From geometry to dynamics: Integrable bilinear discretization of quadratic vector fields

Yuri B. Suris

(Technische Universität Berlin)

Seminar “Geometry, Dynamics and Mechanics”
09.11.2021

Discretization in
Geometry and Dynamics
SFB Transregio 109

Part 1. Generalities

The problem of integrable discretization. Hamiltonian approach (Birkhäuser, 2003)

Consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\} \tag{1}$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot, \cdot\}$. Thus, flow (1) possesses sufficiently many functionally independent integrals $I_k(x)$ in involution.

The *problem of integrable discretization*: find a family of diffeomorphisms $\mathcal{P} \to \mathcal{P}$,

$$\widetilde{x} = \Phi(x; \epsilon), \tag{2}$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

1. The maps (2) *approximate* the flow (1):

$$\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2).$$

2. The maps (2) are *Poisson* w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
3. The maps (2) are *integrable*, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

While integrable lattice systems (like Toda or Volterra lattices) can be discretized in a systematic way (based, e.g., on the r-matrix structure), there is no systematic way to obtain *decent* integrable discretizations for integrable systems of classical mechanics.

Missing in the book: Hirota-Kimura discretizations

- R.Hirota, K.Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K.Kimura, R.Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Reasons for this omission: discretization of the Euler top seemed to be an isolated curiosity; discretization of the Lagrange top seemed to be completely incomprehensible, if not even wrong.

Renewed interest stimulated by a talk by T. Ratiu at the Oberwolfach Workshop “Geometric Integration”, March 2006, who claimed that HK-type discretizations for the Clebsch system and for the Kovalevsky top are also integrable.

Hirota-Kimura or Kahan?

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

$$\dot{x} = Q(x) + Bx + c \quad \rightsquigarrow \quad (\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x})/2 + c,$$

where $B \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* form, and $Q(x, \widetilde{x}) = (Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x}))/2$ is the corresponding symmetric *bilinear* function. Thus,

$$\dot{x}_k \rightsquigarrow (\widetilde{x}_k - x_k)/\epsilon, \quad x_k^2 \rightsquigarrow x_k \widetilde{x}_k, \quad x_j x_k \rightsquigarrow (x_j \widetilde{x}_k + \widetilde{x}_j x_k)/2.$$

Linear w.r.t. $\widetilde{x}$, therefore defines a *rational* map $\widetilde{x} = \Phi_f(x, \epsilon)$. Obvious symmetry: $x \leftrightarrow \widetilde{x}$, $\epsilon \mapsto -\epsilon$, therefore Φ_f *reversible*:

$$\Phi_f^{-1}(x, \epsilon) = \Phi_f(x, -\epsilon).$$

In particular, Φ_f is *birational*, and $\deg \Phi_f = \deg \Phi_f^{-1} = n$.

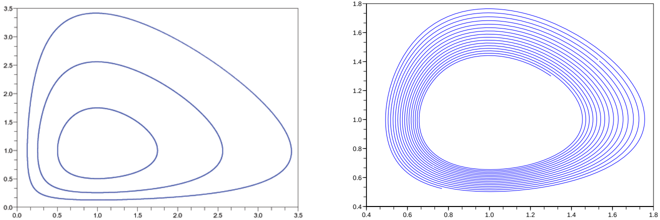
Illustration: Lotka-Volterra system

Kahan's discretization for the Lotka-Volterra system:

$$\begin{cases} \dot{x} = x(1-y), \\ \dot{y} = y(x-1), \end{cases} \quad \rightsquigarrow \quad \begin{cases} \widetilde{x} - x = \epsilon(\widetilde{x} + x) - \epsilon(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = \epsilon(\widetilde{x}y + x\widetilde{y}) - \epsilon(\widetilde{y} + y). \end{cases}$$

Explicitly:

$$\begin{cases} \widetilde{x} = x\, \dfrac{(1+\epsilon)^2 - \epsilon(1+\epsilon)x - \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}, \\[2ex] \widetilde{y} = y\, \dfrac{(1-\epsilon)^2 + \epsilon(1+\epsilon)x + \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}. \end{cases}$$



Left: three orbits of Kahan's discretization with $\epsilon = 0.1$,
right: one orbit of the explicit Euler with $\epsilon = 0.01$.

- J.M. Sanz-Serna. *An unconventional symplectic integrator of W.Kahan.* Applied Numer. Math. 1994, **16**, 245–250.

A sort of an explanation of a non-spiralling behavior: Kahan's discretization is symplectic w.r.t. $dx \wedge dy/(xy)$.

List of integrable HK-discretizations

- M. Petrera, A. Pfadler, Yu. S. *On integrability of Hirota-Kimura type discretizations.* Regular Chaotic Dyn., 2011, **16**, 245–289.

1. Reduced Nahm equations.
2. Euler top.
3. Three-wave interaction system.
4. Periodic Volterra chain of period $N = 3, 4$:
5. Dressing chain with $N = 3$:
6. System of two interacting Euler tops.
7. Lagrange top.
8. Kirchhof and Clebsch cases of rigid body in an ideal fluid.

Integral for non-integrable Kahan discretizations

- E. Celledoni, R.I. McLachlan, B. Owren, G.R.W. Quispel. *Geometric properties of Kahan's method.* J. Phys. A, 2013, **46**, 025201.

Theorem. *Let* $f(x) = J\nabla H(x)$*, with* $J \in so(n)$*, Hamilton function* $H : \mathbb{R}^n \to \mathbb{R}$ *of* $\deg = 3$*. Then* $\Phi_f(x, \epsilon)$ *admits a rational integral:*

$$\widetilde{H}(x, \epsilon) = H(x) + \frac{\epsilon}{3}(\nabla H(x))^{\mathrm{T}} \left(I - \frac{\epsilon}{2} f'(x)\right)^{-1} f(x),$$

and an invariant volume form

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{\det\left(I - \frac{\epsilon}{2} f'(x)\right)}.$$

Degree of denominator $\det(I - \frac{\epsilon}{2} f'(x))$ *is n, degree of numerator of* $\widetilde{H}(x, \epsilon)$ *is* $n + 1$*.*

Part 2. Integrability of planar quadratic birational maps

Why planar?

- Planar algebraic geometry is much simpler.
- Structure of the group of birational maps of $\mathbb{P}^n$ is unknown for $n \geq 3$. For $n = 2$, generated by quadratic maps (M. Noether theorem).
- For $n \geq 3$, many new phenomena. For instance, there does not hold necessarily that $\deg \Phi^{-1} = \deg \Phi$. (Kahan maps have this property and thus are very special!)

Planar birational maps

- Consider a birational map

$$\phi\colon \mathbb{CP}^2 \to \mathbb{CP}^2, \quad [x:y:z] \mapsto [X:Y:Z],$$

X, Y, Z homogeneous polynomials of $\deg = d$ without a non-trivial (polynomial) common factor.

- *Indeterminacy set* (finitely many points, are blown up by ϕ):

$$\mathcal{I}(\phi) = \{X = Y = Z = 0\}.$$

- *Critical set* ($\dim = 1$, is blown down by ϕ):

$$\mathcal{C}(\phi) = \{\det \partial(X,Y,Z)/\partial(x,y,z) = 0\}.$$

Illustration

Standard quadratic involution

$$\sigma : [x : y : z] \mapsto \left[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right] = [yz : xz : xy].$$

Indeterminacy set:

$$\mathcal{I}(\sigma) = \Big\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\Big\}.$$

Critical set:

$$\mathcal{C}(\sigma) = \{xyz = 0\} = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}.$$

Blow up and blow down

Blow up:
σ blows up $[1:0:0]$ to the line $\{x=0\}$. To see this, consider images of points approaching $[1:0:0]$ along a line with the slope $[\lambda:\mu]\in\mathbb{P}^1$:

$$\sigma:[1:\lambda t:\mu t]\mapsto[\lambda\mu t^2:\mu t:\lambda t]=[\lambda\mu t:\mu:\lambda]\xrightarrow[t\to 0]{}[0:\mu:\lambda].$$

Limiting points comprise the line $\{x=0\}\subset\mathbb{P}^2$.

Blow down:
σ blows down the line $\{x=0\}$ to $[1:0:0]$. Indeed,

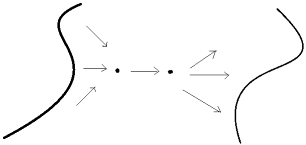
$$\sigma:[0:y:z]\mapsto[yz:0:0]=[1:0:0].$$

Degree lowering and singularity confinement

A component $V \subset \mathcal{C}(\phi)$ is a *degree lowering curve*, if for some $n \in \mathbb{N}$ we have $\phi^n(V) \in \mathcal{I}(\phi)$. A *singularity confinement pattern* is a sequence

$$\mathcal{C}(\phi) \supset V \to \phi(V) \to \cdots \to \phi^n(V) \to \phi^{n+1}(V) \subset \mathcal{C}(\phi^{-1}).$$

A presence of such a curve is necessary and sufficient for $\deg(\phi^n) < (\deg\phi)^n$.



Algebraic entropy

Definition. *Dynamical degree* and *algebraic entropy* of ϕ are

$$\lambda_1(\phi) = \lim_{n\to\infty} (\deg(\phi^n))^{1/n} \le d \quad \text{and} \quad h(\phi) = \log(\lambda_1(\phi)) \le \log(d).$$

Inequalities strict iff there exist degree lowering curves.

How drastic can be the degree drop of iterations ϕ^n?

Definition. A birational map ϕ is *integrable* if $h(\phi) = 0$.

Birational quadratic maps of $\mathbb{P}^2$

A generic birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of deg $= 2$ can be represented as $\phi = A_1 \circ \sigma \circ A_2$, where $A_1, A_2 \in \mathrm{Aut}(\mathbb{P}^2)$, and

$$\sigma : [x : y : z] \to [yz : xz : xy].$$

The variety of such maps has dim $= 14$.

A generic map from this set, not an involution, can be described by a pair of bilinear (Kahan type) relations:

$$\widetilde{x} - x = a_1 + a_2(x + \widetilde{x}) + a_3(y + \widetilde{y}) + a_4 x\widetilde{x} + a_5 y\widetilde{y} + a_6 x\widetilde{y} + a_7 y\widetilde{x},$$
$$\widetilde{y} - y = b_1 + b_2(x + \widetilde{x}) + b_3(y + \widetilde{y}) + b_4 x\widetilde{x} + b_5 y\widetilde{y} + b_6 x\widetilde{y} + b_7 y\widetilde{x}.$$

Singularities of birational quadratic maps of $\mathbb{P}^2$

- Singularities: $\mathcal{I}(\phi) = \{p_1, p_2, p_3\}$, $\mathcal{I}(\phi^{-1}) = \{q_1, q_2, q_3\}$.
- ϕ blows up points p_1, p_2, p_3 to lines $(q_2 q_3), (q_1 q_3), (q_1 q_2)$, resp.
- ϕ blows down lines $(p_2 p_3), (p_1 p_3), (p_1 p_2)$ to points q_1, q_2, q_3, resp.

Lifting to automorphism

Definition. Map ϕ is *confining*, if all three lines $(p_j p_k)$ are *degree lowering* (i.e., participate in *singularity confinement patterns*):

$$(p_j p_k) \to q_i \to \phi(q_i) \to \cdots \to \phi^{n_i - 1}(q_i) = p_{\sigma_i} \to (q_{\sigma_j} q_{\sigma_k}).$$

Orbit data of a confining map ϕ: (n_1, n_2, n_3) and $(\sigma_1, \sigma_2, \sigma_3)$.

A confining map ϕ can be lifted to an automorphism $\hat{\phi}$ of a surface S obtained from $\mathbb{P}^2$ by blowing up all participating points.

Dynamical degree $\lambda_1(\phi)$ can be found as the spectral radius of the action of $\hat{\phi}^*$ on $\mathrm{Pic}(S)$.

Theorem [Bedford, Kim' 2004]. For a confining map, $\lambda_1(\phi)$ depends only on the orbit data associated to ϕ.

Example of integrable planar birational map: Kahan discretization of Hamiltonian systems

For $n = 2$, consider $f(x, y) = J\nabla H(x, y)$, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Φ_f is a birational planar map with an invariant measure and an integral $\Rightarrow$ completely integrable. Integral:

$$\widetilde{H}(x, y, \epsilon) = \frac{C(x, y, \epsilon)}{D(x, y, \epsilon)},$$

where $\deg C = 3$, $\deg D = 2$. Level sets:

$$\mathcal{E}_\lambda = \big\{(x, y) : C(x, y, \epsilon) - \lambda D(x, y, \epsilon) = 0\big\},$$

a pencil of cubic curves, characterized by its nine *base points*. On each invariant curve, Φ_f induces a translation (respective to the addition law on this curve).

Complexification, projectivization

Pencil

$$\bar{\mathcal{E}}_\lambda = \Big\{[x:y:z]\in\mathbb{CP}^2\ :\ \bar{C}(x,y,z,\epsilon)-\lambda z\bar{D}(x,y,z,\epsilon)=0\Big\}.$$

spanned by two curves,

$$\bar{\mathcal{E}}_0 = \Big\{[x:y:z]\in\mathbb{CP}^2\ :\ \bar{C}(x,y,z,\epsilon)=0\Big\},$$

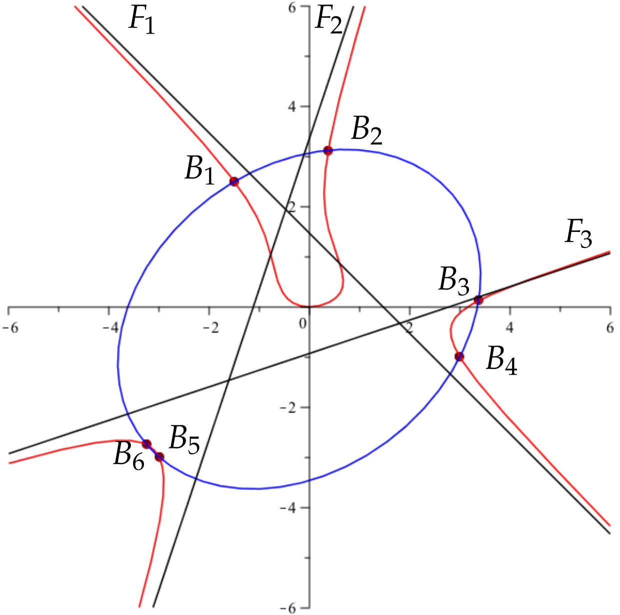
assumed nonsingular, and

$$\bar{\mathcal{E}}_\infty = \Big\{[x:y:z]\in\mathbb{CP}^2\ :\ z\bar{D}(x,y,z,\epsilon)=0\Big\}$$

reducible, consisting of conic $\{\bar{D}(x,y,z,\epsilon)=0\}$ and the line at infinity $\{z=0\}$. Three base points at infinity:

$$\{F_1,F_2,F_3\}=\bar{\mathcal{E}}_0\cap\{z=0\},$$

and six further base points $\{B_1,\dots B_6\}=\bar{\mathcal{E}}_0\cap\{\bar{D}=0\}$.

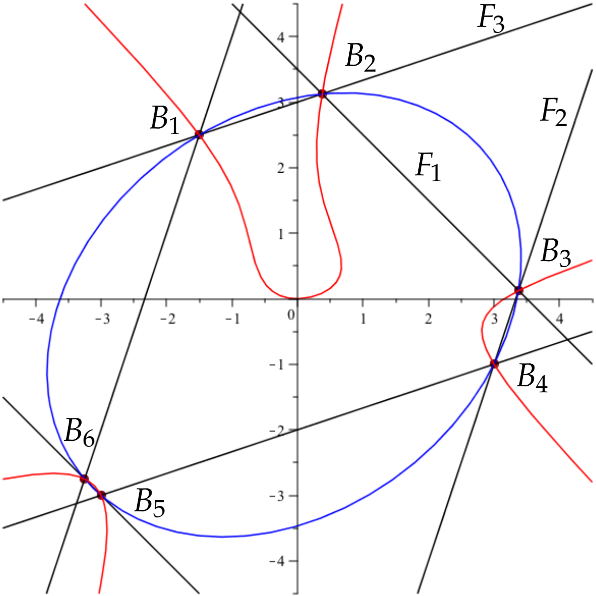


F_1
F_2
F_3
B_1
B_2
B_3
B_4
B_5
B_6
6
4
2
0
-2
-4
-6
-6
-4
-2
2
4
6

Main result

- M. Petrera, J. Smirin, Yu. S. *Geometry of the Kahan discretizations of planar quadratic Hamiltonian systems*. Proc. R. Soc. A **476** (2019) 20180761

Theorem. *A pencil of elliptic curves consists of invariant curves for Kahan's discretization of a planar quadratic Hamiltonian vector field iff the hexagon through the six finite base points has three pairs of parallel sides which pass through the three base points at infinity.*



F_3
B_2
F_2
B_1
F_1
B_3
B_4
B_6
B_5

Manin involutions for cubic curves

Definition. Consider a nonsingular cubic curve $\bar{\mathcal{E}}$ in $\mathbb{CP}^2$.

- For a point $P_0 \in \bar{\mathcal{E}}$, the *Manin involution* $I_{\bar{\mathcal{E}},P_0} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as follows:
  - For $P \neq P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique third intersection point of $\bar{\mathcal{E}}$ with the line $(P_0 P)$;
  - For $P = P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique second intersection point of $\bar{\mathcal{E}}$ with the tangent line to $\bar{\mathcal{E}}$ at $P = P_0$.
- For two distinct points $P_0, P_1 \in \bar{\mathcal{E}}$, the *Manin transformation* $M_{\bar{\mathcal{E}},P_0,P_1} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as

$$M_{\bar{\mathcal{E}},P_0,P_1} = I_{\bar{\mathcal{E}},P_1} \circ I_{\bar{\mathcal{E}},P_0}.$$

With a natural addition law on $\bar{\mathcal{E}}$:

$$I_{\bar{\mathcal{E}},P_0}(P) = -(P_0 + P), \quad M_{\bar{\mathcal{E}},P_0,P_1}(P) = P + P_0 - P_1.$$

Manin involutions for cubic pencils

Definition. Consider a pencil $\mathfrak{E} = \{\bar{\mathcal{E}}_\lambda\}$ of cubic curves in $\mathbb{CP}^2$ with at least one nonsingular member.

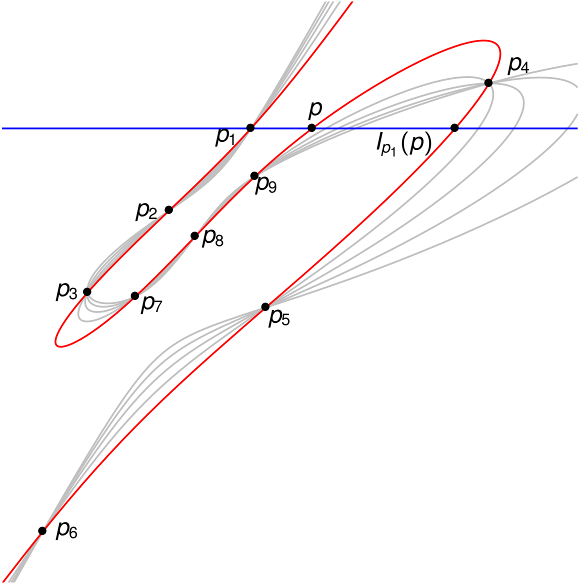
- Let B be a base point of the pencil. The *Manin involution* $I_{\mathfrak{E},B} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as follows. For any $P \in \mathbb{CP}^2$, not a base point of $\mathfrak{E}$, let $\bar{\mathcal{E}}_\lambda$ be the unique curve of $\mathfrak{E}$ through P. Set

$$I_{\mathfrak{E},B}(P) = I_{\bar{\mathcal{E}}_\lambda,B}(P).$$

- Let B_1, B_2 be two distinct base points of the pencil. The *Manin transformation* $M_{\mathfrak{E},B_1,B_2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as

$$M_{\mathfrak{E},B_1,B_2} = I_{\mathfrak{E},B_2} \circ I_{\mathfrak{E},B_1}.$$

Manin involutions for cubic pencils



Direct statement. Proof.

First one shows that Kahan map Φ_f is a Manin transformation in six different ways:

$$\begin{aligned}\Phi_f &= I_{\mathfrak{E},B_1}\circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1}\circ I_{\mathfrak{E},B_4}\\ &= I_{\mathfrak{E},B_5}\circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2}\circ I_{\mathfrak{E},B_2}\\ &= I_{\mathfrak{E},B_3}\circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3}\circ I_{\mathfrak{E},B_6}.\end{aligned}$$

Thus (on any invariant curve of $\mathfrak{E}$):

$$F_1 - B_1 = B_2 - F_2 = F_3 - B_3 = B_4 - F_1 = F_2 - B_5 = B_6 - F_3,$$

and

$$F_1 + F_2 + F_3 = O.$$

Have, e.g.:

$$B_1 + B_2 = F_1 + F_2 = -F_3 \quad\Leftrightarrow\quad B_1 + B_2 + F_3 = O.$$

Thus, line (B_1B_2) passes through F_3. $\square$

Inverse statement. Proof.

Prescribe arbitrary nine coefficients of the side lines of the hexagon (three slopes μ_1, μ_2, μ_3 and six heights $b_1, \ldots, b_6$):

$$\begin{array}{ll} (B_1B_2): y = \mu_3 x + b_1, & (B_4B_5): \ y = \mu_3 x + b_4, \\ (B_2B_3): y = \mu_1 x + b_2, & (B_5B_6): \ y = \mu_1 x + b_5, \\ (B_3B_4): y = \mu_2 x + b_3, & (B_6B_1): \ y = \mu_2 x + b_6. \end{array}$$

This defines nine points $B_1, \ldots, B_6$ and F_1, F_2, F_3, therefore a pencil $\mathfrak{E}$ of cubic curves with those nine base points. Set

$$\begin{aligned} \Phi &= I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1} \circ I_{\mathfrak{E},B_4} \\ &= I_{\mathfrak{E},B_5} \circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2} \circ I_{\mathfrak{E},B_2} \\ &= I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3} \circ I_{\mathfrak{E},B_6}. \end{aligned}$$

This is a birational map of $\mathbb{CP}^2$ of degree 2. Check that this is a Kahan discretization of $f = J\nabla H$ with deg $H = 3$.

Inverse statement. Proof.

Explicit expression:

$$
\begin{aligned}
&H(x,y) = \\
&\frac{2\mu_{12}}{b_{14}\mu_{23}\mu_{13}}\Big(\tfrac{1}{3}(\mu_3 x - y)^3 + \tfrac{1}{2}(b_1+b_4)(\mu_3 x - y)^2 + b_1 b_4(\mu_3 x - y)\Big)\\
&-\frac{2\mu_{23}}{b_{25}\mu_{12}\mu_{13}}\Big(\tfrac{1}{3}(\mu_1 x - y)^3 + \tfrac{1}{2}(b_2+b_5)(\mu_1 x - y)^2 + b_2 b_5(\mu_1 x - y)\Big)\\
&+\frac{2\mu_{13}}{b_{36}\mu_{12}\mu_{23}}\Big(\tfrac{1}{3}(\mu_2 x - y)^3 + \tfrac{1}{2}(b_3+b_6)(\mu_2 x - y)^2 + b_3 b_6(\mu_2 x - y)\Big),
\end{aligned}
$$

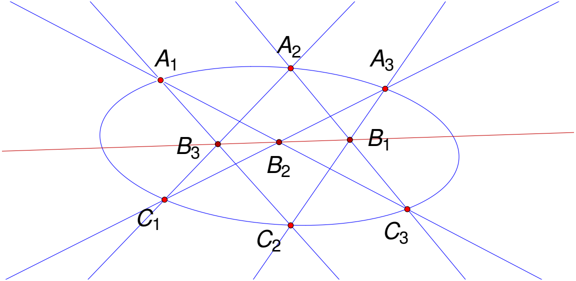
where $b_{ij} = b_i - b_j$, $\mu_{ij} = \mu_i - \mu_j$.

Geometry implies dynamics! □

Projective generalization of Hamiltonian case

Pascal configuration: six points A_1, A_2, A_3, C_1, C_2, C_3 on a conic $\mathcal{C}$, and three intersection points on a line ℓ:

$$B_1 = (A_2C_3)\cap(A_3C_2), \quad B_2 = (A_3C_1)\cap(A_1C_3), \quad B_3 = (A_1C_2)\cap(A_2C_1).$$



Consider pencil $\mathfrak{E}$ of cubic curves passing through the nine points A_i, C_i, B_i (contains a reducible cubic $\mathcal{C} \cup \ell$).

Construction

Theorem [Yu. S.' 2020]. *The map*

$$\begin{aligned} \Phi &= I_{\mathfrak{E},A_1} \circ I_{\mathfrak{E},B_1} = I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},C_1} \\ &= I_{\mathfrak{E},A_2} \circ I_{\mathfrak{E},B_2} = I_{\mathfrak{E},B_2} \circ I_{\mathfrak{E},C_2} \\ &= I_{\mathfrak{E},A_3} \circ I_{\mathfrak{E},B_3} = I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},C_3} \end{aligned}$$

is a birational map **of degree** 2 *with*

- $\mathcal{I}(\Phi) = \{C_1, C_2, C_3\}$, *blown up to lines* $c_1 = (A_2A_3)$, $c_2 = (A_3A_1)$, $c_3 = (A_1A_2)$,
- $\mathcal{C}(\Phi)$ *consisting of three lines* $a_1 = (C_2C_3)$, $a_2 = (C_3C_1)$, $a_3 = (C_2C_3)$, *blown down to points* A_1, A_2, A_3.

Singularity confinement patterns of the map Φ*:*

$$\begin{aligned} (C_2C_3) \to A_1 \to B_1 \to C_1 \to (A_2A_3), \\ (C_3C_1) \to A_2 \to B_2 \to C_2 \to (A_3A_1), \\ (C_1C_2) \to A_3 \to B_3 \to C_3 \to (A_1A_2). \end{aligned}$$

Proof

To show: why the six Manin transformations correspond to one and the same translation on any curve of the pencil:

$$A_1 - B_1 = B_1 - C_1 = A_2 - B_2 = B_2 - C_2 = A_3 - B_3 = B_3 - C_3.$$

Collinearities of Pascal configuration are translated to:

$$\begin{array}{ll} A_2 + C_3 + B_1 = O, & A_3 + C_2 + B_1 = O, \\ A_3 + C_1 + B_2 = O, & A_1 + C_3 + B_2 = O, \\ A_1 + C_2 + B_3 = O, & A_2 + C_1 + B_3 = O, \end{array}$$

and

$$B_1 + B_2 + B_3 = O.$$

$$\text{Now:} \quad \begin{aligned} A_1 + C_1 &= -(C_2 + B_3) - (A_3 + B_2) \\ &= -(A_3 + C_2) - (B_2 + B_3) \;=\; B_1 + B_1, \end{aligned}$$

which proves that $A_1 - B_1 = B_1 - C_1$. Similarly,

$$A_2 + C_1 = -B_3 = B_1 + B_2,$$

which proves that $B_1 - C_1 = A_2 - B_2$.
All other equations follow in the same way. $\square$

An early example

R. Penrose, C. Smith. *A quadratic mapping with invariant cubic curve*. Math. Proc. Camb. Phyl. Soc. **89** (1981), 89–105:

$$\Phi : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_0(x_0 + ax_1 + a^{-1}x_2) \\ x_1(x_1 + ax_2 + a^{-1}x_0) \\ x_2(x_2 + ax_0 + a^{-1}x_1) \end{bmatrix}$$

with

$$A_1 = [0 : 1 : -a], \quad C_1 = [0 : a : -1], \quad B_1 = [0 : 1 : -1]$$

(and others cyclically). Upon a projective transformation sending B_1, B_2, B_3 to infinity, get a Kahan discretization of a Hamiltonian vector field with $H(x, y) = xy(1 - x - y)$ with the time step $\epsilon = (a - 1)/(a + 1)$.

Further examples: $(\gamma_1, \gamma_2, \gamma_3)$-family of 2d quadratic systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{\ell_1^{\gamma_1 - 1} \ell_2^{\gamma_2 - 1} \ell_3^{\gamma_3 - 1}} J \nabla H,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H(x, y) = (\ell_1(x, y))^{\gamma_1} (\ell_2(x, y))^{\gamma_2} (\ell_3(x, y))^{\gamma_3},$$

$\ell_i(x, y) = a_i x + b_i y$ are linear forms, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$.

Origin: reduced Nahm equations for symmetric monopoles [N. Hitchin, N. Manton, M. Murray' 1995]

- Tetrahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$:

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = -2xy, \end{cases} \qquad H_1(x, y) = \frac{y}{3}(3x^2 - y^2).$$

- Octahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$:

$$\begin{cases} \dot{x} = x^2 - 6y^2, \\ \dot{y} = -3xy - 2y^2, \end{cases} \qquad H_2(x, y) = \frac{y}{2}(2x + 3y)(x - y)^2.$$

- Icosahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$:

$$\begin{cases} \dot{x} = 2x^2 - y^2, \\ \dot{y} = -10xy + y^2, \end{cases} \qquad H_3(x, y) = \frac{y}{6}(3x - y)^2(4x + y)^3.$$

In all three cases level curves $H_i(x, y) = c$ are of genus $g = 1$.

The $(\gamma_1, \gamma_2, \gamma_3)$-family: discretization

Hirota-Kimura-Kahan discretizations are integrable [M. Petrera, A. Pfadler, Yu. S.' 2011]:

$$\left\{ \begin{array}{l} \widetilde{x} - x = \epsilon(\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = -\epsilon(\widetilde{x}y + x\widetilde{y}), \end{array} \right.$$

$$\left\{ \begin{array}{l} \widetilde{x} - x = \epsilon(2\widetilde{x}x - 12\widetilde{y}y), \\ \widetilde{y} - y = -\epsilon(3\widetilde{x}y + 3x\widetilde{y} + 4\widetilde{y}y), \end{array} \right.$$

$$\left\{ \begin{array}{l} \widetilde{x} - x = \epsilon(2\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = \epsilon(-5\widetilde{x}y - 5x\widetilde{y} + \widetilde{y}y). \end{array} \right.$$

In all three cases, the map admits an invariant pencil of elliptic curves, of degrees 3, 4, and 6, respectively.

The $(\gamma_1, \gamma_2, \gamma_3)$-family: classification of integrable cases through discretization

Theorem [R. Zander' 2020]. *The only three cases when the Kahan discretization of the* $(\gamma_1, \gamma_2, \gamma_3)$*-system is confining, are* $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1), (1, 1, 2)$*, and* $(1, 2, 3)$*. The orbit data in these three cases are:* $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$ *and, respectively,*

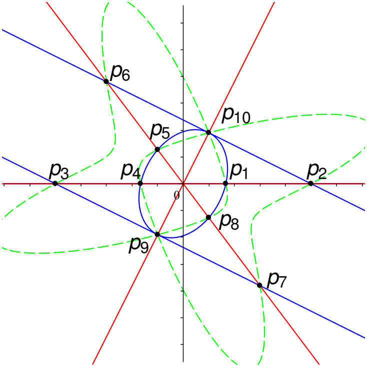
$$(n_1, n_2, n_3) = (3, 3, 3),\ (4, 4, 2),\ \textit{and}\ \ (6, 3, 2).$$

Observe: these (n_1, n_2, n_3) are the only positive integer solutions of

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1.$$

Puzzle: what do lengths of singularity confinement patterns have to do with tilings of the plane by congruent triangles???

Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$



Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$

- Invariant pencil consists of quartic curves with two double points: $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$.
- $\mathcal{I}(\phi) = \{p_4, p_8, p_{10}\}$, $\mathcal{I}(\phi^{-1}) = \{p_1, p_5, p_9\}$.
- Singularity confinement patterns:

$$
\begin{aligned}
&(p_8 p_{10}) \to p_1 \to p_2 \to p_3 \to p_4 \to (p_5 p_9)\\
&(p_4 p_{10}) \to p_5 \to p_6 \to p_7 \to p_8 \to (p_1 p_9)\\
&(p_4 p_8) \to p_9 \to p_{10} \to (p_1 p_5)
\end{aligned}
$$

- What is the geometric representation?

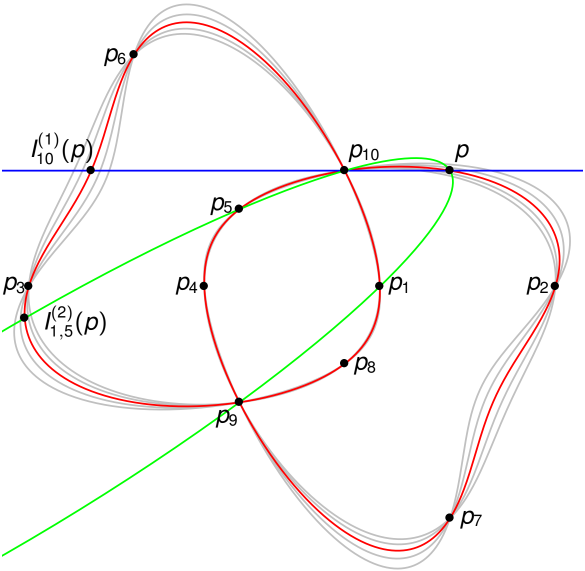
Involutions for quartic pencils with two double points

Manin involutions for $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$:

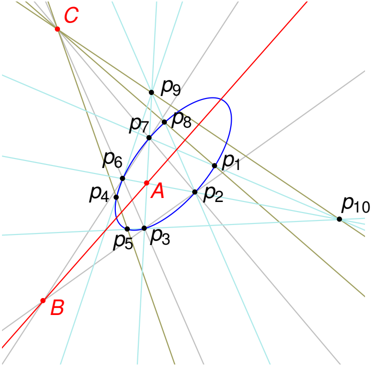
- $I_k^{(1)}$, $k \in \{9, 10\}$: $I_k^{(1)}(p)$ is the third intersection point of the quartic through p with the line (pp_k).
- $I_{i,j}^{(2)}$, $i, j \in \{1, \ldots, 8\}$: $I_{i,j}^{(2)}(p)$ is the sixth intersection point of the quartic through p with the conic through p_9, p_{10}, p_i, p_j, p.

Are derived from Manin involutions for a cubic pencil upon a quadratic Cremona transformation resolving both double points.

Involutions for quartic pencils with two double points



Quadratic Manin maps for special quartic pencils



Geometry of base points of a *projectively symmetric quartic pencil with two double points* $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$.

Quadratic Manin maps for special quartic pencils

Theorem [M. Petrera, Yu. S., K. Wei, R. Zander' 2021].

1. *The projective involution σ can be represented as*

$$\sigma = I^{(2)}_{1,8} = I^{(2)}_{2,7} = I^{(2)}_{3,6} = I^{(2)}_{4,5}.$$

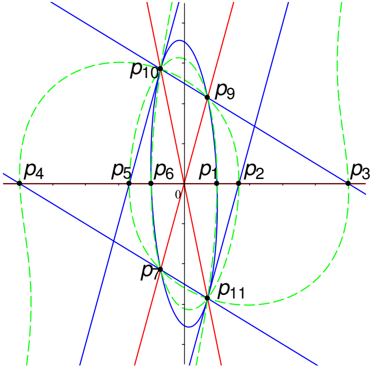
2. *The map*

$$\phi \;=\; I^{(2)}_{i,k} \circ I^{(2)}_{j,k} \;=\; I^{(1)}_{9} \circ \sigma \;=\; \sigma \circ I^{(1)}_{10},$$

$(i,j) \in \{(1,2),(2,3),(3,4),(5,6),(6,7),(7,8)\}$ *and* $k \in \{1,\dots,8\}$ *distinct from* i,j*, is a birational map* **of degree 2**, *with the singularity confinement patterns:*

$$(p_8p_{10}) \to p_1 \to p_2 \to p_3 \to p_4 \to (p_5p_9),$$
$$(p_4p_{10}) \to p_5 \to p_6 \to p_7 \to p_8 \to (p_1p_9),$$
$$(p_4p_8) \to p_9 \to p_{10} \to (p_1p_5).$$

Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$



Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$

- Invariant pencil of sextic curves with 3 double points and 2 triple points: $\mathfrak{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$.
- $\mathcal{I}(\phi) = \{p_6, p_9, p_{11}\}, \mathcal{I}(\phi^{-1}) = \{p_1, p_7, p_{10}\}$.
- Singularity confinement patterns:

$$
\begin{aligned}
&(p_9 p_{11}) \to p_1 \to p_2 \to p_3 \to p_4 \to p_5 \to p_6 \to (p_7 p_{10}),\\
&(p_6 p_{11}) \to p_7 \to p_8 \to p_9 \to (p_1 p_{10}),\\
&(p_6 p_9) \to p_{10} \to p_{11} \to (p_1 p_7).
\end{aligned}
$$

- What is the geometric representation?

Kahan discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$

Manin involutions for $\mathfrak{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$:

- $I_{i,j,k}^{(4)}$, $i, j \in \{1, \ldots, 6\}$, $k \in \{7, 8, 9\}$: e.g., $I_{i,j,9}^{(4)}$ is defined in terms of intersection of $\mathfrak{E}$ with quartics of the pencil
$$\mathcal{P}(4; p_i, p_j, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2).$$
- $I_{i,k}^{(3)}$, $i \in \{1, \ldots, 6\}$, $k \in \{10, 11\}$: e.g., $I_{i,10}^{(3)}$ is defined in terms of intersection of $\mathfrak{E}$ with cubics of the pencil
$$\mathcal{P}(3; p_i, p_7, p_8, p_9, p_{10}^2, p_{11}).$$

Theorem [M. Petrera, Yu. S., K. Wei, R. Zander' 2021]. *The map ϕ can be represented as compositions of (suitably defined) Manin involutions in the following ways:*
$$\phi = I_{i,k,m}^{(4)} \circ I_{j,k,m}^{(4)} = I_{i,n}^{(3)} \circ I_{j,n}^{(3)}$$
for any $(i, j) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$, $k \in \{1, \ldots, 6\} \setminus \{i, j\}$, and $m \in \{7, 8, 9\}$, $n \in \{10, 11\}$.

Part 3. Geometric constructions of integrable 3D birational maps of bidegree 3:3

Hirota-Kimura's discrete time Euler top

$$
\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2). \end{cases}
$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$:

$$
A(x,\epsilon)\widetilde{x} = x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix},
$$

imply a rational map, which is *reversible* (therefore birational):

$$
\widetilde{x} = \Phi(x,\epsilon) = A^{-1}(x,\epsilon)x, \quad \Phi^{-1}(x,\epsilon) = \Phi(x,-\epsilon).
$$

- Explicit formulas:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where $\Delta(x,\epsilon) = \det A(x,\epsilon)$

$$= 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3.$$

- Two independent integrals:

$$I_3(x,\epsilon) = \frac{1-\epsilon^2\alpha_2\alpha_3 x_1^2}{1-\epsilon^2\alpha_3\alpha_1 x_2^2}, \quad I_1(x,\epsilon) = \frac{1-\epsilon^2\alpha_3\alpha_1 x_2^2}{1-\epsilon^2\alpha_1\alpha_2 x_3^2}.$$

- Invariant volume form:

$$\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)}, \quad \phi(x) = 1-\epsilon^2\alpha_i\alpha_j x_k^2$$

  and bi-Hamiltonian structure found in:

- M. Petrera, Yu. S. *On the Hamiltonian structure of the Hirota-Kimura discretization of the Euler top.* Math. Nachr., 2010, **283**, 1654–1663.

Geometry of the discrete time Euler top

Space $\mathbb{P}^3$ is foliated by joint level sets of two integrals of dET, each being a spatial elliptic curve – an intersection of two quadrics

$$C = \mathcal{H}(\lambda) \cap \mathcal{Z}(\mu),$$

where

$$\mathcal{H}(\lambda) = \left\{ H_{12}(x,\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2} = \lambda \right\}$$

is a hyperboloid, while

$$\mathcal{Z}(\mu) = \left\{ I_3(x,\epsilon) = \frac{1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2}{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2} = \mu \right\}$$

is a cylinder.

Involutions on quadrics $\mathcal{H}$, $\mathcal{Z}$

- Through any point $x \in C$ there pass two straight line generators ℓ_1, ℓ_2 of $\mathcal{H}(\lambda)$. Their direction vectors are *rational* w.r.t. x. Denote by $i_1(x)$, resp. $i_2(x)$, the second intersection point of ℓ_1, resp. of ℓ_2, with C. This defines two involutions $i_1, i_2 : C \to C$, which lead to two birational maps $i_1, i_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.
- Similarly, through any point $x \in C$ there pass a straight line generator of $\mathcal{Z}(\mu)$ (parallel to x_3-axis). Denote by $\sigma(x)$ its second intersection point with C, so that

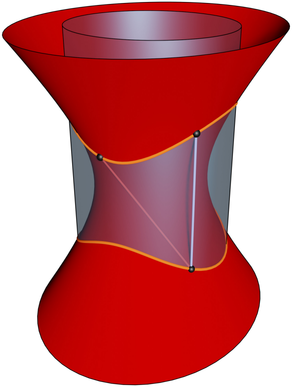
$$\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3).$$

  This is a linear projective involution $\sigma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

HK discrete time Euler top from involutions

Theorem [N. Smeenk, Yu. S.' 2020]. *The discrete time Euler top can be represented as*

$$\Phi = \sigma \circ i_1 = i_2 \circ \sigma.$$



General construction: separable pencils

Consider $\mathbb{P}^3$ with homogeneous coordinates $[x_1 : x_2 : x_3 : x_4]$. Consider two *separable pencils of quadrics*,

$$Q(\lambda) = \{X_1 X_2 - \lambda X_3 X_4 = 0\}, \quad P(\mu) = \{U_1 U_2 - \mu U_3 U_4 = 0\},$$

where X_j and U_j are two quadruples of independent linear forms on $\mathbb{C}^4$. Pencil $Q(\lambda)$ (say) consists of all quadrics through the *base set* consisting of four lines $\{X_1 = X_3 = 0\}$, $\{X_1 = X_4 = 0\}$, $\{X_2 = X_3 = 0\}$, $\{X_2 = X_4 = 0\}$.

Space $\mathbb{P}^3$ is foliated by elliptic curves $C = Q(\lambda) \cap P(\mu)$.

Two straight line generators of $Q(\lambda)$ through $[X_1 : X_2 : X_3 : X_4]$:

$$\ell_1 = \Big\{[X_1 : (1+t)X_2 : (1+t)X_3 : X_4] : t \in \mathbb{P}^1\Big\},$$

and

$$\ell_2 = \Big\{[(1+t)X_1 : X_2 : (1+t)X_3 : X_4] : t \in \mathbb{P}^1\Big\}.$$

General construction: involution along generators

Theorem [J. Alonso, Yu. S., K. Wei' 2021].

• *Involution* $i_1 : C \to C$ *along generators* ℓ_1 *of* $Q(\lambda)$ *is given by*

$$i_1 : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 T_2 : X_2 T_0 : X_3 T_0 : X_4 T_2],$$

where T_0, T_2 *are polynomials of* $\deg = 4$. *Thus,* $i_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *is of* $\deg = 5$.

• *If four pairwise non-intersecting lines*

$$\{X_1 = X_4 = 0\},\ \{X_2 = X_3 = 0\},\ \{U_1 = U_3 = 0\},\ \{U_2 = U_4 = 0\}$$

lie on a quadric, say $Q_0 = 0$, *then* T_2 *and* T_0 *are divisible by* Q_0, *and*

$$i_1 : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 Q_2 : X_2 Q_1 : X_3 Q_1 : X_4 Q_2],$$

with certain quadratic polynomials Q_1, Q_2. *Thus,* $i_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *is of* $\deg = 3$.

General construction: combine with a linear involution

If both pencils $Q(\lambda)$ and $P(\mu)$ are invariant under a linear projective involution $\sigma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, we obtain a birational map

$$\phi = \sigma \circ i_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

of $\deg = 3$, with two integrals of motion,

$$\frac{X_1(x)X_2(x)}{X_3(x)X_4(x)} = \lambda \quad \text{and} \quad \frac{U_1(x)U_2(x)}{U_3(x)U_4(x)} = \mu.$$

Example: discrete time Zhukovsky-Volterra gyrostat

Theorem [J. Alonso, Yu. S., K. Wei' 2021]. *Set*

$$
\begin{aligned}
X_1 &= \sqrt{\alpha_1}x_2 - \sqrt{\alpha_2}x_1 - (\beta_1/\sqrt{\alpha_2})x_4, \\
X_2 &= \sqrt{\alpha_1}x_2 + \sqrt{\alpha_2}x_1 + (\beta_1/\sqrt{\alpha_2})x_4, \\
X_3 &= x_4 - \epsilon\sqrt{\alpha_1\alpha_2}x_3, \\
X_4 &= x_4 + \epsilon\sqrt{\alpha_1\alpha_2}x_3,
\end{aligned}
$$

$$
\begin{aligned}
U_1 &= \sqrt{\alpha_1}x_3 - \sqrt{\alpha_3}x_1 + (\beta_1/\sqrt{\alpha_3})x_4, \\
U_2 &= \sqrt{\alpha_1}x_3 + \sqrt{\alpha_3}x_1 - (\beta_1/\sqrt{\alpha_3})x_4, \\
U_3 &= x_4 - \epsilon\sqrt{\alpha_3\alpha_1}x_2, \\
U_4 &= x_4 + \epsilon\sqrt{\alpha_3\alpha_1}x_2,
\end{aligned}
$$

defining two separable pencils invariant under $\sigma : x_3 \to -x_3$.

Then the above construction leads to the map given, in the inhomogeneous coordinates, by

$$\phi : \begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1) + \epsilon\beta_1(\widetilde{x}_3 + x_3), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2) - \epsilon\beta_1(\widetilde{x}_2 + x_2), \end{cases}$$

with two integrals of motion,

$$H_2(\varepsilon) = \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1 + \dfrac{\beta_1^2}{\alpha_3}}{1 - \varepsilon^2\alpha_3\alpha_1 x_2^2},$$

$$H_3(\varepsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1 - \dfrac{\beta_1^2}{\alpha_2}}{1 - \varepsilon^2\alpha_1\alpha_2 x_3^2}.$$

This is the HK discretization of a Zhukovsky-Volterra gyrostat,

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1 + \beta_1 x_3, \\ \dot{x}_3 = \alpha_3 x_1 x_2 - \beta_1 x_2, \end{cases}$$

a system with two polynomial integrals of motion

$$H_2 = \alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1,$$
$$H_3 = \alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1.$$

Summary

- Classification of integrable cases of Kahan discretization for the $(\gamma_1, \gamma_2, \gamma_3)$-family.
- Geometric construction of Manin involutions for pencils of elliptic curves of degree 4 and 6.
- Integrable Kahan discretizations for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$ are Manin maps for pencils of elliptic curves of degree 3, 4, 6, resp.
- Special geometry of base points ensures $\deg = 2$ for certain Manin maps.
- Special geometry of base sets of two separable pencils of quadrics ensures $\deg = 3$ for certain integrable birational maps in 3D defined via involutions along generators.