The ultimate frontier of KAM theory (in finite dim.)

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joint work with LUCA BIASCO
-2015

$$H_\varepsilon = \frac{1}{2}|y|^2 + \varepsilon f(x) \qquad f \text{ analytic}$$

$y \in \mathbb{R}^d$
$x \in \mathbb{T}^d$

$$\boxed{\dot{y} = \omega = \partial_y H_0 .}$$

classical K.A.M. theory
1954 1963

more positive set of initial data $\rightsquigarrow$ quasi periodic motions $\varepsilon \neq 0$

Neishtadt, Pöschel

most of the phase-space is filled by Lagrangian (primary) invariant tori for $\phi^t_{H_\varepsilon}$, $\varepsilon < \bar{\varepsilon}$

Phase-space $\mathcal{M} = B_1 \times \mathbb{T}^d$ $\quad B_1 = \{ y \in \mathbb{R}^d \mid |y| < 1 \}$.

KAM $\exists$ a set of $\underline{\text{meas.}(\mathcal{N}) < \sqrt{\varepsilon}}$ s.t.

$\forall$ $(y,x) \in \mathcal{M} \setminus \mathcal{N}$

$\overline{\phi^t_{H_\varepsilon}(y,x)}$ = Lagr. inv. torus

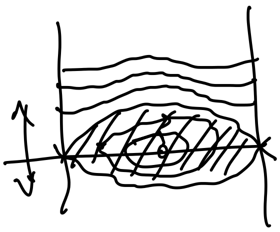
and the flow is conjugated to $\theta \to \theta + \underline{\omega} t$

ω is Diophantine $|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau}$ $\forall k \in \mathbb{Z}^d \setminus \{0\}$
$\tau, \gamma > 0$
$(\tau > d-1)$

If you consider Lagrangian graphs over $\mathbb{T}^d$ (primary tori) then the meas. estimate is optimal.

Trivial example, $d=1$ $\quad \frac{1}{2} y^2 + \varepsilon(\cos x - 1)$

$\sim \sqrt{\varepsilon}$



meas $\sim \sqrt{\varepsilon}$

Question: What is the measure of the set $\mathcal{M} \setminus \mathcal{L}_\varepsilon$? for generic potentials f

$\mathcal{L}_\varepsilon = \{ (y,x) \in \mathcal{M} \mid \phi^t_{H_\varepsilon}(y,x) \text{ span } d\text{-dim. Lagrangian tori with transitive flow} \}$.

Euristic argument.

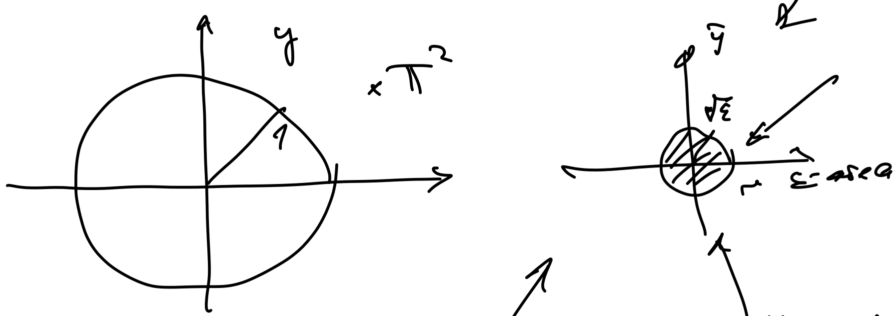
$d=2$ $\quad H = \frac{1}{2}(y_1^2+y_2^2) + \varepsilon f(x_1,x_2)$ $\quad (x_1$

$\to$ $y = \sqrt{\varepsilon}\,\bar y$ & divide the Hamilt. by $\frac{1}{\sqrt{\varepsilon}}$

$x = \bar x$ & rescaling of time you get

$$\bar H = \frac{1}{2}(\bar y_1^2 + \bar y_2^2) + f(\bar x_1, \bar x_2)$$

free of parameters !!



$d \geq 2$

corresponds in higher dim $d>2$ to neighb. of double resonances

$k \cdot y = 0 \quad k \in \mathbb{Z}^d,\ k \neq 0$

$k \cdot \omega$

$\{|(1,0)\cdot y| < \sqrt{\varepsilon}\} \cap \{|(0,1)\cdot y| < \sqrt{\varepsilon}\}$

One expects in this region to have $O(1)$ zones free of invariant Tori.

Arnold-Kozlov-Neishtadt 2006 - Ency. Springer.

Thm (Biasco-C) Let $d, s>0$. Let $f \in \mathbb{A}^u_s$ (is open and dense in $\{f$ r-analytic with $\|f\|_s = \sup_k |f_k| e^{|k|s} = 1\}$

$\exists\ \varepsilon_0, a > 0 \mid \forall\ 0<\varepsilon<\varepsilon_0$

$$\mathrm{meas}\,(\mathcal{M} \setminus \mathcal{K}_\varepsilon) \leq c\,\varepsilon\,|\log\varepsilon|^a .$$

all inv. lagr. Tori

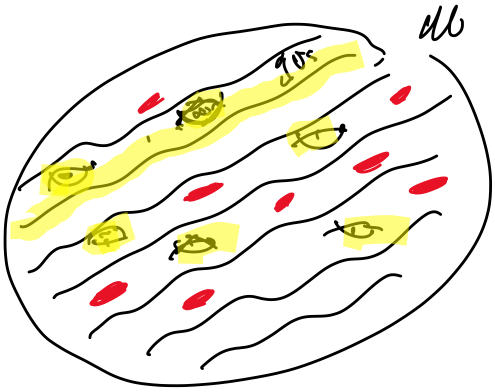
More precisely, $\exists$ a neigh $\mathcal{D}_\varepsilon$ of double resonances set

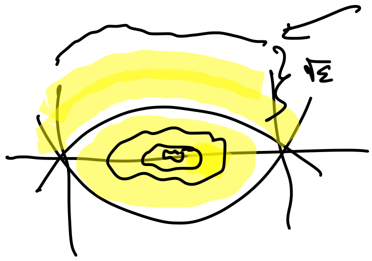
of meas $\varepsilon|\log\varepsilon|^a$

and an (almost) exponentially small set $\mathcal{E}_\varepsilon$ NON PERTURBATIVE SET

(meas $\mathcal{E} \leq \varepsilon^{|\log\varepsilon|}$) s.t. ...

$\left((B_1 \times \mathbb{T}^n) \setminus \mathcal{N}_\varepsilon \right) \setminus \mathcal{E}_\varepsilon$ is filled by KAM tori with flow conj to $\theta \to \theta + \omega t$ with ω almost-Liouville $|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau}$ $\gamma \sim e^{-\frac{1}{\varepsilon^b}}$.





The class $\mathbb{P}^n_s$

$n = d$

$\mathbb{P}^n_s \; \{ f : \|f\|_s = 1 \;|\;$ (i) $\lim_{\substack{|k| \to +\infty \\ k \in \mathcal{G}^n}} |f_k| \, e^{|k| s} \, |k|^n > 0$

(ii) $\forall \; k \in \mathcal{G}^n$

generators of maximal 1-d lattices in $\mathbb{Z}^d$

m.c.d.$(k_j) = 1$

$$(\pi_{\mathbb{Z}k} f)(\theta) = \sum_{j \in \mathbb{Z}} f_{jk} \, e^{ij\theta}$$

is Morse with different critical values

Remark. $\mathcal{E}_\varepsilon$ $\qquad \mathcal{N}_\varepsilon \subseteq \bar{\mathcal{N}}_\varepsilon$ any meas $\bar{\mathcal{N}}_\varepsilon = \varepsilon^b$ any $b < 1$.

$$\Rightarrow \quad \text{meas}(\mathcal{E}_\varepsilon) \leq e^{-\frac{1}{\varepsilon^c}} \quad \text{some } c.$$

On $\mathcal{N}_\varepsilon$ — Step 1

$$y \in B_1^n = D_0 \,\dot\cup\, D_1 \,\dot\cup\, D_2$$

let $K := (\log \varepsilon)^2$

$\alpha := \sqrt{\varepsilon} \, K^\nu_1$ $\left(\nu = \frac{9}{2}n + 2\right)$

$$D_0 = \{ y \in B_1 \;|\; \min_{k \in \mathcal{G}^n} |y \cdot k| \geq \alpha \}.$$

$$= \qquad \cdots \, g_{1,K} \atop |k| \le K,$$

$$D_1 = \bigcup_{k \in \mathcal{G}^n_{1,K}} \overbrace{\{ y \in B_1 \mid \underline{|y \cdot k| < \alpha} \quad \& \quad \text{no other indep. resonance up to order } \underline{6}K \}}^{D_1^k}$$

index of simple resonances.

$|k| \le (\log \varepsilon)^2$

(related Nondinearity 2020)

$$\mathcal{N}_\varepsilon := D_2 = B_1 \setminus (D_0 \cup D_1)$$

low resonant, simple resonant.

Easy : $\text{meas } \mathcal{N}_\varepsilon \le c \, \varepsilon K^c = \varepsilon (\log \varepsilon)^a$

non-pert. fct.

Easy also : apply KAM to D_0 $\rightsquigarrow$ primary tori in $D_0 \times \mathbb{T}^n$ up to an (almost)-exp. small fct.

Dynamics in $\underline{D_1 \times \mathbb{T}^d}$?

$\forall \; k \in \mathcal{G}^n_{1,K}$ can do (refined) averaging theory:

$$\Psi_k : D_1^k \times \mathbb{T}^d \circlearrowleft$$

$$H_\varepsilon \circ \Psi_k = \boxed{\frac{|y|^2}{2} + \varepsilon \left(g_0^k(y) + g^k(y, \underbrace{k \cdot x}_{\theta}) \right.} + \left. f^k(y,x) \right)$$

$$|g_0^k| \, , \, |g^k - \pi_{\mathbb{Z}k} f| \le \frac{1}{|\log \varepsilon|^a} = \frac{1}{K^a}$$

$$|f^k| \le e^{-2Ks}$$

$\theta \in \mathbb{T}^1$.

Secular Hamiltonian is a 1-deg. of freedom system with parameter (adiabatic actions).

Strategy is obvious : • integrate the secular system $\to$ (Arnold-Liouville)

- check Kolmogorov non-deg. of the new int. part.
- apply KAM.

conjugate to

$\forall \quad |k| \le K = (\log \varepsilon)^2$

standard pendulum-like Hamilt.

$$\boxed{(1+\mu(p_1,q_1))\left(P_1 - \bar{P}_1(\hat{p})\right)^2 + G_0(\hat{p}) + G(\hat{p}, q_1)}$$

$\hat{p} = (P_2, \dots, P_n)$

(p_1, q_1) $\qquad p = (P_1, \hat{p}) = (P_1, P_2, \dots, P_n)$

① → topological obstructions.

② → (*) Kolm. non-deg. is false (use analyticity)

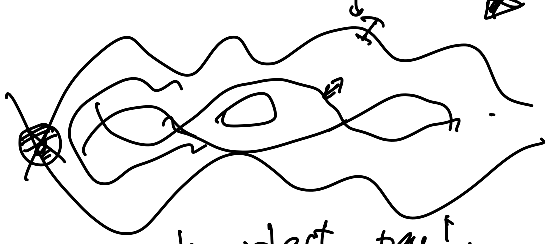
→ uniform application of KAM.

$$\boxed{K_\varepsilon \ge |k|} > N(f) \ge 1$$

scalar part $\quad \boxed{g^k = \cos(\theta + \theta^k) + F^k}$

$|F^k| < \frac{1}{8}$

$1 \le |k| \le N(f)$.
indep. of ε



hardest part.

$$\boxed{I(\lambda) \sim a(\lambda) + b(\lambda)\, \lambda \log \lambda}$$

analytic.