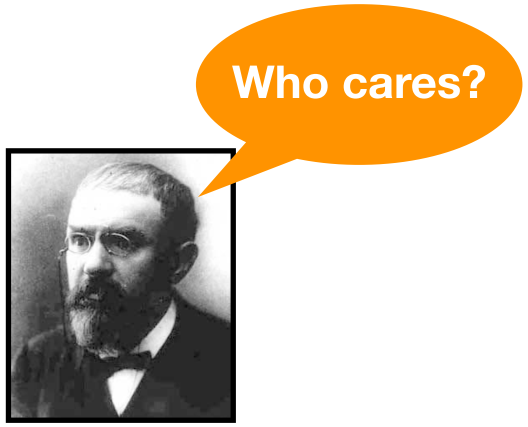
where $u = u(x,t)$ is the velocity, $p(x,t) = P(x,t)/\rho$ is the pressure scaled by the density, ν is the kinematic viscosity and $f = f(x,t)$ is an external forcing term.

Navier (1822) *Stokes (1845)*

Millennium Prize problem

In three space dimensions and time, given an initial velocity field and identically zero forcing term, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.

From a dynamical systems perspective, this may not be the most important question.



Henri Poincaré

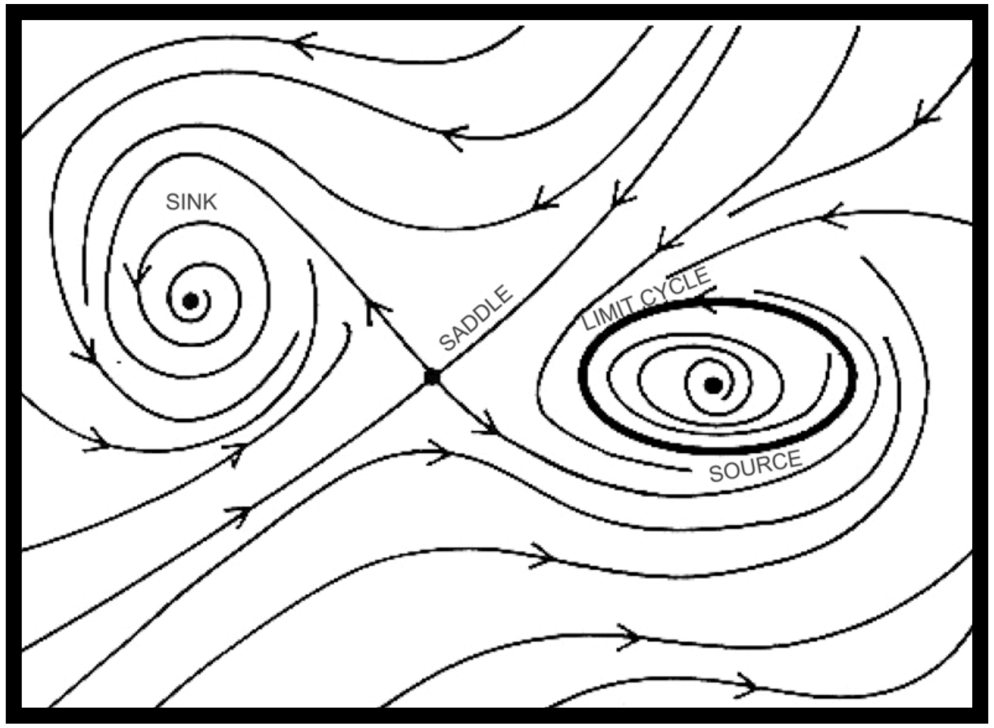
What shall we care about then ?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré

Compact invariant sets

Exploit smoothness, boundedness and low dimensionality.

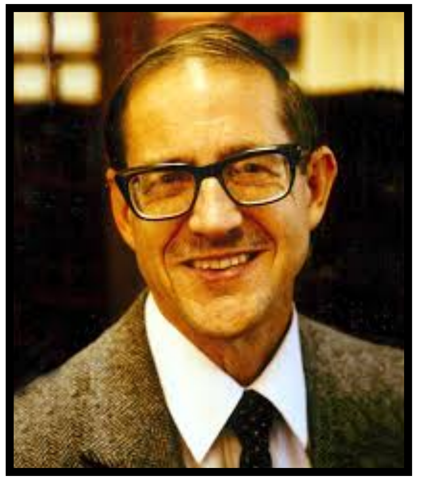


- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.
- Global attractors.

$$\mathcal{F}(x) = 0$$

In 1959, James Serrin published two papers on the existence and stability of certain solutions to the Navier-Stokes equations in the limit of large viscosity.

- Existence of globally stable equilibrium solutions;
- Existence of periodic solutions on a three-dimensional bounded domain subject to time-periodic boundary data and body forces.



James Serrin

Many authors followed Serrin in studying the periodically forced (non-autonomous) Navier-Stokes system dominated by viscosity.

- [**Kaniel & Shinbrot, 1967**] Existence of periodic strong solutions for small time-periodic forcing f (for 3D bounded domains with fixed boundaries);
- [**Takeshita, 1969**] Existence of periodic strong solutions for any time-periodic forcing f (for 2D bounded domains with fixed boundaries);
- many more proofs of existence of periodic orbits for non-autonomous NS [**Teramoto, Maremonti, Kozono & Nakao, Kato, Farwig & Okabe, Hsia**]

- Our understanding of periodic flows in response to time-periodic forcing is rather advanced.

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• The same cannot be said about spontaneous periodic motions, that is periodic flows driven by a time-independent forcing.

• The regular vortex shedding in the wake of a cylinder, for instance, arises in the absence of a body force and as a consequence of the nonlinearity in NS, not by virtue of the advection being dominated by viscous damping.

Goal: Develop a general (computer-assisted) approach to prove existence of spontaneous periodic orbits in the Navier-Stokes flow for some time-independent f.

Major difficulty: the Navier-Stokes equations are nonlinear and infinite dimensional.

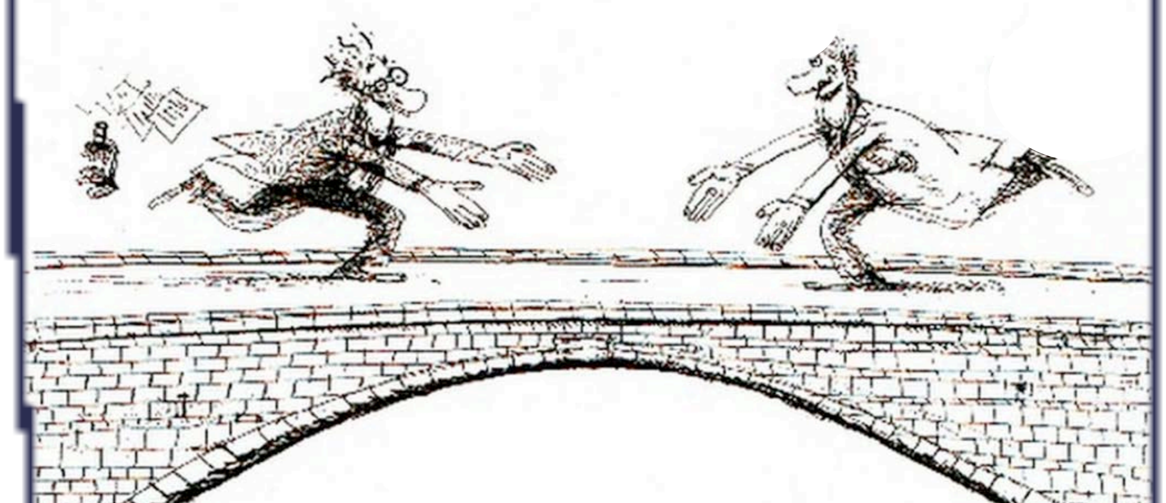
How do mathematicians study nonlinear equations ?

Pen and Paper Analysis

&

Computations

- Nonlinear Analysis
- Topological Methods
- Functional Analysis
- Algebraic Topology
- Calculus of Variations
- Morse-Conley Theory
- Floer Homology
- Symplectic Geometry
- Leray-Schauder Degree
- Bifurcation Theory
- Banach Algebras
- Fixed Point Theory
- Fourier series



Computer-Assisted Proofs

- Scientific Computing
- Numerical Methods
- Approximation Theory
- Numerical Linear Algebra
- Iterative Methods
- Galerkin Approximations
- Continuation Methods
- Fast Fourier Transform (FFT)
- Algorithms
- MATLAB
- C++
- Programming
- Interval Arithmetics

<u>Goal</u>: Develop a general (computer-assisted) approach to prove existence of spontaneous periodic orbits in the Navier-Stokes flow for some time-independent f.

Computer-assisted proofs (CAPs) in dynamics

The main idea is to construct algorithms that provide an approximate solution to a problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

This field draws inspiration from the ideas in

- Scientific computing
- Functional analysis
- Approximation theory
- Nonlinear analysis
- Numerical analysis
- Topological methods

Early pioneer works

Cesari [1964] ***Functional analysis and Galerkin's method.***
Lanford [1982] ***A computer-assisted proof of the Feigenbaum conjectures.***
Mischaikow & Mrozek [1995] ***Chaos in the Lorenz equations.***
Tucker [1999] ***The Lorenz attractor exists.***

A functional analytic approach to CAPs in dynamics

A general nonlinear problem

$$\boxed{\mathcal{F}(x) = 0}$$

The unknown x could be a

- **solution to an initial value problem of an ODE**
- **periodic orbit of an ODE**
- **local (un)stable manifold of a fixed point of an ODE**
- **normal bundle of a periodic orbit of an ODE**
- **local (un)stable manifold of a periodic orbit of an ODE**
- **connecting orbit of an ODE**
- **periodic orbit of a functional delay equation**
- **critical point of an action functional**
- **solution to a boundary value problem**
- **steady state of a PDE**
- **bifurcation equilibrium point of a PDE**
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A general nonlinear problem **to solve in a Banach space**

$$\boxed{\mathcal{F}(x) = 0}$$

$$X$$

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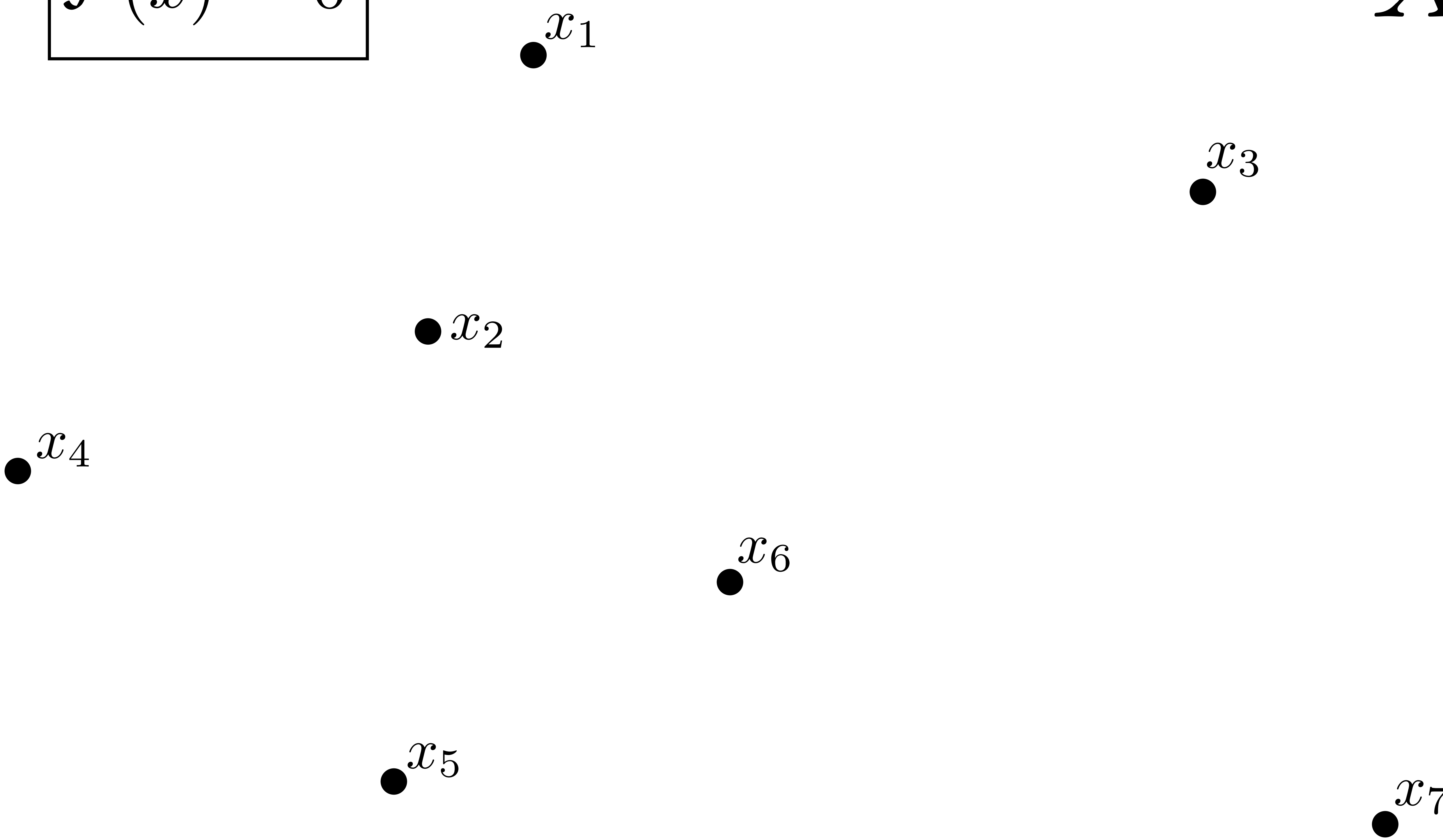
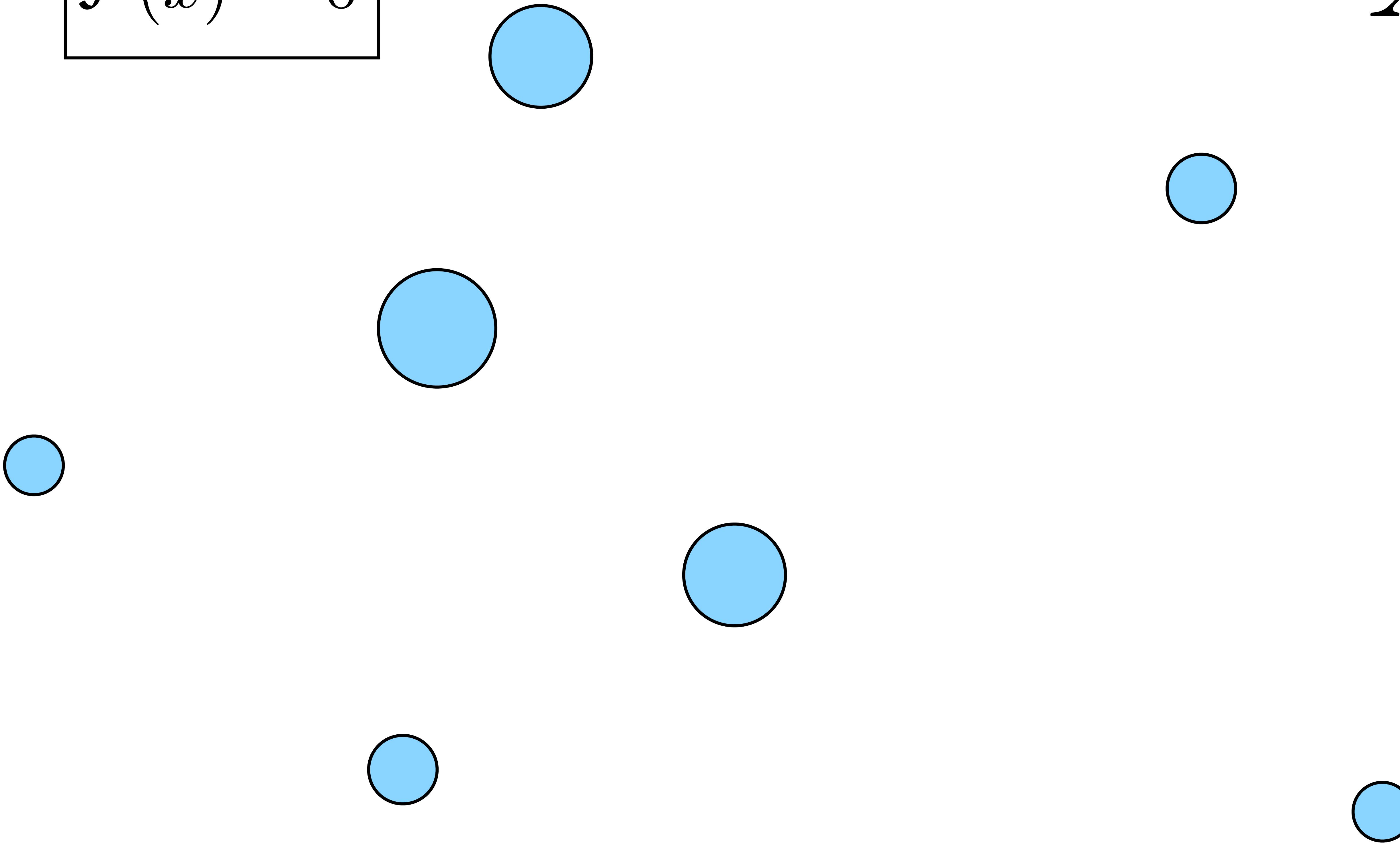


Impossible to compute exactly !

A general nonlinear problem **to solve in a Banach space** X

$$\boxed{\mathcal{F}(x) = 0}$$



Alternative: find small balls in which it is demonstrated (in a mathematically rigorous sense) that a unique solution exists.

How to find these small isolating balls ?

1. Let $\bar{x}$ a numerical approximation of $\mathcal{F}(x) = 0$ in X computed using a finite dimensional reduction.

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6. Find $r > 0$ such that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping (tool : **radii polynomials**).

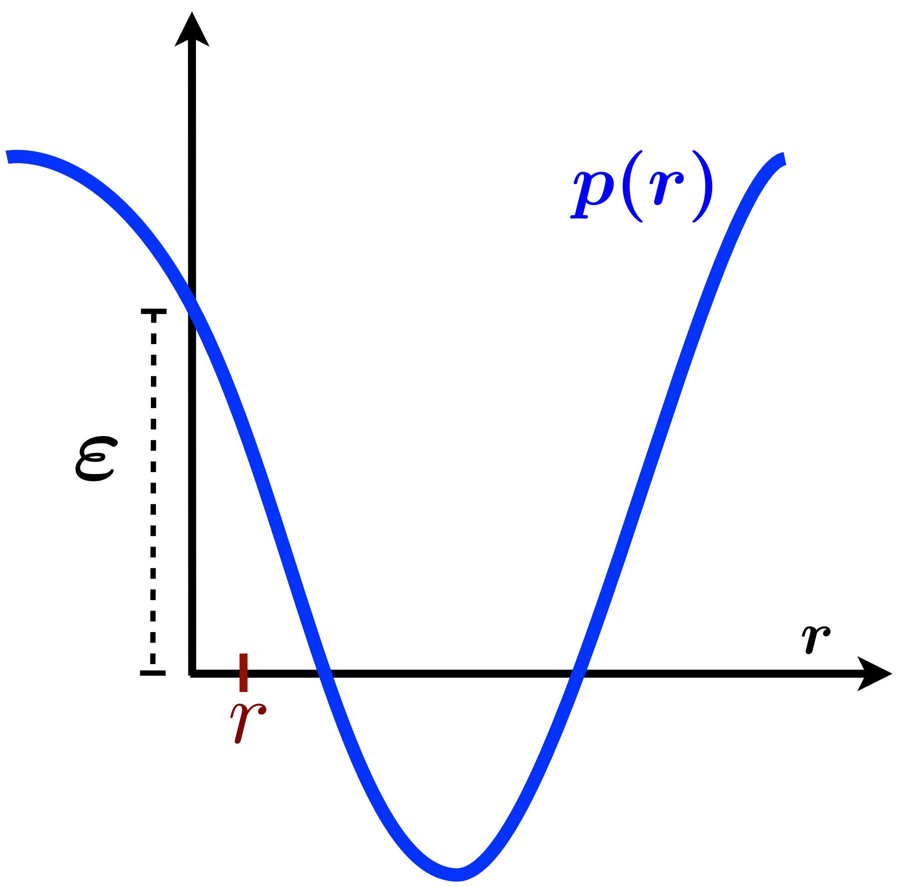
Theorem : Let $T : X \to X$ defined by $T(x) = x - A\mathcal{F}(x)$ with $T \in C^1(X)$. Let $r > 0$ and consider bounds ε and $\kappa = \kappa(r)$ satisfying

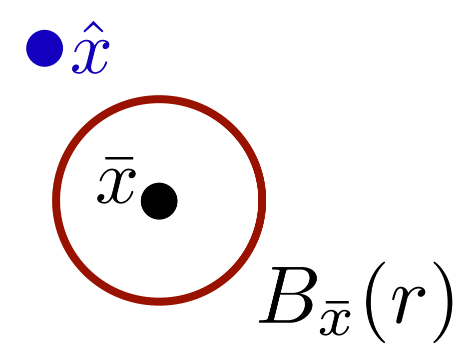
$$\begin{aligned} \|T(\bar{x}) - \bar{x}\|_X &= \|A\mathcal{F}(\bar{x})\|_X \le \varepsilon \\ \sup_{w \in B_{\bar{x}}(r)} \|DT(w)\|_X &= \sup_{w \in B_{\bar{x}}(r)} \|I - A \cdot D\mathcal{F}(w)\|_X \le \kappa(r). \end{aligned}$$

If

$$p(r) \stackrel{\text{def}}{=} \varepsilon + r\kappa(r) - r < 0 \quad \textbf{(radii polynomial)}$$

then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction with Lipschitz constant $\kappa(r) < 1$. Moreover A is injective and therefore $\mathcal{F} = 0$ has a unique solution in $B_{\bar{x}}(r)$.





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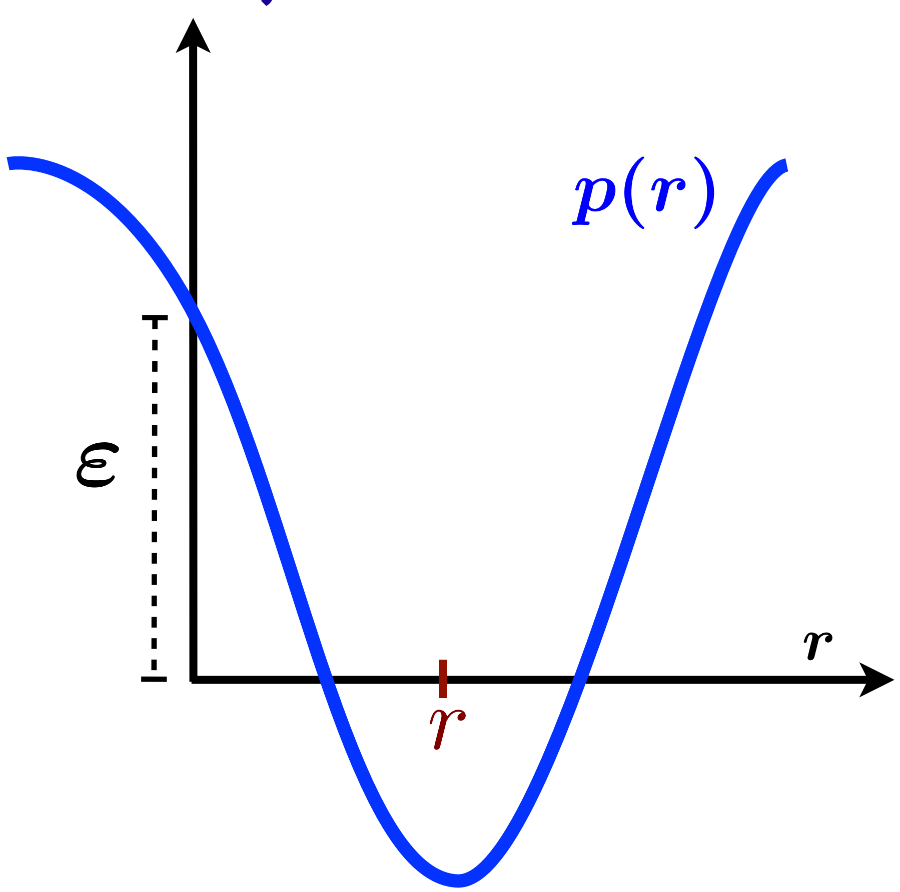
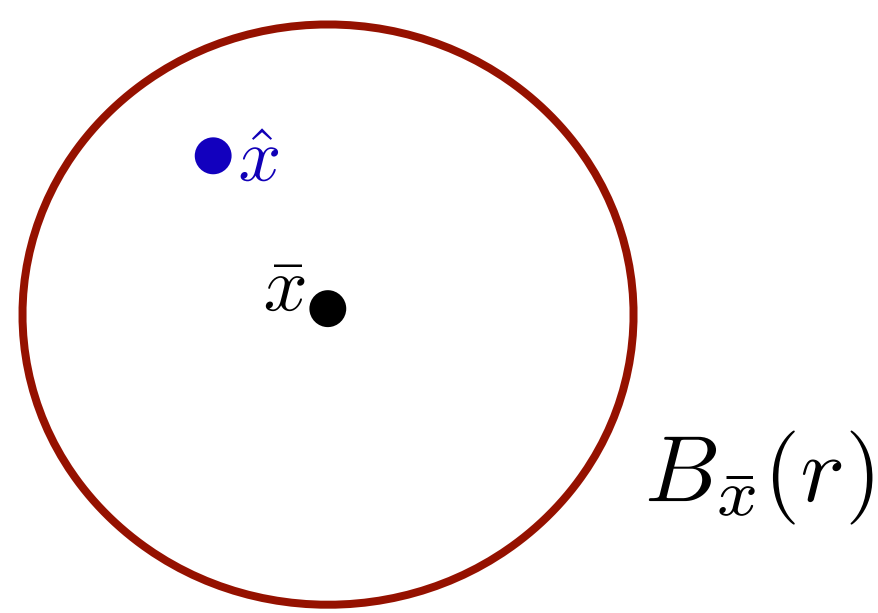
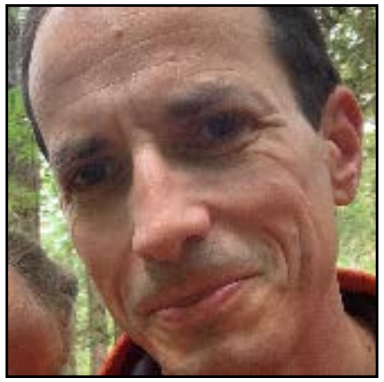




Spontaneous periodic orbits in the Navier-Stokes flow

<u>Joint work with</u>



J.B. van den Berg
VU Amsterdam

Maxime Breden
École Polytechnique

Lennaert van Veen
Ontario Tech

A zero-finding problem for periodic orbits in NS

Applying the curl operator to Navier-Stokes yields the vorticity equation

$$\partial_t \omega - \nu \Delta \omega + \text{nonlinear terms} = f^\omega \quad \text{on } \mathbb{T}^3 \times \mathbb{R},$$

where $\omega \stackrel{\text{def}}{=} \nabla \times u$ and $f^\omega \stackrel{\text{def}}{=} \nabla \times f$.

Plugging the space-time Fourier expansion of the vorticity

$$\omega(x,t) = \sum_{n \in \mathbb{Z}^4} \omega_n e^{i(\tilde{n} \cdot x + n_4 \Omega t)}, \quad \tilde{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3,$$

in the vorticity equation yields having to solve the zero-finding problem

$$F_n(W) \stackrel{\text{def}}{=} i\Omega n_4 \omega_n + \nu \tilde{n}^2 \omega_n - f_n^\omega + \text{nonlinear terms} = 0,$$

where Ω is the time-frequency, $\tilde{n}^2 \stackrel{\text{def}}{=} n_1^2 + n_2^2 + n_3^2$, and

$$W = \begin{pmatrix} \Omega \\ (\omega_n)_{n \in \mathbb{Z}^4 \setminus \{0\}} \end{pmatrix}.$$

A zero-finding problem for periodic orbits in NS

Lemma: Let W be such that the vorticity ω is analytic. Assume that $F(W)=0$ and $\nabla\cdot\omega=0$. Assume also that f does not depend on time and has space average zero. Define $u=M\omega$ (that is u solves $\omega=\nabla\times u$). Then there exists a pressure function $p:\mathbb{T}^3\times\mathbb{R}\to\mathbb{R}$ such that (u,p) is a $\frac{2\pi}{\Omega}$-periodic solution of NS.

$$\mathcal{F}(W)=\begin{pmatrix} F_{☾}(W) \\ (F_n(W))_{n\in\mathbb{Z}^4_*}\end{pmatrix}=0$$

$$F_{☾}(W)=\int_0^{\frac{2\pi}{\Omega}}\int_{\mathbb{T}^3}\omega(x,t)\cdot\partial_t\hat{\omega}(x,t)\,dxdt=0$$

(phase condition)

$F_n(W)\overset{\text{def}}{=} i\Omega n_4\omega_n+\nu\tilde{n}^2\omega_n-f_n^\omega$ + nonlinear terms

Given $\eta\geq 1$, denote the weighted ℓ^1 Banach algebra (under discrete convolution)

$$\ell^1_\eta(\mathbb{C})\overset{\text{def}}{=}\left\{a\in\mathbb{C}^{\mathbb{Z}^4_*}:\|a\|_{\ell^1_\eta}\overset{\text{def}}{=}\sum_{n\in\mathbb{Z}^4_*}|a_n|\eta^{|n_1|+\cdots+|n_4|}<\infty\right\}.$$

Banach space : $X=\mathbb{C}\times\left(\ell^1_\eta(\mathbb{C})\right)^3$ with norm

$$\|W\|_X=|\Omega|+\|\omega_1\|_{\ell^1_\eta}+\|\omega_2\|_{\ell^1_\eta}+\|\omega_3\|_{\ell^1_\eta}.$$

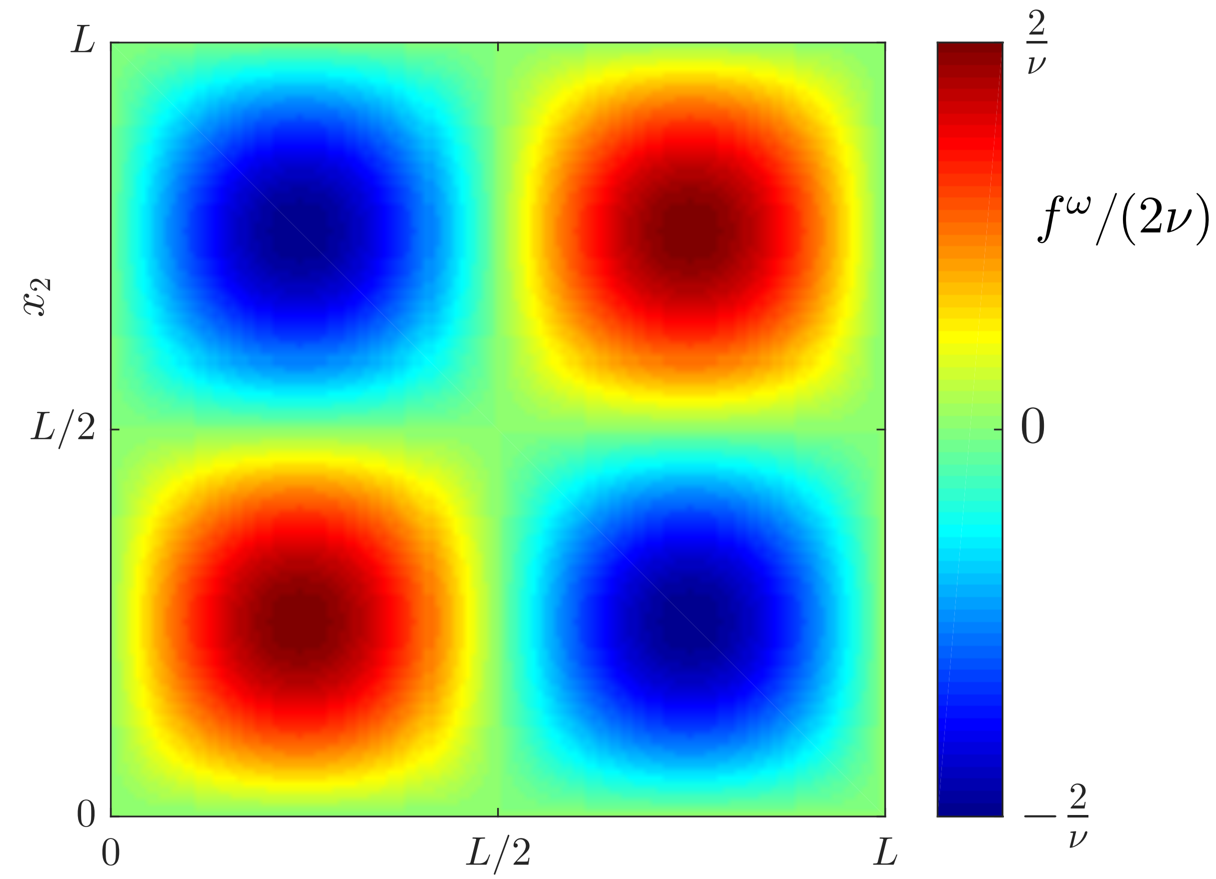
Spontaneous periodic orbits in the Navier-Stokes flow

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f, & \text{on } \mathbb{T}^3 \text{ of size } L = 2\pi \\ \nabla \cdot u = 0. \end{cases}$$

Taylor-Green (time-independent) forcing term

$$f = \begin{pmatrix} 2 \sin x_1 \cos x_2 \\ -2 \cos x_1 \sin x_2 \\ 0 \end{pmatrix}$$

$$f^\omega \stackrel{\text{def}}{=} \nabla \times f = \begin{pmatrix} 0 \\ 0 \\ 4 \sin x_1 \sin x_2 \end{pmatrix}$$



Density plot of the vertical vorticity of $f^\omega/(2\nu)$. Red corresponds to vortices with a counter clockwise rotation and blue clockwise.

The autonomous Navier-Stokes equations under this time-independent forcing term admit a viscous equilibrium solution for which we have the analytic expression

$$u^* = \frac{1}{2\nu} f, \qquad p^* = \frac{1}{4\nu^2} \left(\cos 2x_1 + \cos 2x_2 \right).$$

Spontaneous periodic orbits in the Navier-Stokes flow

$$F_n(W) \stackrel{\text{def}}{=} i\Omega n_4 \omega_n + \nu \tilde{n}^2 \omega_n - f_n^\omega + \text{nonlinear terms}$$

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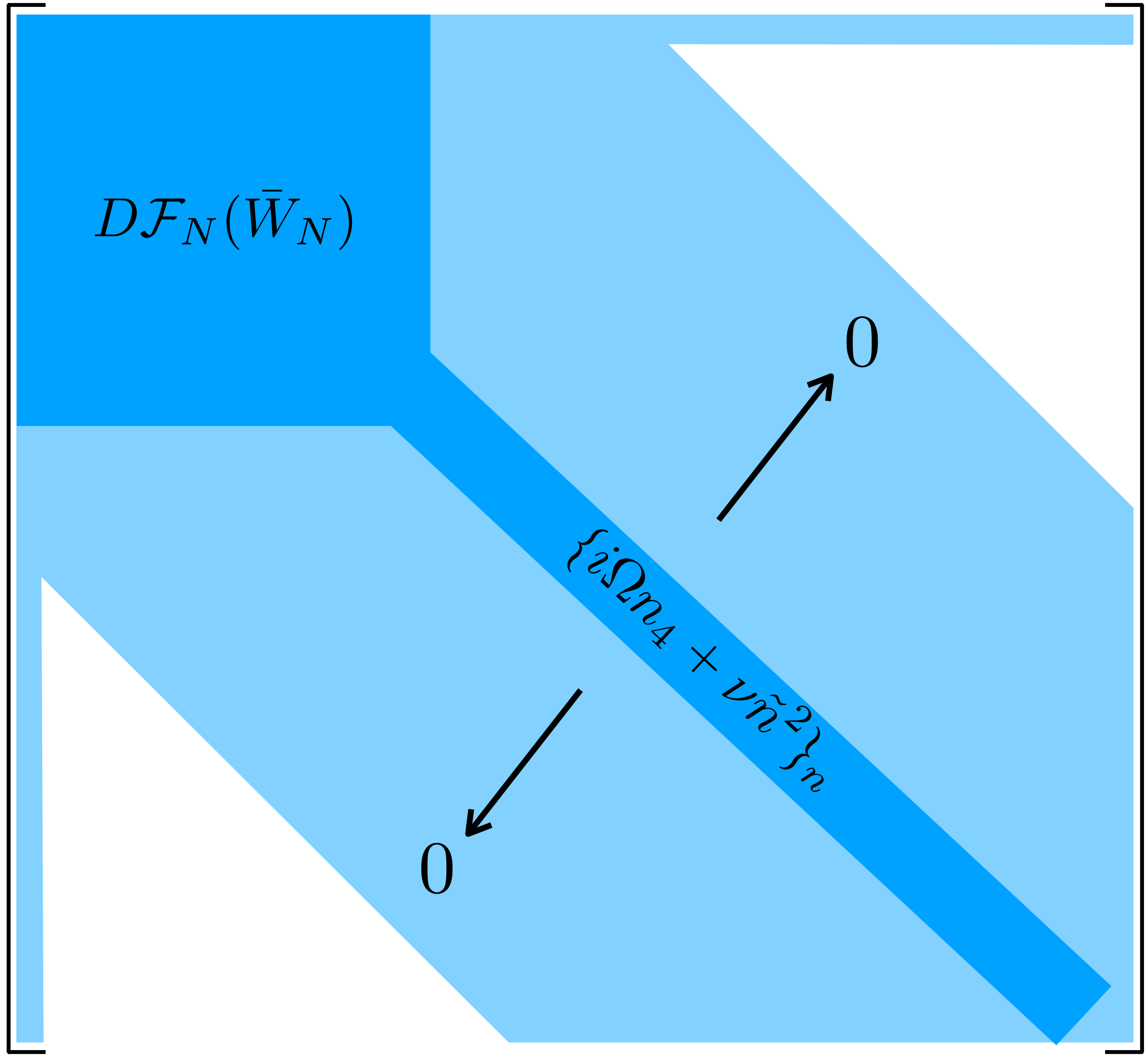

Regularity implies decay

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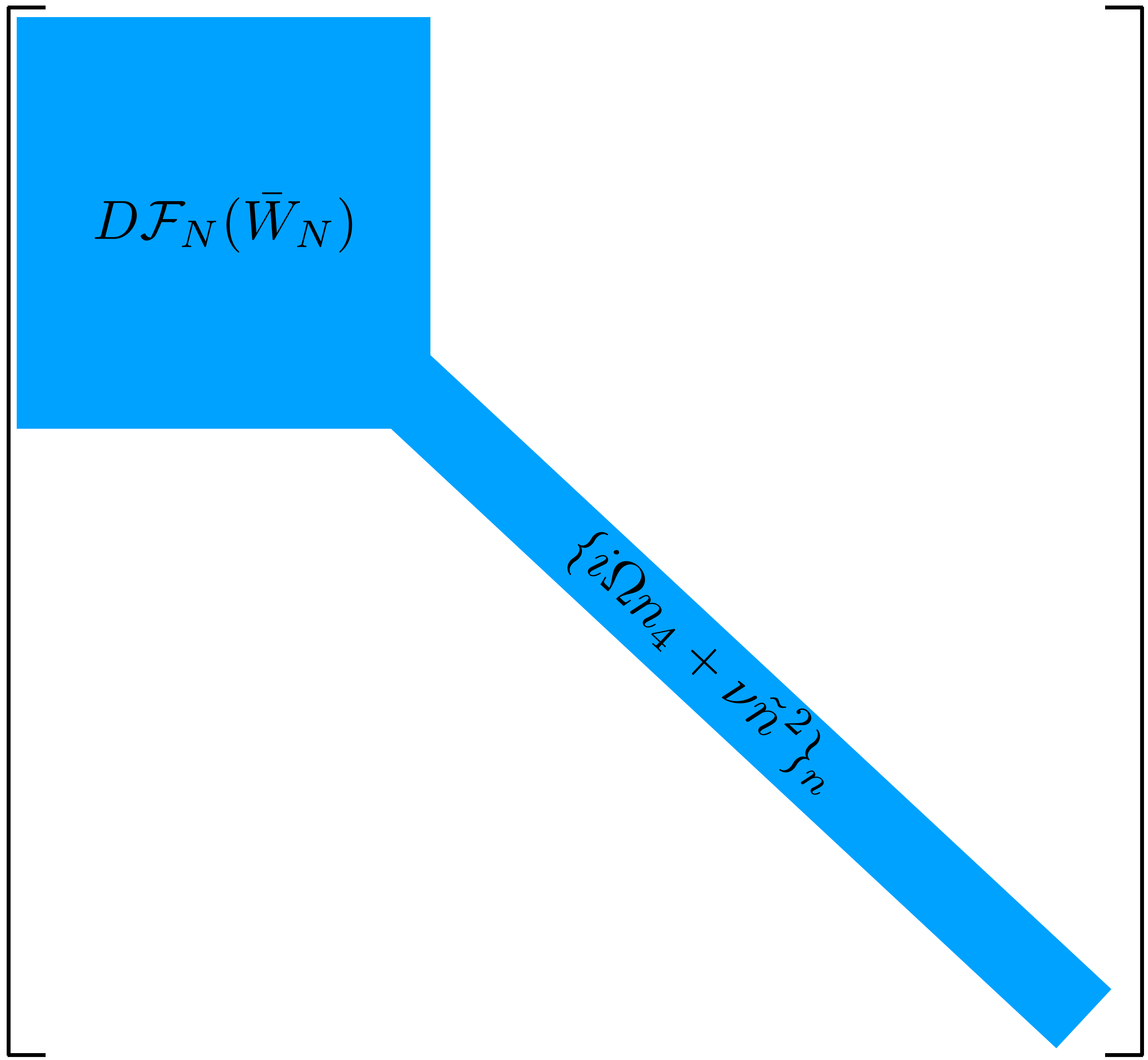
$D\mathcal{F}(\bar{W}) \approx$



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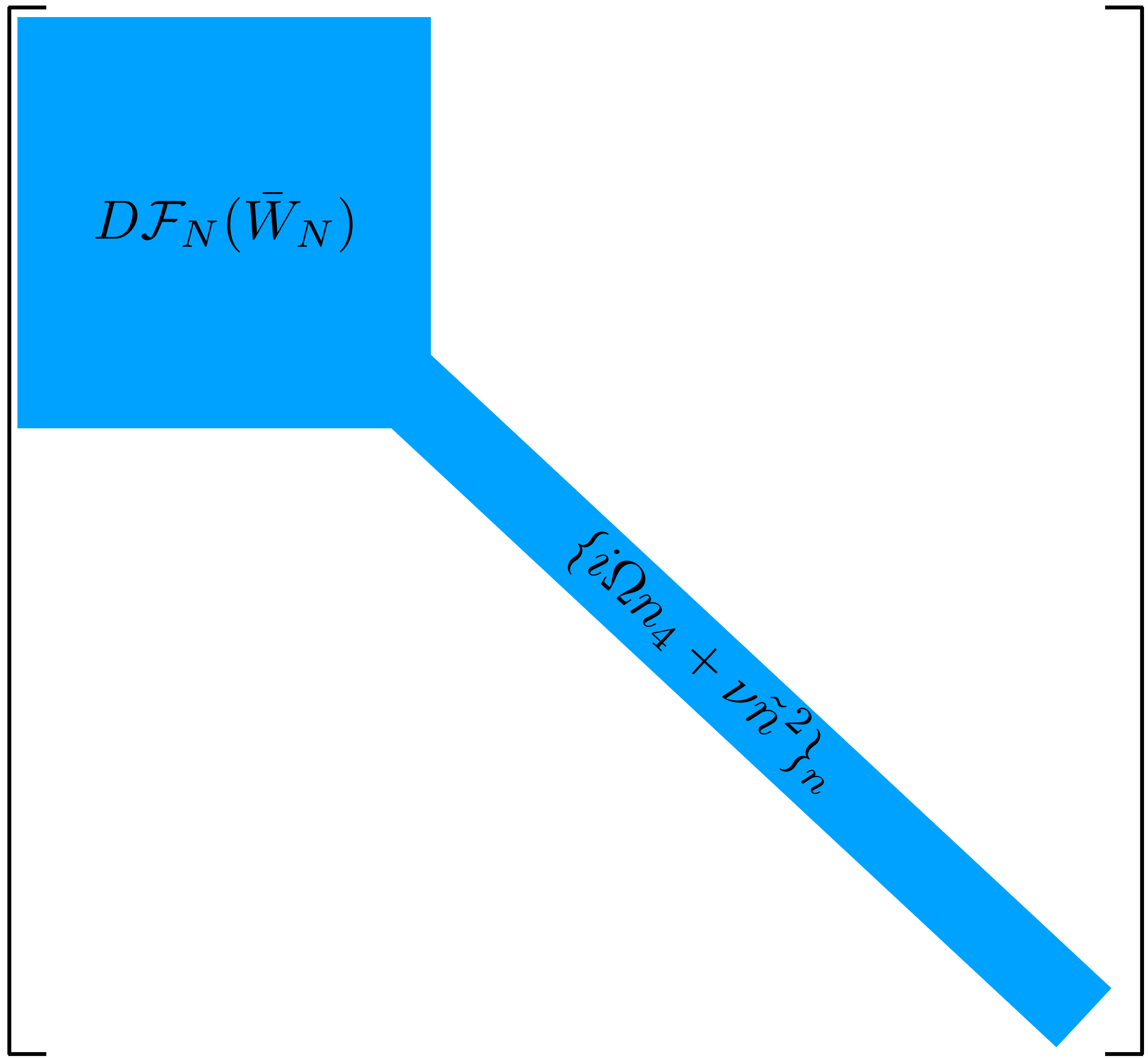

$$A_N \approx D\mathcal{F}_N(\bar{W}_N)^{-1}$$

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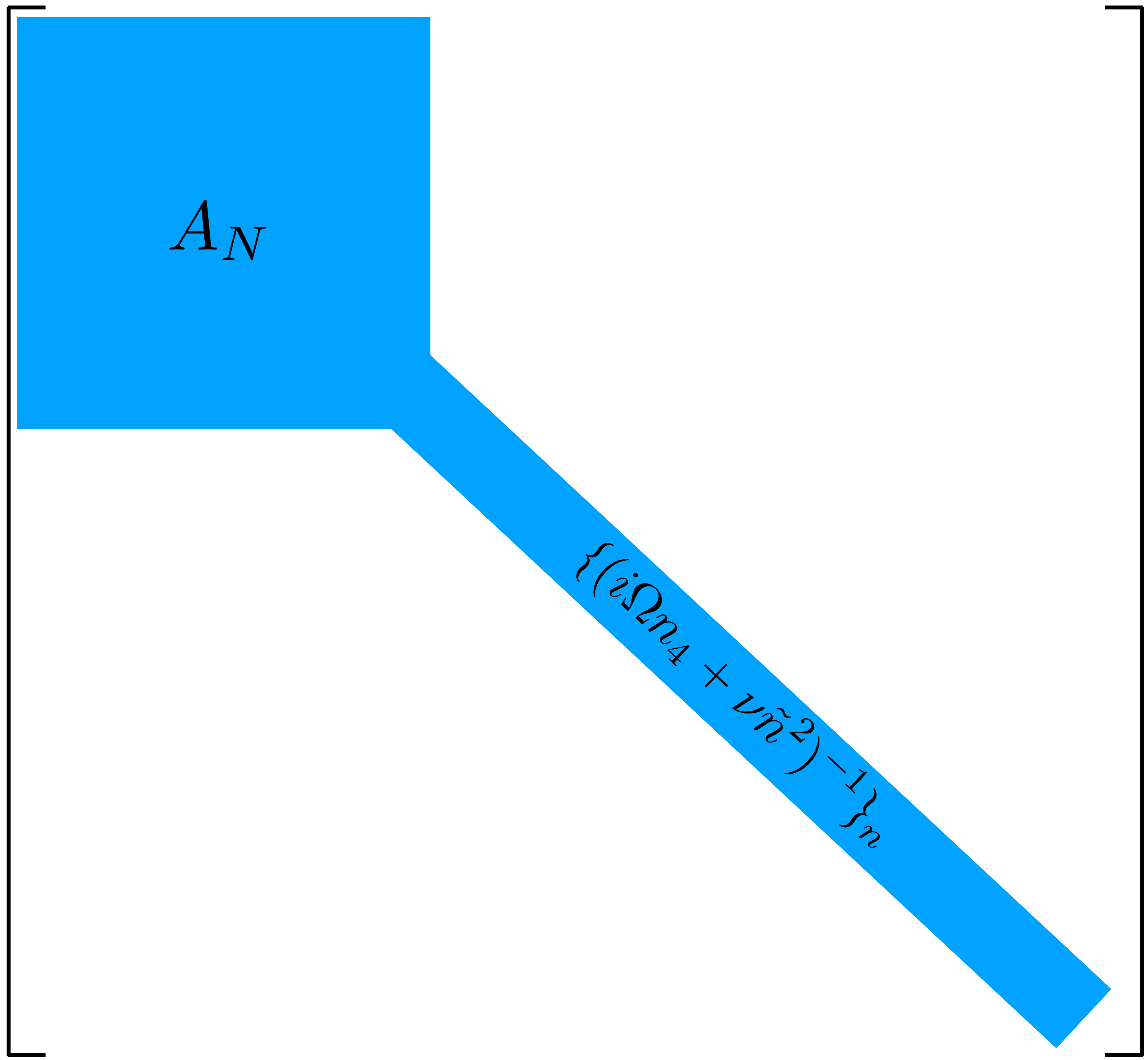


A functional analytic approach to CAPs in dynamics

Theorem : Let $r > 0$ and consider ε and $\kappa = \kappa(r)$ be such that

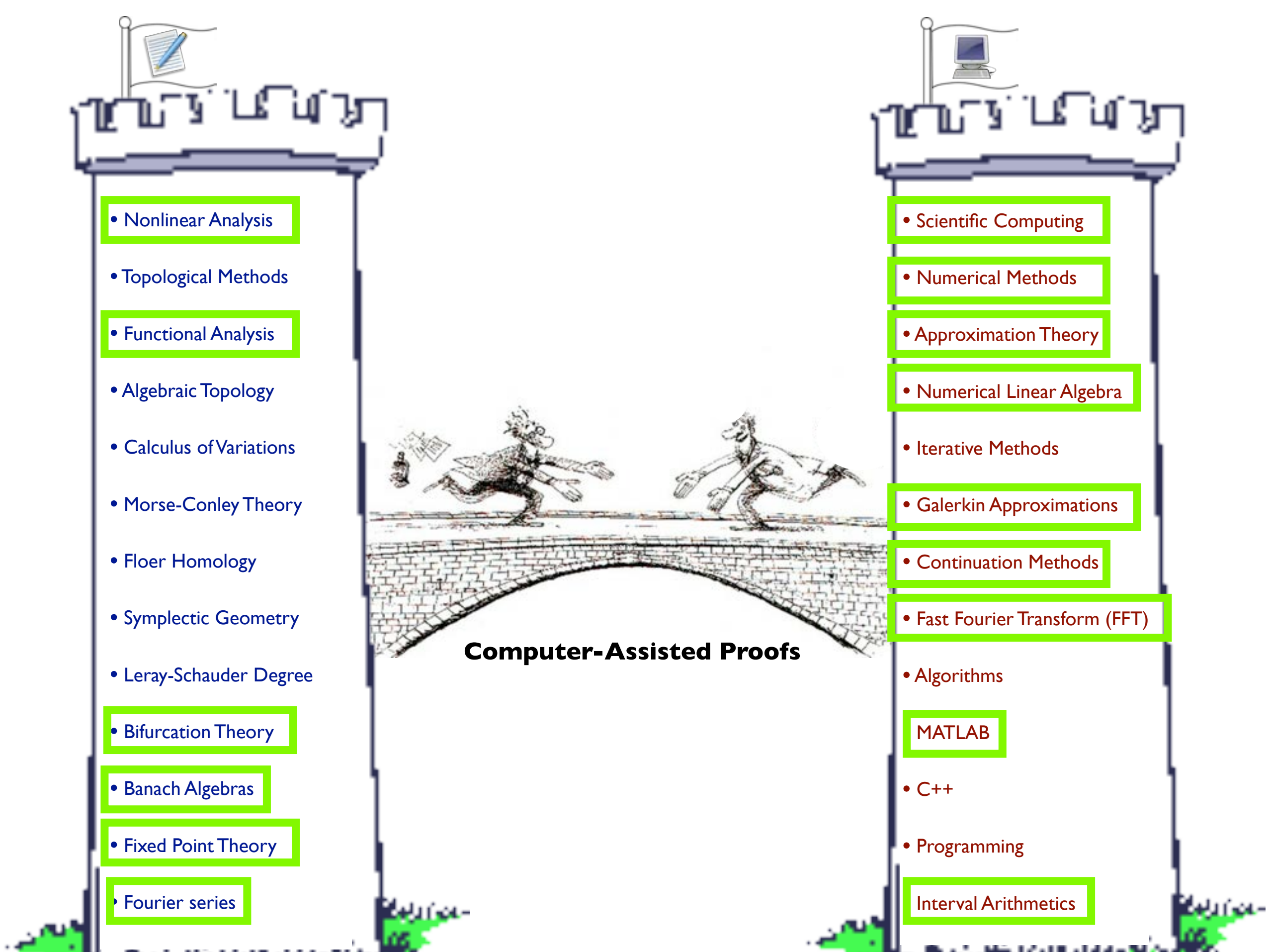
$$\|A\mathcal{F}(\bar{W})\|_X \le \varepsilon$$

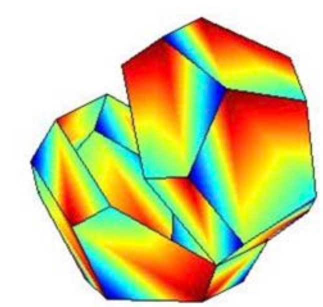
$$\sup_{Z \in \overline{B_{\bar{W}}(r)}} \|I - A\, D\mathcal{F}(Z)\|_{B(X)} \le \kappa(r).$$

Define the *radii polynomial*

$$p(r) \stackrel{\text{def}}{=} \varepsilon + r\kappa(r) - r.$$

If $\exists\, r_0 > 0$ such that $p(r_0) < 0$, then $\exists\,!\ \tilde{W} \in B_{\bar{W}}(r)$ satisfying $\mathcal{F}(\tilde{W}) = 0$.

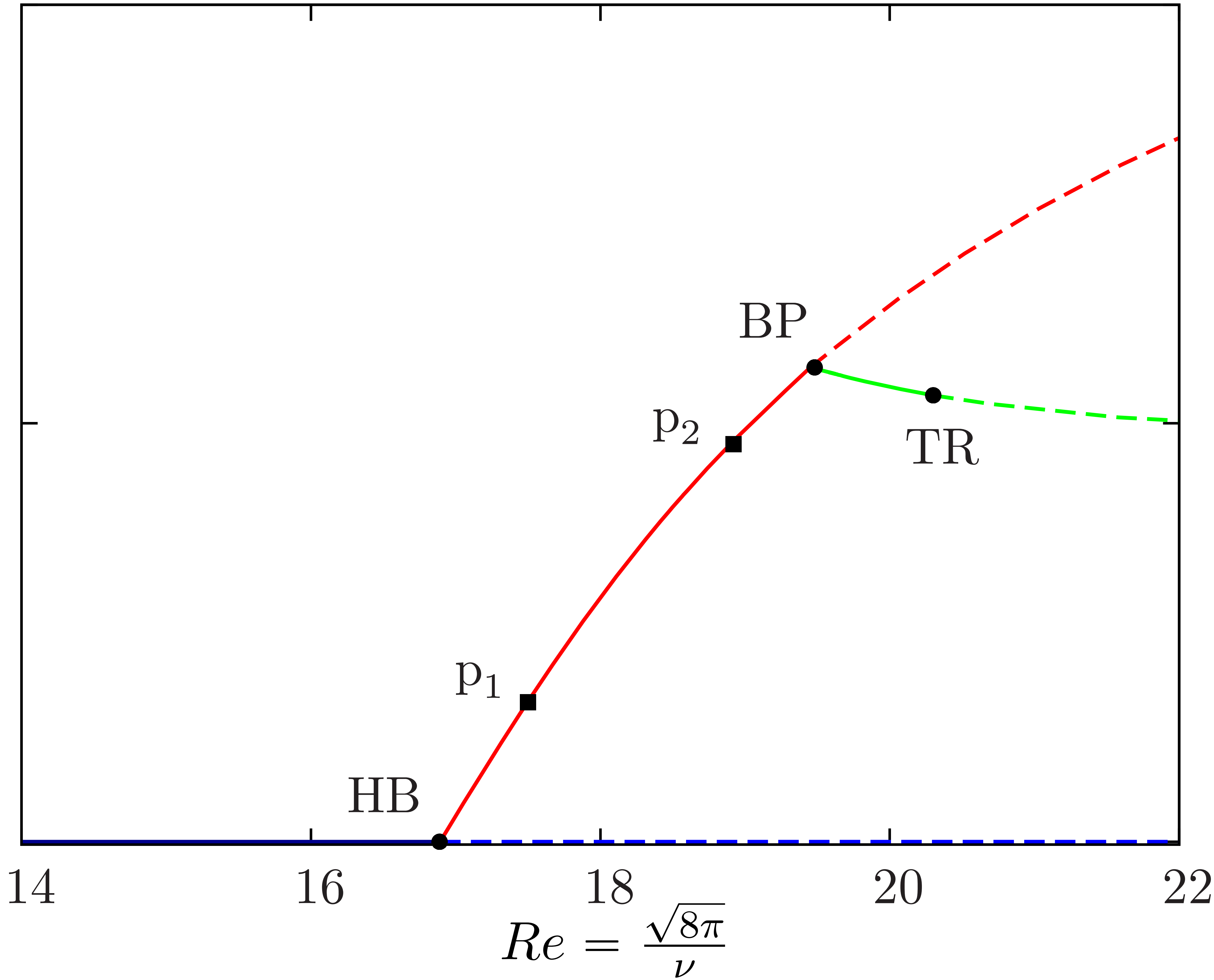




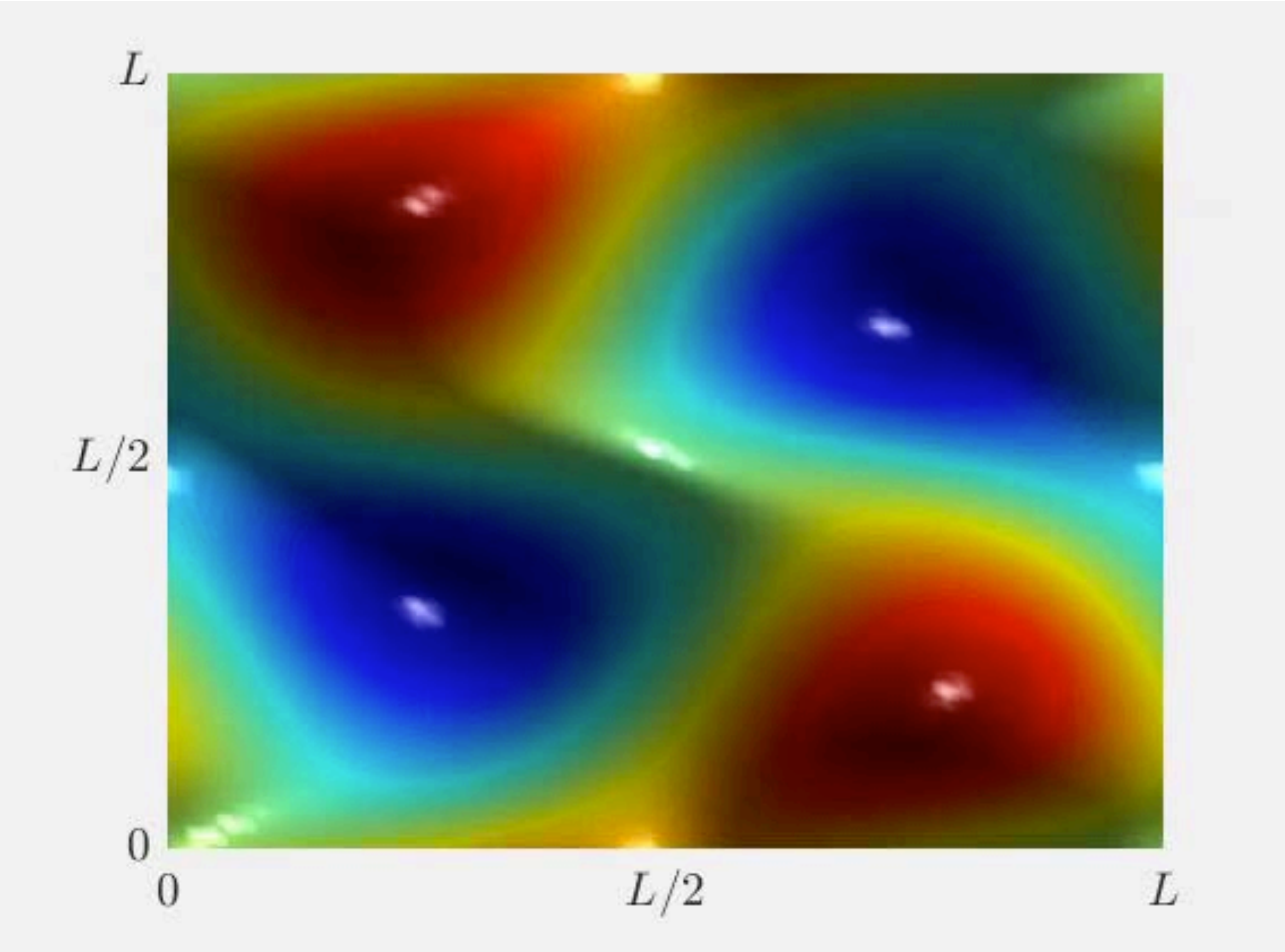
INTLAB - INTerval LABoratory

The Matlab/Octave toolbox for Reliable Computing Version 12

Spontaneous periodic orbits in the Navier-Stokes flow



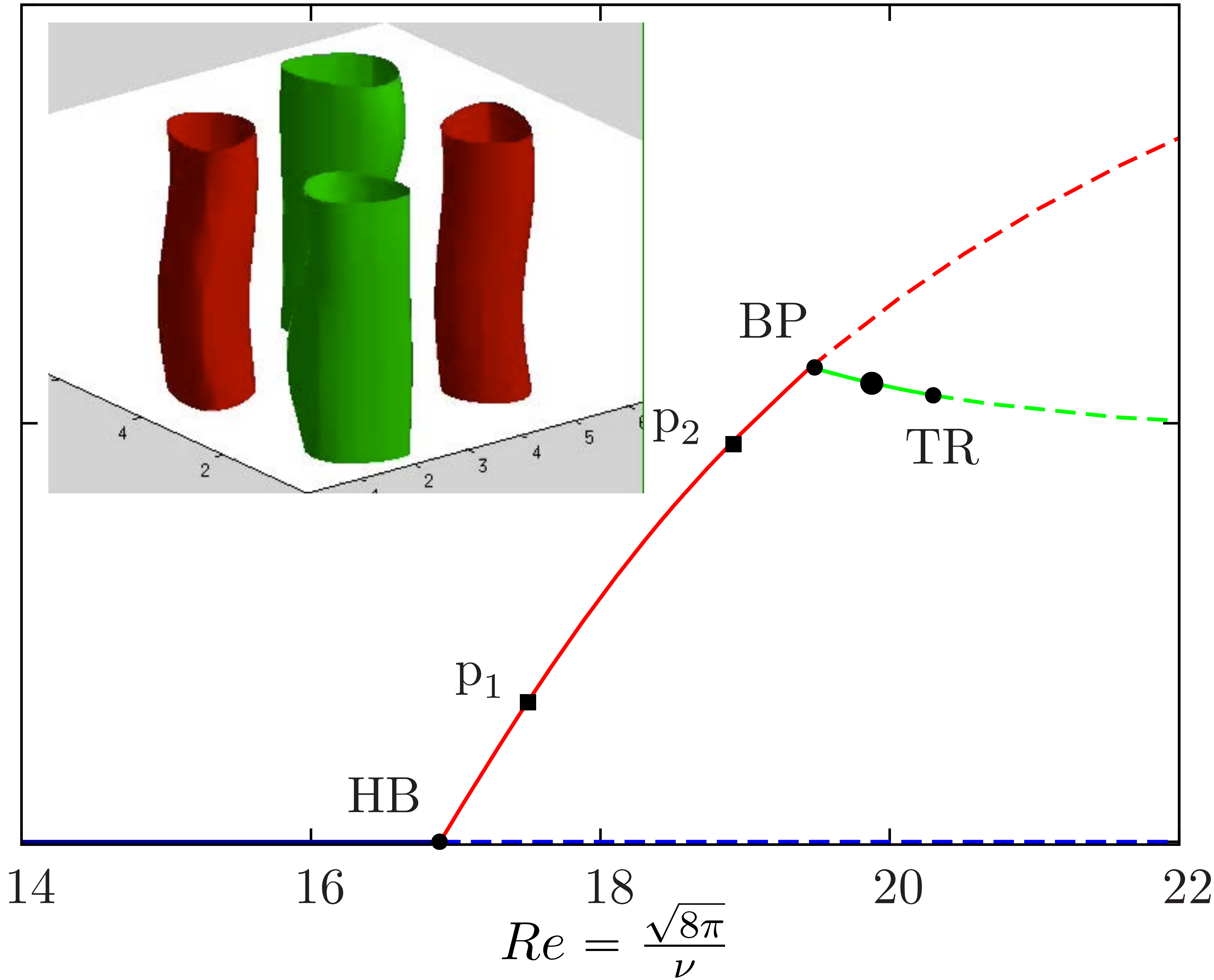
Theorem: Consider NS defined on the three-torus $\mathbb{T}^3$ (with size length $L = 2\pi$) and consider the Taylor-Green time-independent forcing term. Let $\nu = 0.265$ and $(\bar{u}, \bar{p})$ be a numerical solution computed with $N_{x_1} = N_{x_2} = 21$, $N_{x_3} = 0$ and $N_t = 16$ Fourier coefficients. Let $r = 2.2491 \cdot 10^{-6}$. There exists a $\frac{2\pi}{\Omega}$-periodic solution (u, p) of NS with $|\Omega - \bar{\Omega}| \leqslant r$ and $\|u - \bar{u}\|_{C^0} \leqslant r$.



	η	N_{x_1}	N_{x_2}	N_{x_3}	N_t	$N^\dagger$	$\widetilde{N}$	RAM (GB)	CPU days
p_1	1	17	17	0	11	130	265	10	6
p_2	1	21	21	0	16	210	425	110	95

Spontaneous Periodic Orbits in the Navier–Stokes Flow. *J. Nonlinear Sci.* 31 (2021), no. 2, Paper No. 41

Future work: a fully 3D spontaneous periodic orbit



Other Recent Applications

Equilibria of PDEs

Joint work with

Rustum Choksi
McGill

Gabriel Martine
McGill

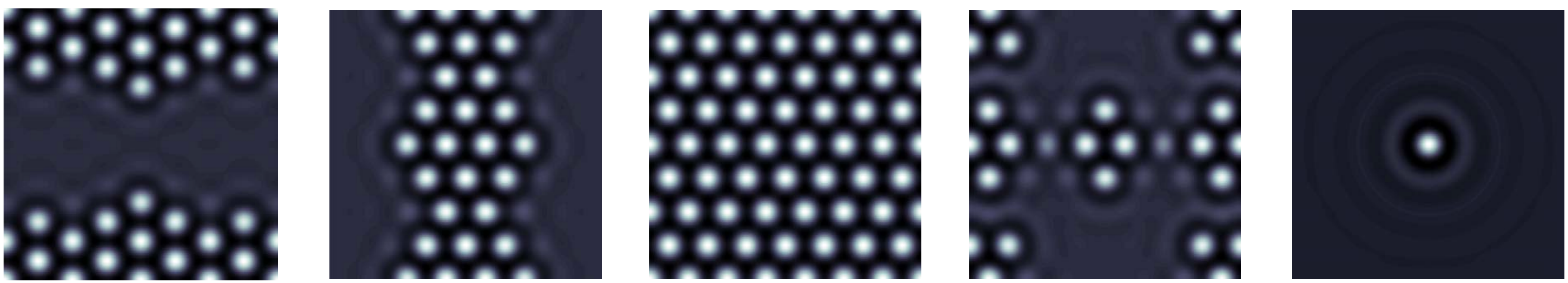
2D Phase-Field-Crystal Model

$$\psi_t = \nabla^2 \left(\left(\nabla^2 + 1\right)^2 \psi + \psi^3 - \beta\psi \right)$$

$$\Omega = \left[0, \frac{4\pi}{\sqrt{3}} N_x\right] \times [0, 4\pi N_y]$$

$N_x, N_y \in \mathbb{N}$: number of atoms lined up in the x, y-axes

Steady states in the localized patterns regime



$\beta = 0.6$, $(N_x, N_y) = (7, 4)$

Microscopic patterns in the 2D phase-field-crystal model. *Preprint.*

Equilibria of PDEs

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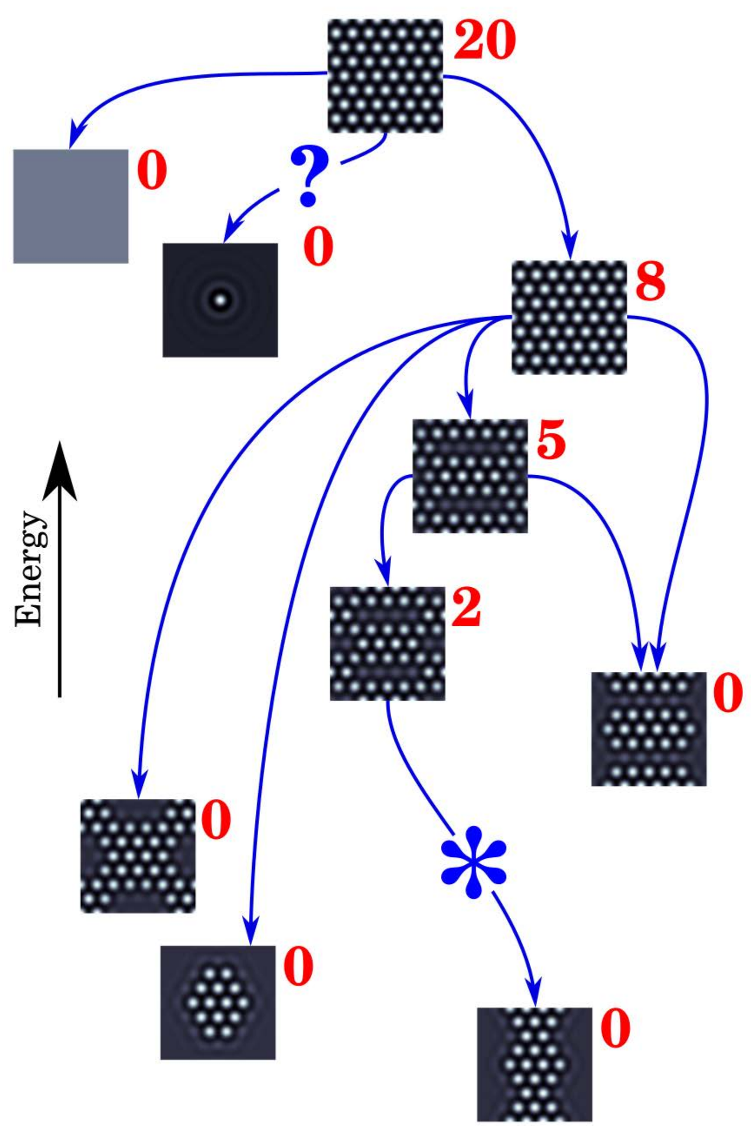
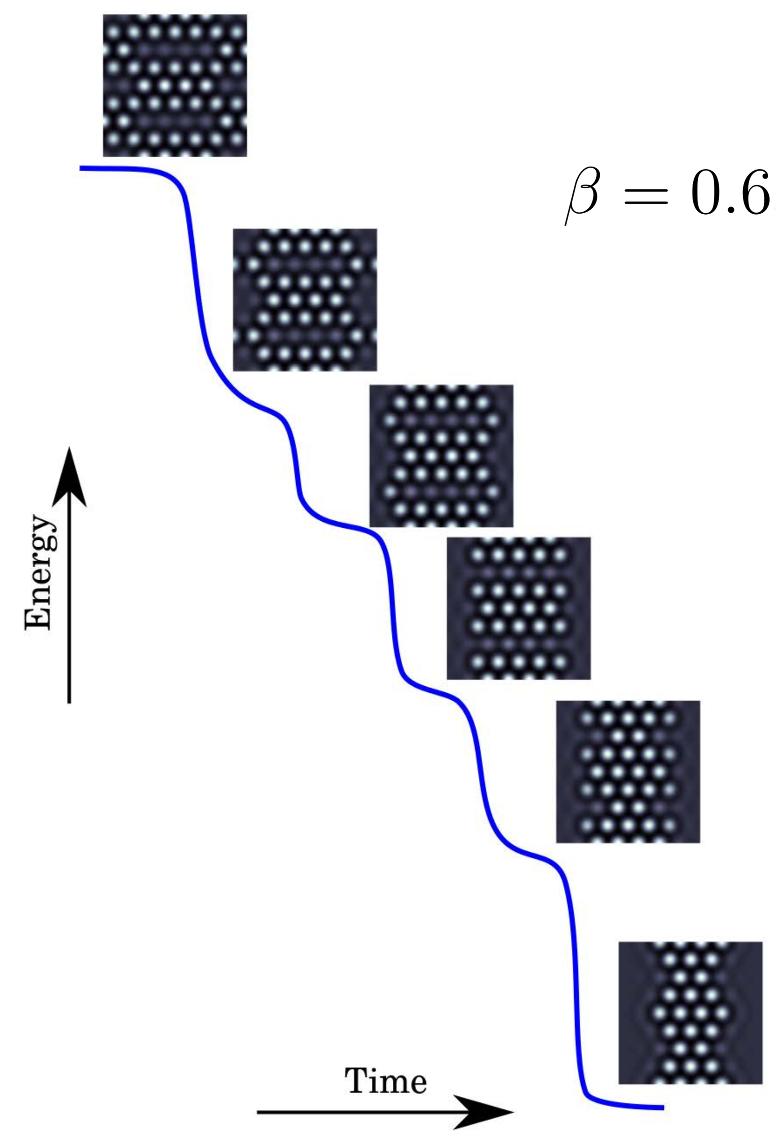
$$\Omega = \left[0, \frac{4\pi}{\sqrt{3}} N_x \right] \times [0, 4\pi N_y]$$

$N_x, N_y \in \mathbb{N}$: number of atoms lined up in the x, y-axes





$\beta = 0.6$, $(N_x, N_y) = (7, 4)$

Equilibria of PDEs

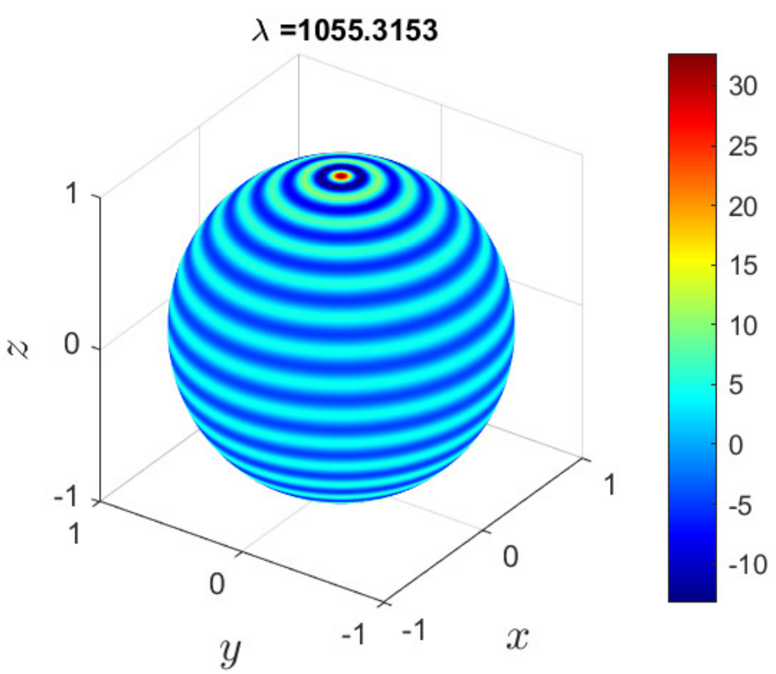
Joint work with

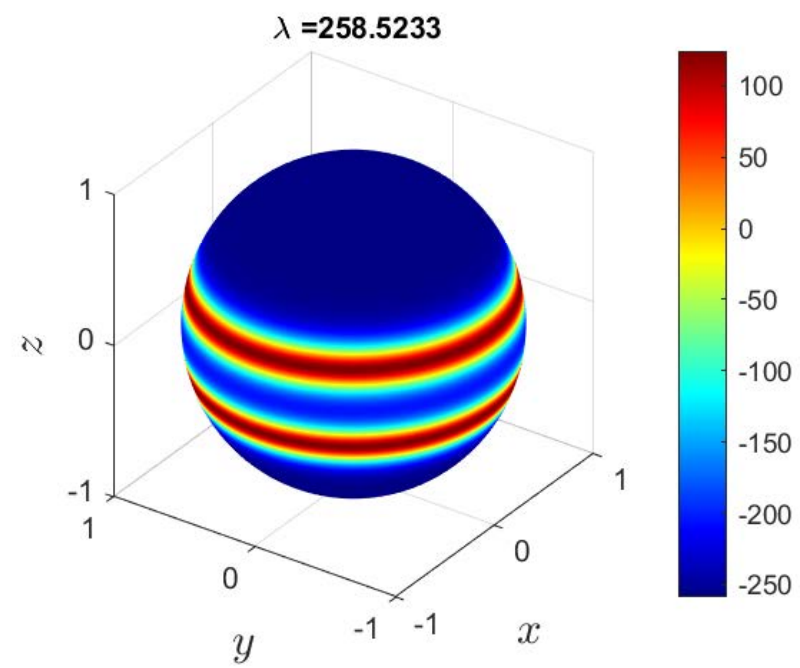
J.B. van den Berg
VU Amsterdam

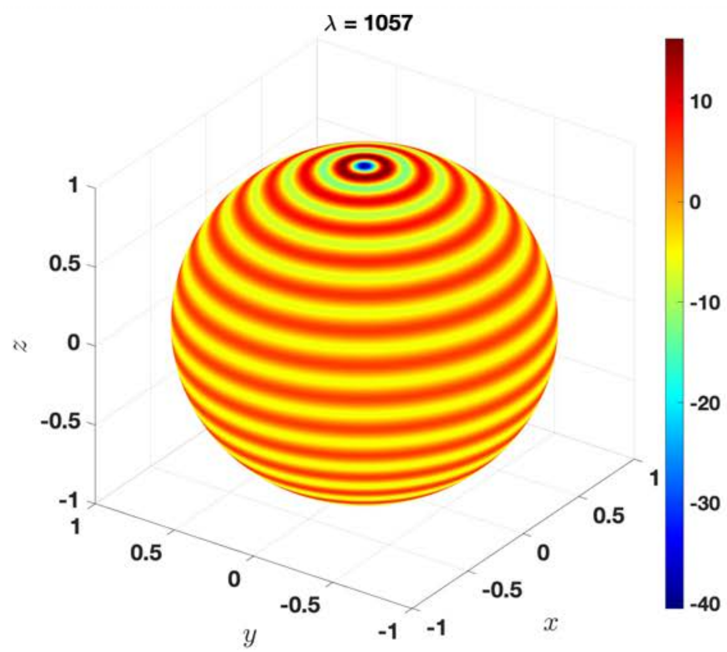
Gabriel Duchesne
McGill

A nonlinear Laplace-Beltrami equation on the sphere

$$u_t = \Delta u + \lambda u + u^2$$





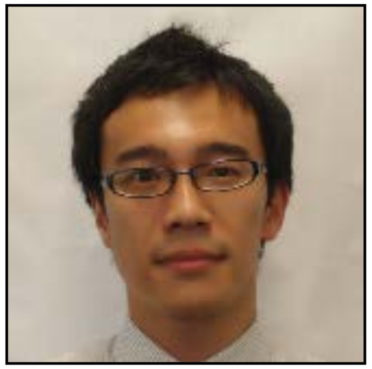


Rotation invariant patterns for a nonlinear Laplace-Beltrami equation: a Taylor-Chebyshev series approach. *Preprint.*

Global dynamics in the nonlinear Schrödinger equation

Joint work with

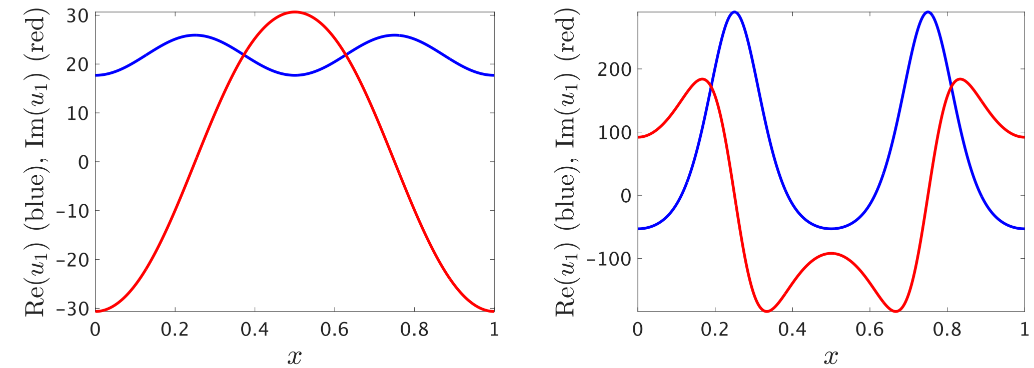
Jonathan Jaquette
Boston U.



Akitoshi Takayasu
U. of Tsukuba

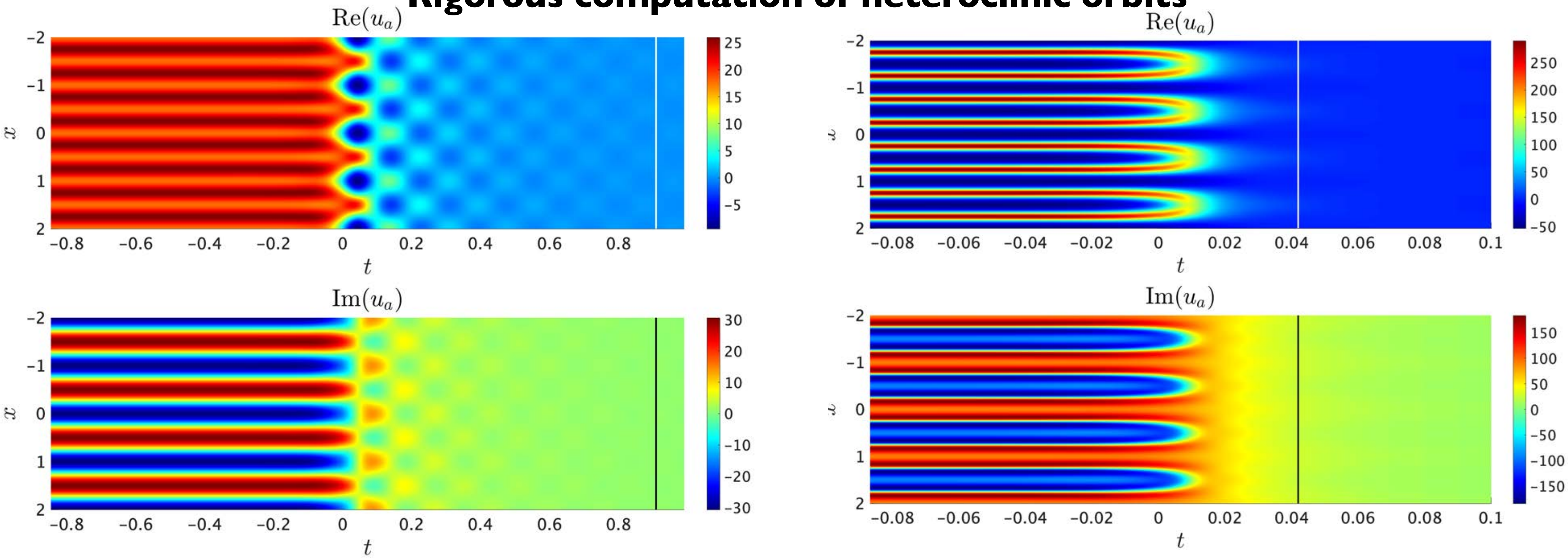
$$-iu_t = \triangle u + u^2$$

$$x \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$$



Nontrivial steady states

Rigorous computation of heteroclinic orbits



Global dynamics in nonconservative nonlinear Schrödinger equations. *Preprint, 2021.*

MS188 Computer-Assisted Mathematical Proofs in Nonlinear Dynamics (Thursday, May 27th)
8:00AM--8:25AM Akitoshi Takayasu - Rigorous Integrator for Dissipative PDEs using the Chebyshev-Fourier Spectral Method

Periodic orbits in the ill-posed Boussinesq equation

$$u_{tt} = u_{yy} + \lambda u_{yyyy} + (u^2)_{yy}, \quad \lambda > 0$$
$$u = u(t,y) \in \mathbb{R}, \; y \in [0,1], \quad u(t,0) = u(t,1)$$

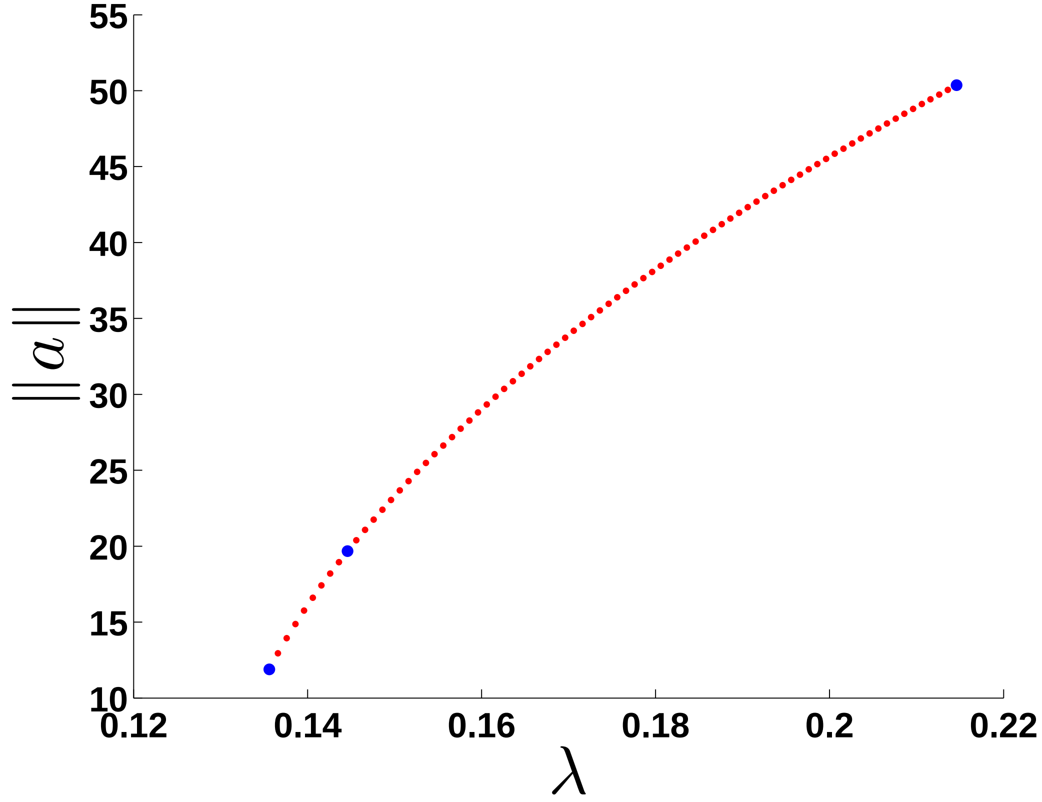
This "bad" version of Boussinesq arises in the study of water waves. Specifically, it is used to describe a two-dimensional flow of a body of water over a flat bottom with air above the water, assuming that the water waves have small amplitudes and the water is shallow.

Joint work with

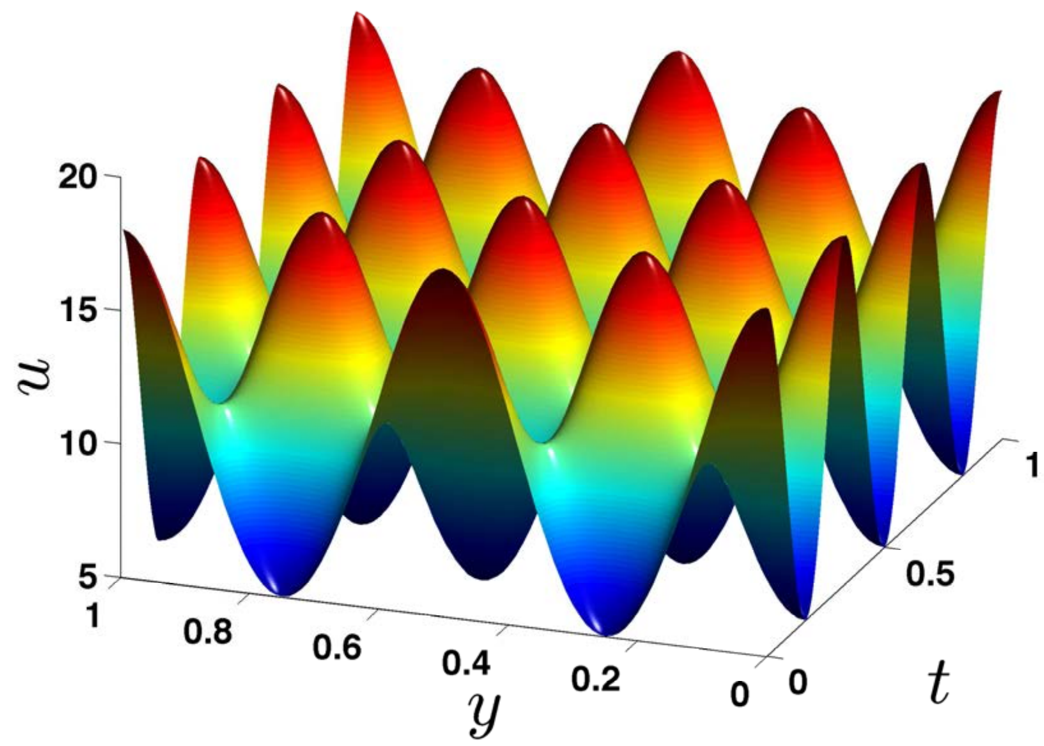
Marcio Gameiro
U. Sao Paulo

Roberto Castelli
VU Amsterdam

Periodic orbits in the ill-posed Boussinesq equation

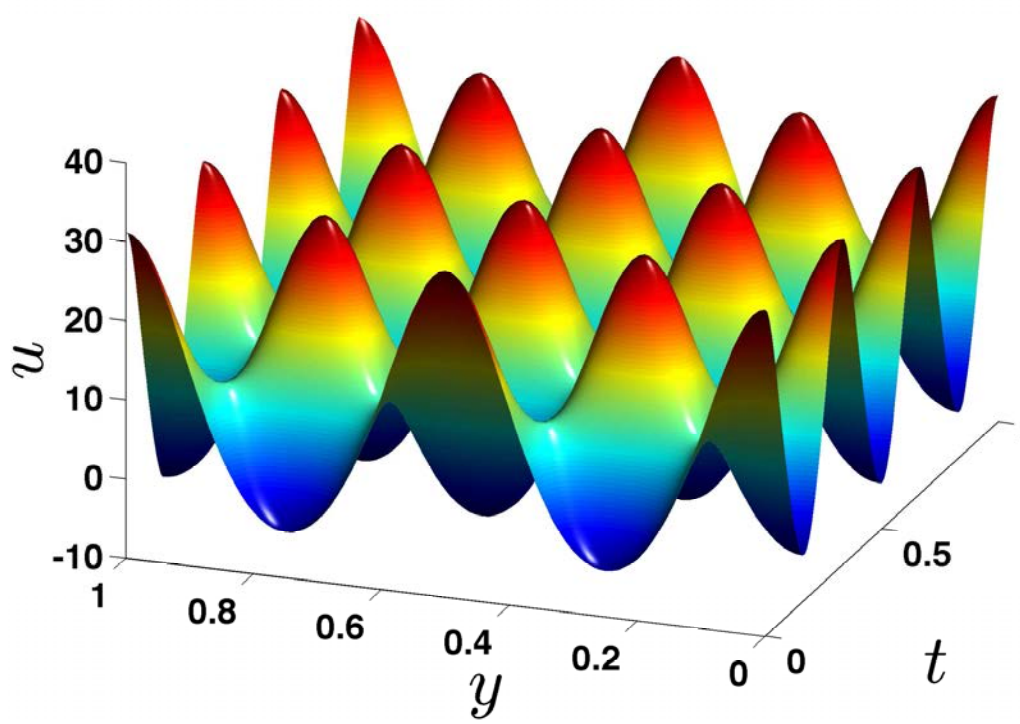


(a) Branch of solutions.



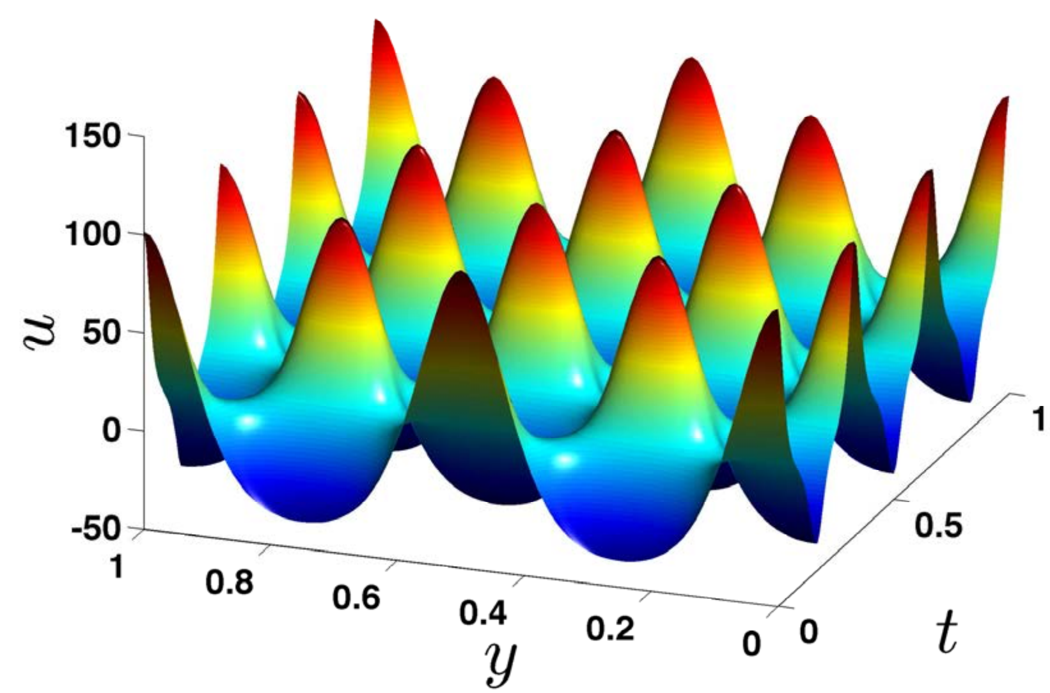
(b) $\lambda = 0.1356$

$m_1 = m_2 = 30,\ r = 2.16093 \times 10^{-10}$



(c) $\lambda = 0.1446$

$m_1 = m_2 = 61,\ r = 4.37571 \times 10^{-11}$

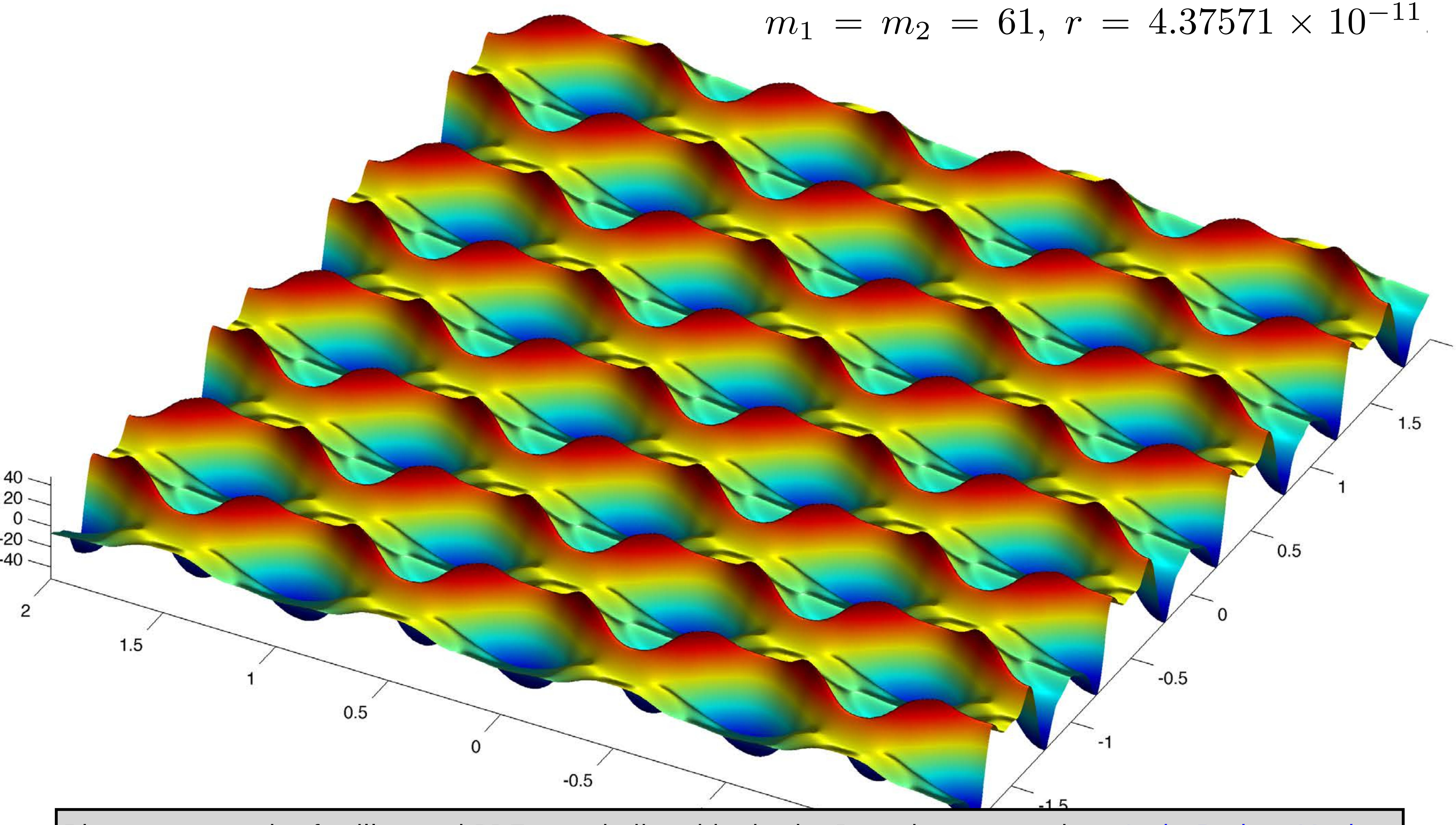


(d) $\lambda = 0.2146$

$m_1 = m_2 = 69,\ r = 3.03211 \times 10^{-4}$

Periodic orbits in the ill-posed Boussinesq equation

$m_1 = m_2 = 61,\ r = 4.37571 \times 10^{-11}$



Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation. *Arch. Ration. Mech. Anal.* 228 (2018), no. 1, 129–157.

Coexistence of hexagons and rolls

Joint work with

J.B. van den Berg
VU Amsterdam

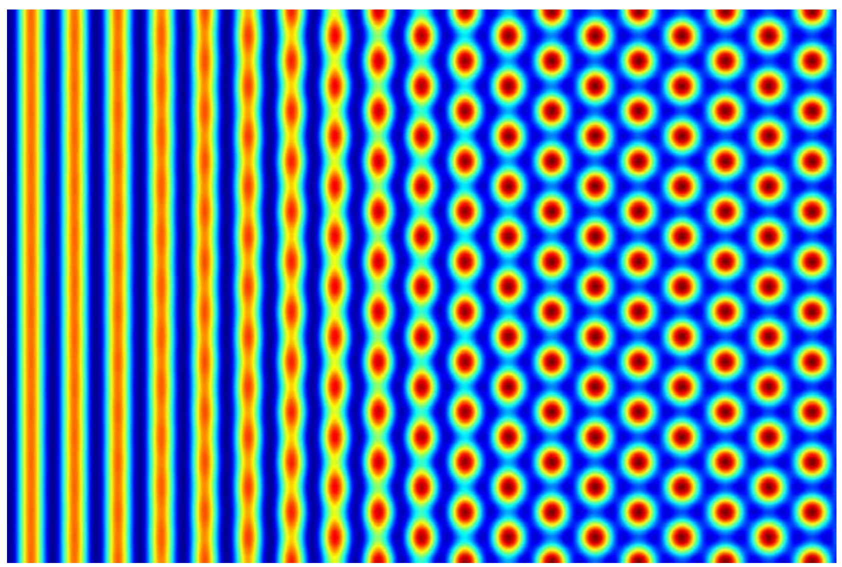
A. Deschênes
Laval

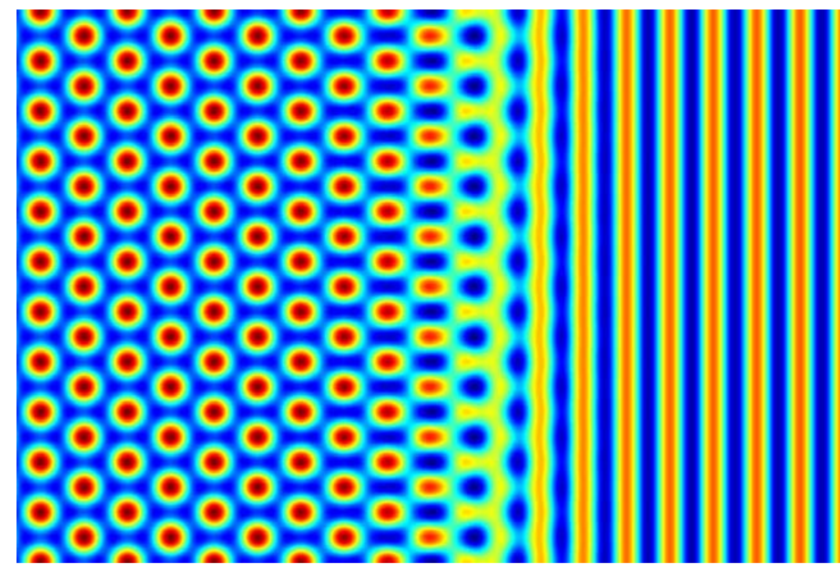
J.D. Mireles James
FAU

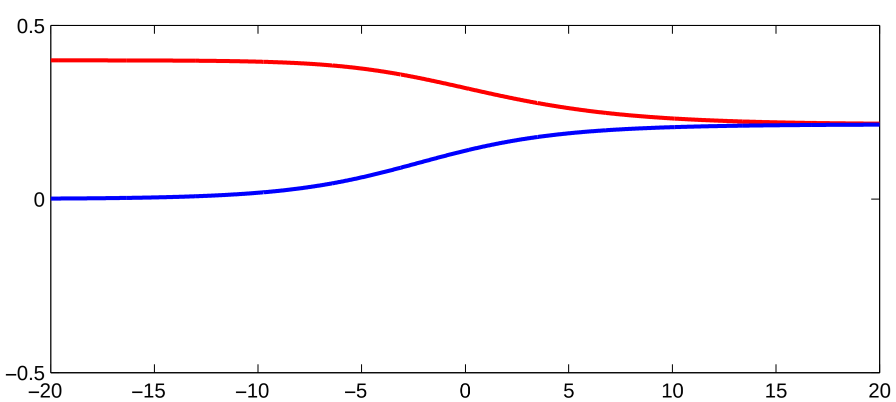
$$\partial_t \mathrm{U} = -(1+\Delta)^2\mathrm{U} + \mu\mathrm{U} - \beta|\nabla\mathrm{U}|^2 - \mathrm{U}^3$$

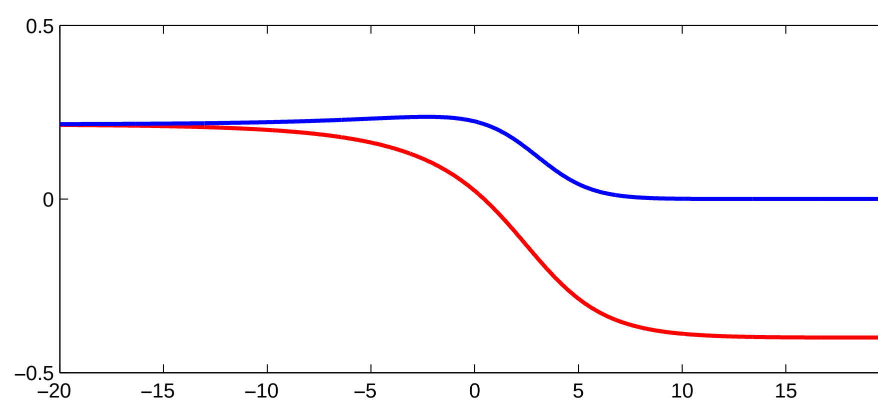
$$\mathrm{U} = \mathrm{U}(t,x) \in \mathbb{R},\ t \geq 0,\ x \in \mathbb{R}^2$$

This equation generalizes the Swift-Hohenberg equation and the additional term $\beta|\nabla\mathrm{U}|^2$, reminiscent of the Kuramoto-Sivashinsky equation, breaks the up-down symmetry $\mathrm{U} \mapsto -\mathrm{U}$ for $\beta \neq 0$.









Stationary coexistence of hexagons and rolls via rigorous computations. *SIAM J. Appl. Dyn. Syst.* 14 (2015), no. 2, 942–979.

Periodic orbits in the Mackey-Glass equation

Joint work with

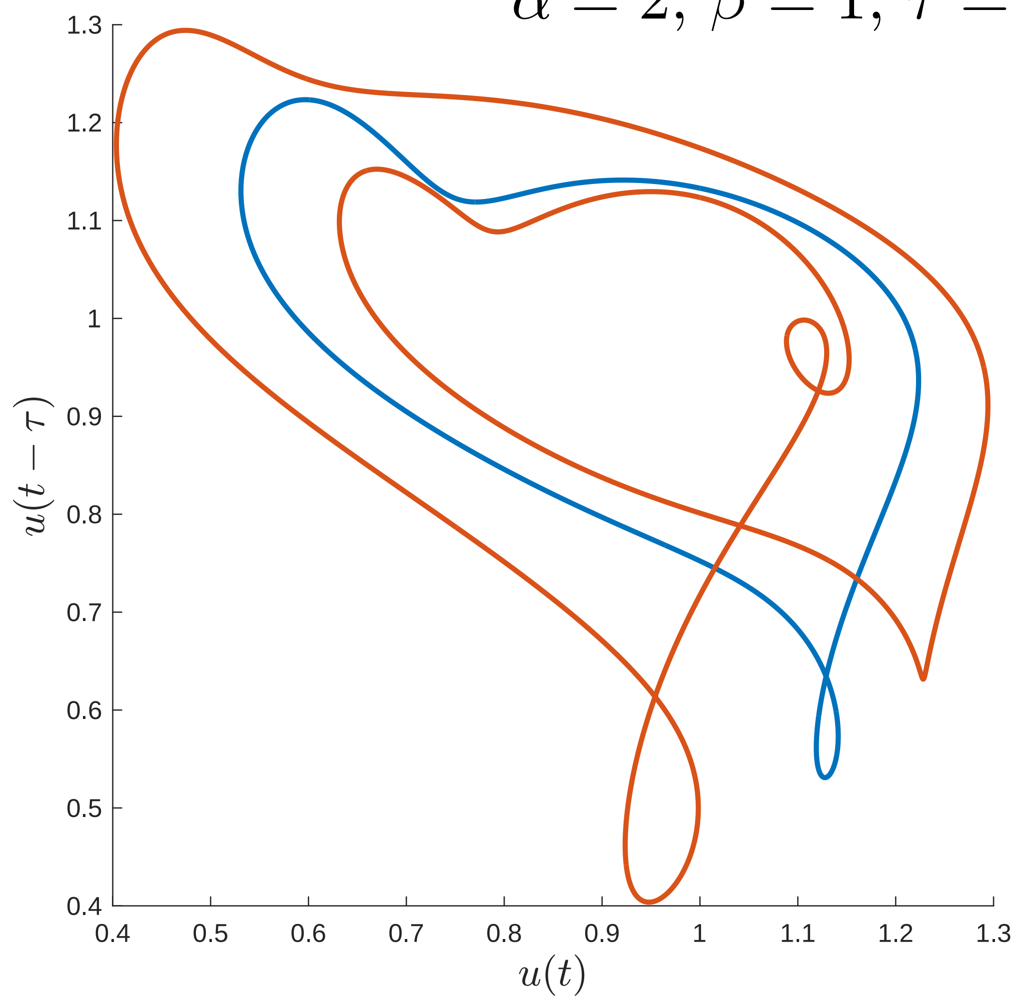
J.B. van den Berg
VU Amsterdam

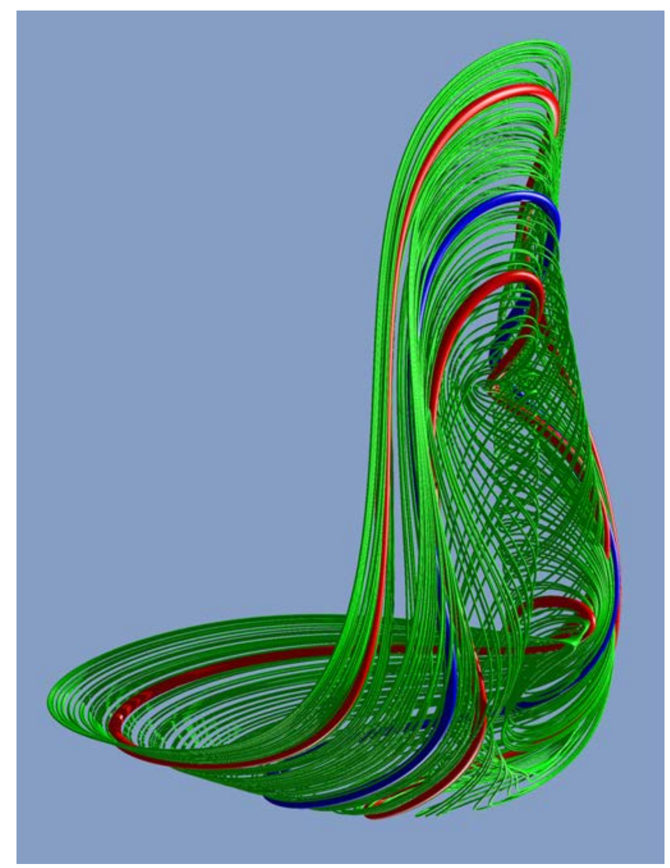
C. Groothedde
VU Amsterdam

$$u'(t) = -\beta u(t) + \alpha \frac{u(t-\tau)}{1 + u(t-\tau)^\rho}.$$

Models the concentration of white blood cells in a subject.

$\alpha = 2,\ \beta = 1,\ \tau = 2$ and $\rho = 9.65$





A general method for computer-assisted proofs of periodic solutions in delay differential problems. Journal of Dynamics and Differential Equations, 2021.

Torus-knot choreographies in the n-body problem

Definition: A *choreography* is a periodic solution of the gravitational n-body problem where n equal masses follow the same path.

Joint work with

R. Calleja
UNAM

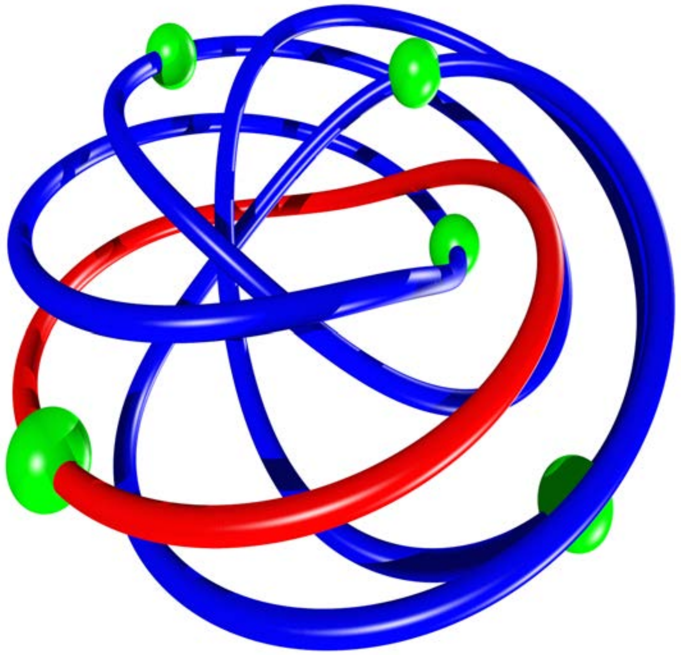
C. Garcia-Azpeitia
UNAM

J.D. Mireles James
FAU

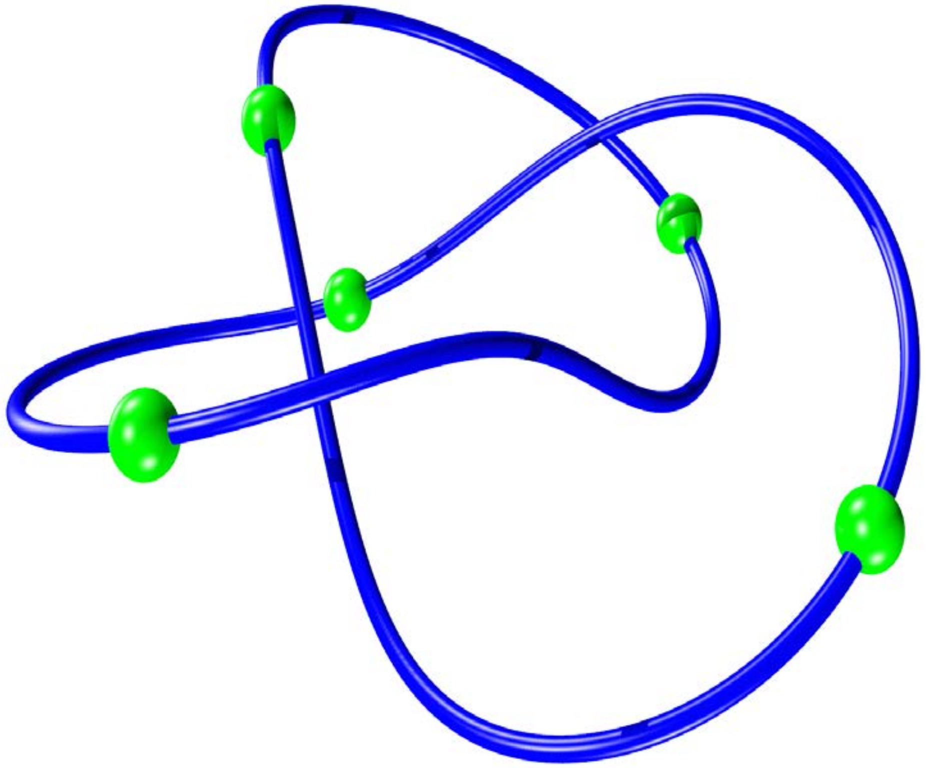
The equations for the generating body $u_n = (w,z) \in \mathbb{C}\times\mathbb{R}$ are reduced to the system of delay differential equations with multiple delays

$$\ddot{w}(t) + 2\sqrt{s_1}i\dot{w}(t) = s_1 w(t) - \sum_{j=1}^{n-1} \frac{w(t) - e^{ij\zeta}w(t+jk\zeta)}{\left(|w(t)-e^{ij\zeta}w(t+jk\zeta)|^2 + |z(t)-z(t+jk\zeta)|^2\right)^{3/2}}$$

$$\ddot{z}(t) = -\sum_{j=1}^{n-1} \frac{z(t)-z(t+jk\zeta)}{\left(|w(t)-e^{ij\zeta}w(t+jk\zeta)|^2 + |z(t)-z(t+jk\zeta)|^2\right)^{3/2}}.$$



Rotating coordinates



Inertial coordinates

Torus knot choreographies in the n-body problem. *Nonlinearity* 34 (2021), no. 1, 313–349.

Questions that guide the research in the field

- Understand the global dynamics of ODEs, PDEs and delay equations
- Compute rigorously compact invariant sets
- Develop computational tools for equilibria, periodic orbits, stable and unstable manifolds, homoclinic and heteroclinic orbits, solutions to BVP, travelling waves, fronts, radial solutions, invariant tori, etc
- Develop rigorous methods to study the stability of the above objects
- Obtain theorems about existence of symbolic dynamics
- Combine the rigorous computations with topology (e.g. Morse-Conley-Floer theory) to obtain forcing theorems
- Study energy landscapes / compute local minimizers of functionals
- Chaos / turbulence in infinite dimensional dynamical systems
- Develop tools to compute Morse-Floer homology


Thank you
McGill
UNIVERSITY
NSERC
CRSNG