Dirac structures for infinite dimensional geometries and applications

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Plan:

Intro and definitions
infinite dimensions and reduction
Applications

Intro

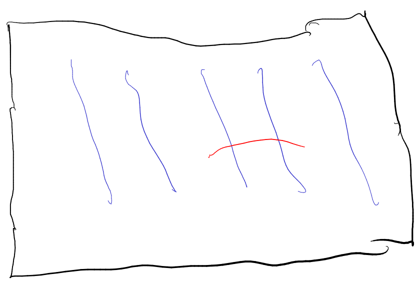
brackets and constraints

(structure ?) $M \hookrightarrow P$ symplectic / Poisson

Dirac ($\sim$ 50): "The Hamiltonian method"

Ex: M coisotropic (1st class)

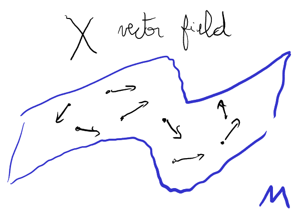
M



Question: structure on M leading to "Hamiltonian" type description

Def: Relations between vectors and covectors

Idea: Poisson brackets / sympl. structures provide a way of relating ODE's and functions.



$\iota_X \omega = dH$ $\qquad$ H function

$X = \{H, \cdot\}$

Key properties:

- infinitesimal nature $\qquad X|_x \sim dH|_x$ at each point x
- skew symmetry $\qquad$ (conservation of energy)
- integrability $\left(\begin{matrix} d\omega = 0 \\ \text{Jacobi } \{,\} \end{matrix} \right)$ $\qquad$ (conservation of area / volume, involutivity of conserved quantities)

Recall: functions $f: A \to B$, $a \mapsto b$ / relations $R \subset A \times B$, $a \sim b$, $f \rightsquigarrow R = g(f)$

(Partial) Def: A Dirac structure on M is a relation $\sim_x$ for each $x \in M$ between vectors and covectors at x,

$$L_x \subseteq T_x M \times T_x^* M$$

such that:

(infinitesimal nature) • L_x is linear subspace $T_x M \oplus T_x^* M$

Regularity: $\coprod_x L_x$ defines smooth subbundle of $TM \oplus T^*M$

(skew symmetry) • if $X \sim_x \alpha \Longrightarrow \alpha(X) = 0$

equiv: $L_x \subseteq L_x^\perp$, for $\langle (X,\alpha), (Y,\beta) \rangle = \alpha(Y) + \beta(X)$ on $TM \oplus T^*M$

• $\dim(L_x) = \dim M \quad \forall x$

equiv: $L_x = L_x^\perp$

Integrability: a background bracket

vector fields : Lie bracket $X, Y \mapsto [X, Y]$

forms : de Rham d, Lie derivatives, contraction

T. Courant
Weinstein
Dorfman

$$(X, \alpha) \in \mathfrak{X}(M) \oplus \Omega^1(M) \ni (Y, \beta)$$

$$[\![(X,\alpha), (Y,\beta)]\!] = \left([X,Y], \mathcal{L}_X \beta - \iota_Y d\alpha \right)$$

Rmk: not Lie!

$\hookrightarrow$ $(TM \oplus T^*M, \langle\,\rangle, [\![\,,\,]\!])$

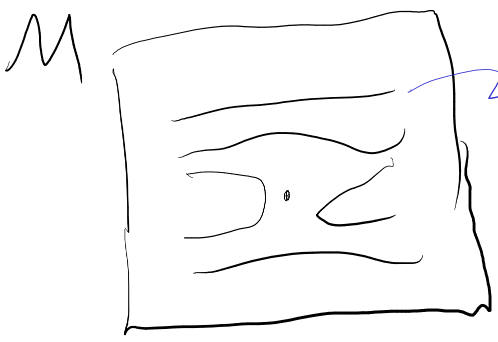
"Courant algebroid"

Last "integrability" axiom for L:

- sections of $L \to M$ are closed under $[\![\,,\,]\!]$

Underlying geometry: foliation with presymplectic forms

L inherits "Lie algebroid" structure



$\to$ singular foliation

each leaf $\mathcal{O} \hookrightarrow M$ inherits a 2-form $\omega_{\mathcal{O}}$

$d\omega_{\mathcal{O}} = 0$

Examples

old and new

- (pre) symplectic : $X \sim_x \alpha \iff \iota_X \omega = \alpha$
- Poisson $\left(\{f,g\} = \pi(df, dg)\right)$: $X \sim_x \alpha \iff X = \pi(\alpha, \cdot)$

the relation is a graph

Rmk: - integrability axiom reduces to $d\omega = 0$ $\wedge$ Jacobi for $\{,\}$

- $L = \mathrm{gr}(\omega^b : TM \to T^*M) \iff L \cap T^*M = 0$

$L = \mathrm{gr}(\pi^\# : T^*M \to TM) \iff L \cap TM = 0$

- $D \hookrightarrow TM$ distribution : $L = D \oplus D^0$ $\quad (X \sim \alpha \iff X \in D \wedge \alpha|_D = 0)$

integrability of L $\iff$ involutivity of D

(new)
- Cartan-Dirac structure : [Alekseev, Severa, Strobl ~ 90's]

$M = H$ Lie group, $\mathfrak{h} = \mathrm{Lie}(H)$ given "$\cdot$" Ad-invariant symmetric non-deg.

$X \sim \alpha \iff \exists \xi \in \mathfrak{g}, \; X = \xi^R - \xi^L \wedge \alpha = \frac{1}{2}(\theta^L + \theta^R) \cdot \xi$

$\hookrightarrow$ infinitesimal conjugacy action

Maurer-Cartan

Important: not "integrable" but "twisted-integrable"

with twist given by $\phi \in \Omega^3(G)$ Cartan 3-form

$\phi = \frac{1}{12} k(\theta^L, [\theta^L, \theta^L])$

[Severa; Strobl, Weinstein, ...]

twists : $[\![(X,\alpha),(Y,\beta)]\!]_\phi = \left([X,Y], L_X\beta - \iota_Y d\alpha + \iota_X \iota_Y \phi\right)$, $\quad d\omega_0 = i_0^* H$

- In general, $M \xhookrightarrow{i} (P, L_P) \rightsquigarrow$ induces L on M

$X \sim_x \alpha \iff \exists \hat{\alpha} \in \Omega^1(P), \; \alpha = i^* \hat{\alpha} \wedge D_x i(X) \underset{i(x)}{\sim} \hat{\alpha}$

Care : smoothness $x \mapsto L_x$ $\rightsquigarrow$ L Dirac on M ("Pull back")

Uses — dynamics and brackets

$$A\dot{x} = f(x)$$

Dynamics: given L, $H : M \to \mathbb{R}$

$x(t) \in M$, $\left[\ \dot{x}(t) \overset{L}{\sim} dH\big|_{x(t)}\ \right]$

$\rightsquigarrow$ Rmk: implicit differential equations of "Hamiltonian" type

↳ [Cendra et al]

Rmk: conservation of energy

skew sym axiom

$\dot{x} \sim dH$

$$\frac{d}{dt}\left(H(x(t))\right) = dH(\dot{x}) = 0$$

Rmk: $x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}$

restriction to leaves

Brackets: $f : M \to \mathbb{R}$ called "admissible" for L if $\boxed{\exists\, X_f : X_f\big|_x \overset{L}{\sim} df\big|_x,\ \forall x \in M}$

f, g admissible $\rightsquigarrow$ $\boxed{\{f, g\} := \mathcal{L}_{X_f}\, g}$ $\rightsquigarrow$ properties of a Poisson bracket!

Infinite dimensional structures

motivation and examples

$\mathcal{H}$ Hilbert space $\left(\mathcal{H}^* \underset{I}{\simeq} \mathcal{H} \right)$

$\partial : D \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ unbounded operator

∂ is <u>closed</u> $\iff$ $\underset{\{(v, I\partial v) : v \in D\}}{\underset{\shortparallel}{gr(\partial)}} \subseteq \mathcal{H} \oplus \mathcal{H}^*$ closed subspace

∂ is skew adjoint $\iff$ $L = gr(\partial) \subseteq \mathcal{H} \oplus \mathcal{H}^*$ satisfies

$gr(\partial)^\perp = gr(\partial)$ with respect to $\langle , \rangle$

Idea: $\mathcal{H} \rightsquigarrow T_x M$, M ∞ dim (Hilbert) manifold

$(x \mapsto \partial_x)$ geometric structure encoded as Dirac st. on M

Technical comments

- The definition of Dirac str. on Hilbert mfld M is analogous

- The involutive distribution underlying L might not be integrable

  when it is, each $\mathcal{O} \hookrightarrow L$ leaf carries 2-form $\omega_{\mathcal{O}}$

Def: a <u>weak Poisson structure</u> on M is a Dirac structure $L \subset TM \oplus T^*M$ such that $L \cap TM = 0$

Rmk: if $\dim M < \infty \Rightarrow L + TM = TM \oplus T^*M$

$\Rightarrow \exists! \, \pi \in \mathfrak{X}^2(M) : L = \mathrm{gr}(\pi)$

if $\dim M = \infty$, may be $L + TM \subsetneq TM \oplus T^*M$

If $\mathcal{O}$ is a leaf of L weak Poisson $\Rightarrow$ $\omega_{\mathcal{O}}$ is weak symplectic str. on $\mathcal{O}$

($\omega_{\mathcal{O}}^\flat : T\mathcal{O} \to T^*\mathcal{O}$ injective)

Rmk: in examples with M ∞-dim, one has a candidate for Poisson $\{F, G\}$ (def by $x \mapsto \partial_x$) but is not defined $\forall f, g$ on M! $\rightsquigarrow$ $\boxed{\text{best seen as Dirac st.}}$

(H. Bursztyn)

Example

$$M = \{ x : [0,1] \to \mathbb{R}^n \text{ class } L^2 \}$$

or Sobolev H_r, $r \geq 0$

[Le Gorrec, Zwart, Maschke, 2005]
related to boundary control problems

$T_x M \simeq M$, $T^*_x M \hookleftarrow M$

$F_x := \int_I x \cdot - \longleftarrow\!\shortmid\, x$

$$L_x = \{ (v', F_v) \in T_x M \oplus T^*_x M : v : I \to \mathbb{R}^n \text{ class } H^1,\ v(0) = v(1) \}$$

$\Rightarrow$ L defines a weak Poisson str on M

Rmks : • at each x, the same operator $\partial v = v'$, $L_x \simeq gr(\partial)\ \forall x$

• $L \cap TM = 0$ but $L + TM \neq TM \oplus T^*M$
($H^1 \subseteq L^2$ dense but $\neq$) $\rightsquigarrow$ no bivector !

• linear functional $F_y(x) = \int_I y \cdot x$ param. by $y \in H^1(S^1, \mathbb{R}^n)$ is admissible for L and

$$\{F_x, F_y\} = F_{x \cdot y'} = -F_{x' \cdot y}$$

• Leaves of L : affine fibers of $M \longrightarrow \mathbb{R}$
$x \longmapsto \int_I x$

Formally: $\{,\}$ is induced by dual of Lie algebra
$H^1(S^1, \mathbb{R}^n) \oplus \mathbb{R} =: \widehat{Lh}$

$$[(x,1),(y,1)] = \left(0, \int_{S^1} x' \cdot y\right)$$

$M \simeq (\widehat{Lh})^*$, h abelian Lie alg.

Reduction

similar to coisotropic reduction in symplectic geom

Rmk: geometry vs supergeometry
[Roytenberg]

Pointwise $\quad C_x \subseteq T_xM \oplus T_x^*M$

coisotropic for $\langle , \rangle$: $C_x \supseteq C_x^\perp$

$\Rightarrow$ $\boxed{\hat{A}_x := C_x / C_x^\perp \text{ inherits } \langle , \rangle}$ ("reduced Courant algebroid")

if $L_x \subseteq T_xM \oplus T_x^*M$ "linear" Dirac and (∞ dim) $L_x + C_x$ closed

$\Rightarrow$ $\boxed{\tilde{L}_x = \dfrac{L_x \cap C_x}{L_x \cap C_x^\perp} \subseteq \hat{A}_x \text{ linear Dirac}}$

Thm 1 (AC, Gualtieri, Meinrenken): let $M \longrightarrow M/G$ principal bundle

and $\boxed{\begin{array}{l} \mathfrak{g} = \mathrm{Lie}\, G \xrightarrow{\rho} \mathfrak{X}(M) \oplus \Omega^1(M) \\ \xi \longmapsto (\xi_M, \alpha_\xi) \end{array}}$ "extended action" [Bursztyn Cavalcanti Gualtieri]

s.t. $\langle \rho(\xi), \rho(\xi) \rangle = 0 \ \wedge$ preserves brackets $([\![,]\!])$

Then:

- $C = \rho(M \times \mathfrak{g})^\perp$ coisotropic and $\hat{A} \to M$ is G-basic inducing

$\boxed{A_{red} := \hat{A}/G \to M/G = \tilde{M}}$ "reduced Courant algd"

- A_{red} inherits a unique Courant algd str. (s.t. $q_{red}: A \dashrightarrow A_{red}$ morph)

and $A_{red} \simeq T\tilde{M} \oplus T^*\tilde{M}$ as "exact" Courant algebroids (possibly twisted $\phi \in \Omega^3_{cl}(\tilde{M})$)

- if L Dirac on M and $L + C$ is closed

$\Rightarrow$ $\boxed{L_{red} := \tilde{L}/G \text{ defines a Dirac str. on } \tilde{M}}$

Ex: suff. $L = gr(\pi)$ Poisson is G-invariant $\Rightarrow$ $\tilde{\pi}$ Poisson on M/G

$q: M \to M/G$
$q^*\tilde{\pi}(df, dg) := \pi(q^*df, q^*dg)$

Choose trivial extension $\rho(\xi) = (\xi_M, 0) \rightsquigarrow A_{red} = T\tilde{M} \oplus T^*\tilde{M} \quad (H = 0)$

$[gr(\pi)]_{red} = gr(\tilde{\pi})$

Morphisms and reduction

Def: a morphism $R : \overbrace{TM_1 \oplus T^*M_1}^{\mathbb{A}_1} \dashrightarrow \overbrace{TM_2 \oplus T^*M_2}^{\mathbb{A}_2}$

is a Dirac structure $R \subset \overline{\mathbb{A}}_1 \times \mathbb{A}_2$ with support in $gr(\varphi : M_1 \to M_2)$

$\overline{\mathbb{A}}_1$: $-\langle , \rangle$

(proj R to $TM_1 \times TM_2$ contained $Tgr(\varphi)$)

composition: composit. of relations + (as dim) transversality condition

Ex: "full" morphisms $R \equiv R_{\varphi,\omega}$

projection of R to $T(gr\varphi)$ is onto $\Longleftrightarrow$ $\exists\ \omega \in \Omega^2(M_1)$:

$(X_1, \alpha_1) \overset{R}{\sim} (X_2, \alpha_2) \Leftrightarrow X_2 = D\varphi(X_1) \wedge \alpha_1 = \varphi^*\alpha_2 + i_{X_1}\omega$

$(\mathbb{A}, L)$ "Manin Pairs"

R can generalize moment maps

[Burztyn, Iglesias, Severa]

Def morph. of pairs $(\mathbb{A}_1, L_1) \overset{R}{\dashrightarrow} (\mathbb{A}_2, L_2)$

s.t.: $L_2\big|_{\phi(x_1)} = R\big|_{(x_1,\phi(x_1))} \circ L_1\big|_{x_1}$

and this composition is transverse

For L_2 = Cartan-Dirac,

$R : (TM \oplus T^*M, TM) \dashrightarrow (\mathbb{A}_H, L_H)$

which is full recovers

quasi-Hamiltonian spaces

[Alekseev, Malkin, Meinrenken]

Thm 2 [CGM] if $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ morph. and we have

extended actions on $\mathbb{A}_j$ which are R-related $\Longrightarrow$ R can be reduced so that

$$\begin{array}{ccc} \mathbb{A}_1 & \overset{R}{\dashrightarrow} & \mathbb{A}_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ \mathbb{A}_{1,red} & \underset{R_{red}}{\dashrightarrow} & \mathbb{A}_{2,red} \end{array}$$

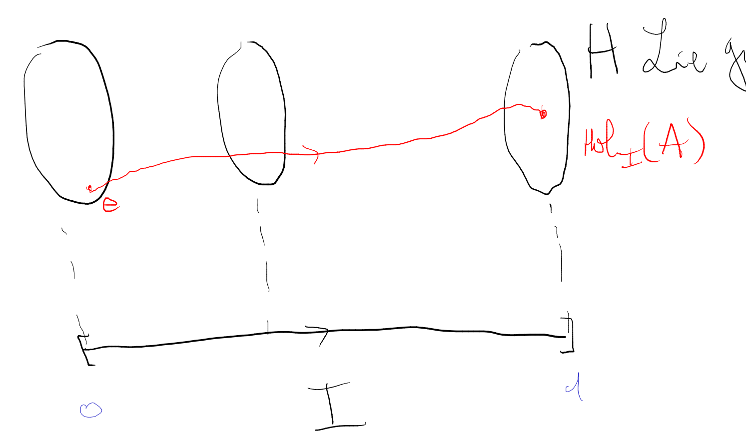
commutes.

Moreover, if R was morph. of pairs $(\mathbb{A}_1, L_1) \dashrightarrow (\mathbb{A}_2, L_2)$ $\wedge$ [extra conditions]

$\Rightarrow$ R_{red} is a morph of pairs $(\mathbb{A}_{1,red}, L_{1,red}) \dashrightarrow (\mathbb{A}_{2,red}, L_{2,red})$

An application

gauge theoretic constructions in dim 1 and 2



$H \times I \to I$

$\mathcal{M} = \mathcal{A}$ = space of principal connections $\simeq \Omega^1_{H_{r\geq 0}}(I, \mathfrak{h})$

$\mathfrak{h} = \mathrm{Lie}\, H$ has invariant "$\cdot$"

$$\mathcal{A} \xrightarrow{Hol_I} H \qquad \text{Holonomy fibration}$$

$\mathcal{A} \ni x \equiv A \rightsquigarrow \partial_x \equiv \partial_A : \Omega^0_{H_{r+1}}(I, \mathfrak{h}) \longrightarrow T_x\mathcal{A}$

$v \longmapsto dv + ad_A v$

infinitesimal gauge transf. of A by v

Def $L_x = \{ (\partial_x v, F_v) : v \in \Omega^0_{H_{r+1}}(I, \mathfrak{h}),\ v(0) = v(1) \} \subset T_x\mathcal{A} \oplus T^*_x\mathcal{A}$

Prop: L defines a Dirac structure on $\mathcal{A}$ which is weak-Poisson

Rmks:
- admissible $\{,\}$ reproduces linear Poisson $\mathcal{A} \subseteq (L\mathfrak{h})^*$ $\rightsquigarrow$ Domain $\{,\}$ does not include functions of Holonomy
- $\{,\}$ is gauge invariant, if heuristically push forward $\mathcal{A} \to H$ you get a bracket not Jacobi! [Alekseev, Kosmann-S, Meinrenken]
- leaves of L are orbits of gauge action
  $\omega_{\mathcal{O}}|_A (v_{\mathcal{O}}, v'_{\mathcal{O}}) = \int_I v \cdot \partial_A v'$
- other boundary conditions $B \subseteq \mathfrak{h} \oplus \bar{\mathfrak{h}}$ lagrangian sub-Lie alg $\rightsquigarrow L_B$ weak Poisson on $\mathcal{A}$

Reduction: from infinite dimensional affine to finite dimensional non-linear

$$G := \underset{H_{r+1}}{\mathrm{Map}}(I, H)_{\partial I} \circlearrowright \mathcal{A} \longrightarrow H$$

trivial on ∂I

extension: $\rho(\xi)|_A = (\partial_A \xi, F_\xi)$ $\qquad F_\xi(y) = \int_I \xi \cdot y$

trivial on ∂I: $\xi(0) = \xi(1) = 0$

$\Longrightarrow \quad C^\perp = \{ (\partial_A v, F_v) : v \in \Omega^1_{H_{r+1}}(I, \mathfrak{h}) \}$

no condition on $v|_{\partial I}$

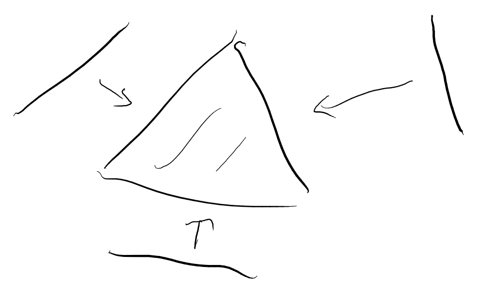
Thm 3 (CGM): reduction by the extended action along holonomy fibration yields

- $(T\mathcal{A} \oplus T^*\mathcal{A})_{red} \simeq TH \oplus T^*H$ twisted by Cartan 3-form $\phi \in \Omega^3(H)$
- $\boxed{L_{red} \simeq \text{Cartan-Dirac}}$
- (red of morphisms) $\rightsquigarrow$ 1:1 correspondence between hamiltonian spaces for L (Hamiltonian loop group LH-spaces) and hamiltonian spaces for Cartan-Dirac
  $\hookrightarrow$ q Ham spaces [Alekseev, Malkin, Meinrenken]

Surfaces

$(\mathcal{M}_\Sigma \to \mathcal{A}, \omega_\Sigma) \rightsquigarrow$ Hamiltonian space for L $\rightsquigarrow$ reduction $\rightsquigarrow$ q Ham spaces from Σ

Triangle



$(\mathcal{M}_{\Delta^2} \to \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \omega_{\Delta^2})$

$\xrightarrow{\text{reduction}}$ multiplicative properties of Cartan-Dirac

Possible Applications

"Hamiltonian" discretization of continuous systems

Continuous system : ∞ - dim phase space $(M, \underbrace{\text{sympl./Poisson}}_{L})$ + Energy functional

"discretization" as a quotient

idea: quotient by ∞ dim transformations preserving finite dim data (eg: lattice)

$(\tilde{M}, L_{red})$ finite dim discretized model

also discretizing energy function should get a "Hamiltonian" discretization

eg: conservation of energy, restriction to leaves, ...

Thank you!