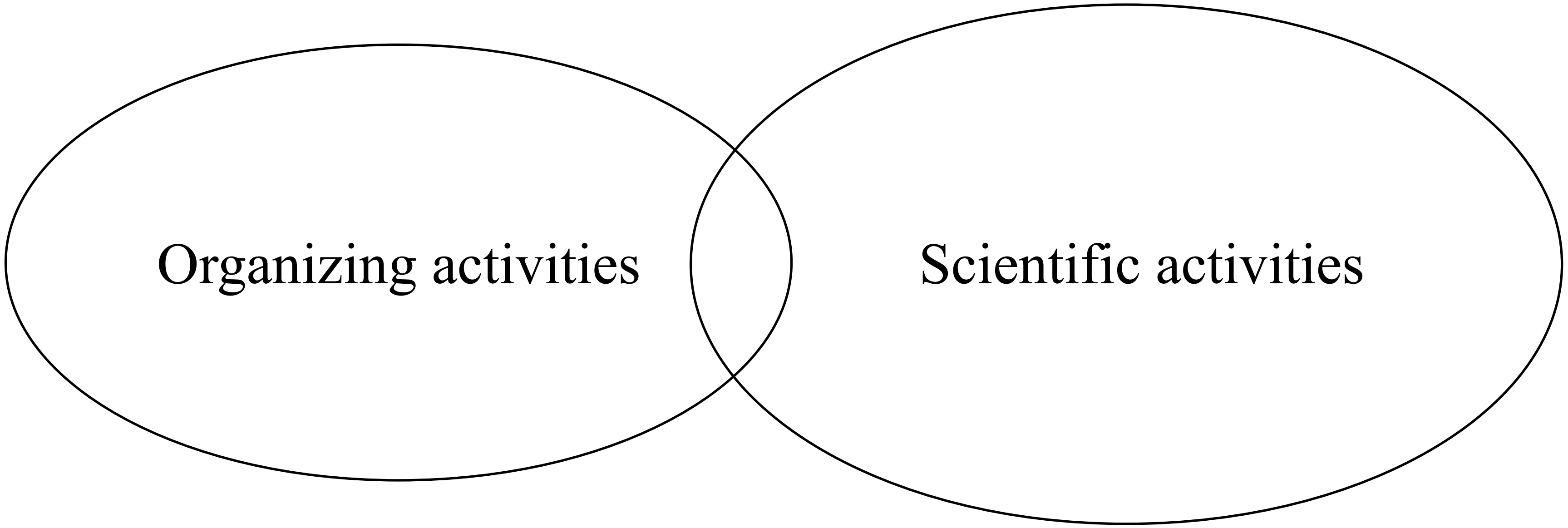
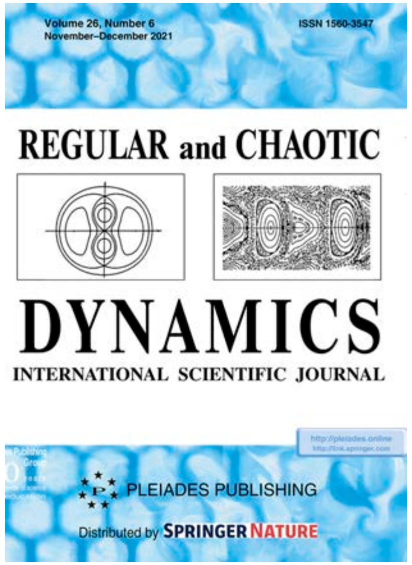
Some notes on A. Borisov path in mechanics

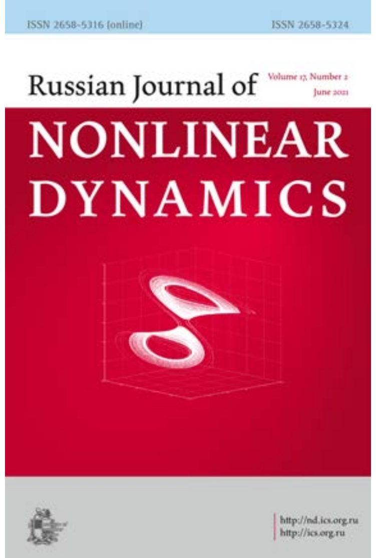


Institute of Computer Science in Izhevsk

- 1997 Laboratory of Dynamical Chaos and Nonlinearity, Udmurt State University (UdSU).
- 2002 Institute of Computer Science
- 2010 Laboratory of Nonlinear Analysis and Design of New Types of Vehicles
- 2020 Institute of Computer Science belongs to the Ural Mathematical Center

- Regular and Chaotic Dynamics (RCD) 1996 (Russian) and 1998 (English)
- Russian Journal of Nonlinear Dynamics 2005 (Russian) and 2018 (English)
- Computer Research and Modeling 2009 (Russian & English)





Alexey Borisov always took a great interest in the history of science. He regarded the results of classical works as fundamental and read them in the original. This interest was so great that he managed to do what even large Russian publishing houses could not: get the fundamental classical works of A.B. Basset, C. Caratheodory, E.J.Cartan, J.G.Darboux, J.H.Jellett, S.Lie, W. Thomson and of many others translated into Russian and publish them.

Conferences Geometry, Dynamics, Integrable Systems (in collaboration with V. Dragovich et al)

- Belgrade, Serbia, 2 – 7 September 2008;
- Belgrade – Fruska Gora, Serbia, 7 – 13 September 2010;
- Lisbon – Sintra, Portugal, 10 – 16 September 2011;
- Izhevsk, Russia, 10 – 14 June 2013;
- Trieste, Italy, 16 – 17 June 2014;
- Izhevsk, Russia, 2 – 5 June 2016;
- Moscow, Russia, 5 – 9 June, 2018;
- Zlatibor, Serbia, June 2022.

IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures, Turbulence,
Moscow, Russia, August 25–30, 2006 (in collaboration with M. Sokolovskiy et al)

IUTAM Symposium on From Mechanical to Biological Systems - an Integrated Approach,
Izhevsk, Russia, 5 – 10 June 2012

Conference Scientific Heritage of Sergey A. Chaplygin:
nonholonomic mechanics, vortex structures and hydrodynamics, Cheboksary, Russia, 2 - 6 June 2019

Music

One cannot but mention another outstanding hobby of Alexey Borisov. Many will remember him as a professor and a musician in one person, with his numerous concerts, masterly play, creative charisma and charm. Alexey Borisov loved the accordion very much and did all he could to popularize this instrument. For many years in succession he held international accordion music festivals entitled "The Mystery of Accordeon" with famous accordion players of Russia and Europe, such as Dmitry Dmitrienko, Emi Dragoi, Valery Kovtun, Ilya Ryskov, William Sabatier, Maria Selezneva, and Roman Zhbanov, and violinist Ksenia Blagovich.

As a result of this hobby, the publishing center started another line of activity, namely, publication of printed music and music manuals.

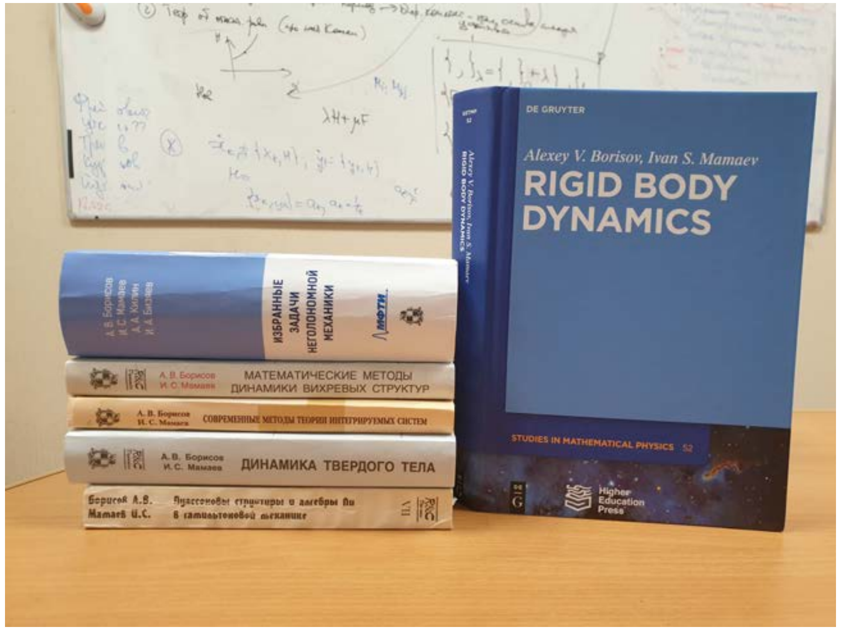
Scientific

Areas of mechanics

- Rigid body dynamics
- Celestial mechanics in spaces of constant curvature
- Figures of equilibrium and the evolution of self-gravitating ellipsoids;
- Vortex dynamics
- Nonholonomic mechanics

Methods of analysis

- Poisson structures
- Integrable and superintegrable systems
- Explicit integration, reduction, and isomorphisms
- Bifurcation analysis and stability
- Numerical methods of investigating dynamics



Jacobi about Euler’s work:

“Euler’s work has the great merit that it presents, wherever possible, all cases in which problems can be solved completely using given methods and means...Therefore, his examples always show a complete content of his method according to the state of science of that time and, as a rule, when it is possible to add a new example to Euler’s examples, it is an enrichment of science, for it rarely happened that a case solvable by his methods escaped his attention. . . ”

P. Halmos:

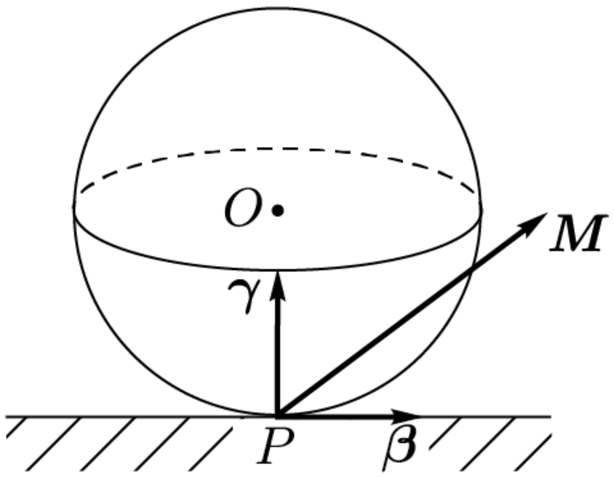
“The heart of mathematics consists of concrete examples and concrete problems. Big general theories are usually afterthoughts based on small but profound insights; the insights themselves come from concrete special cases. . . ”

E. Zehnder about J. Moser:

“These notes owe much to Jurgen Moser’s deep insight into dynamical systems and his broad view of mathematics. They also reflect his specific approach to mathematics by singling out inspiring typical phenomena rather than designing abstract theories. . . ”

Borisov A. V., Mamaev I. S., Bizyaev I. A., Historical and Critical Review of the Development of Nonholonomic Mechanics: the Classical Period, Regular and Chaotic Dynamics, 2016, vol. 21, no. 4, pp. 455-476

Chaplygin sphere Rolling Problem Is Hamiltonian



The condition that there be no slipping at the point of contact imposes the following nonholonomic constraint on the system: $\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0$

The equations of motion of the system have the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega},$$

$$\boldsymbol{M} = \mathrm{I}\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathrm{I}_Q \boldsymbol{\omega}, \quad D = mR^2, \; \mathbf{J} = \mathbf{I} + D\mathbf{E}$$

The system has the following integrals of motion:

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{ const.}$$

Also, the system possesses an invariant measure $\rho_\mu d\boldsymbol{M} d\boldsymbol{\gamma}$ with density

$$\rho_\mu = (\det \mathbf{I}_Q)^{-1/2} = \left[\det \mathbf{J}\left(1 - D\left(\gamma, \mathbf{J}^{-1}\gamma\right)\right)\right]^{-1/2}$$

Borisov A. V., Mamaev I. S., Chaplygin's Ball Rolling Problem Is Hamiltonian, Mathematical Notes, 2001, vol. 70, no. 5, pp. 720-723

Two skew-symmetric representation

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} \qquad \dot{\boldsymbol{M}} = (\boldsymbol{M} - g\boldsymbol{\gamma}) \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}$$

$$\dot{\boldsymbol{\gamma}} = g^{-1}\boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}} \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}$$

$$H = \frac{1}{2}(\mathbf{J}^{-1}\boldsymbol{M}, \boldsymbol{M}) + \frac{1}{2}g(\mathbf{J}^{-1}\boldsymbol{M}, \boldsymbol{\gamma}), \quad g = \frac{D(\mathbf{J}^{-1}\boldsymbol{M}, \boldsymbol{\gamma})}{1 - D(\mathbf{J}^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})}$$

The equations of motion can be represented in conformal Hamiltonian form as

$$\frac{dM_k}{d\tau} = \{H, M_k\}, \quad \frac{d\gamma_k}{d\tau} = \{H, \gamma_k\} \qquad \rho_\mu dt = d\tau, \quad \boldsymbol{L} = \rho_\mu \boldsymbol{M}$$

$$\{L_i, L_j\} = \varepsilon_{ijk}\big(L_k - D(L, \gamma)\rho_\mu^2 J_i J_j \gamma_k\big), \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0$$

Hierarchy of nonholonomic dynamics rolling of body on plane

Suppose that the rigid body rolls without sliding (i.e. the velocity of contact point P is equal to zero) on the fixed surface represented by plane. The first part of equations of motion is the vector dynamical equation of kinetic moment behavior in time with respect to the contact point P. This equation is represented for arbitrary shapes of body and surface in the form

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + m\dot{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{M}_Q \qquad \dot{\gamma} = \gamma \times \boldsymbol{\omega}$$

where $\tilde{\mathbf{I}} = \mathbf{I} + m\boldsymbol{r}^2\mathbf{E} - m\boldsymbol{r} \otimes \boldsymbol{r}$ is the tensor of inertia relative to the point of contact, and $\boldsymbol{M}_Q$ is the moment of external forces relative to the point of contact. Furthermore, these equations must be supplemented with algebraic relations relating the normal γ to the vector $\boldsymbol{r}$ with the help of the Gaussian projection:

$$\gamma = -\frac{\nabla f}{|\nabla f|}$$

where $f(\boldsymbol{r}) = 0$ is the equation defining the surface of the body, and $\nabla f = \left(\frac{\partial f}{\partial r_1}, \frac{\partial f}{\partial r_2}, \frac{\partial f}{\partial r_3}\right)$. In the case of a convex body this equation is uniquely solvable, so we set $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\gamma})$.

For equations we always have the energy integral and the geometrical integral

$$E = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad F_1 = \gamma^2 = 1$$

We consider all the known cases of existence of additional first integrals (one or two at once) and the cases of existence of invariant measure.

Borisov A. V., Mamaev I. S., The rolling motion of a rigid body on a plane and a sphere. Hierarchy of dynamics, Regular and Chaotic Dynamics, 2002, vol. 7, no. 2, pp. 177-200

Before we consider the following cases of the body's motion, we shall present some general construction that let us to establish relations between equations to some point one-to-one map in three-dimensional space. We present the computer analysis of this map using the numerical integration of the indicated system at the fixed value of energy. Using this method we can find out and give a visual interpretation to various possibilities of existence of measure and integrals in their various combinations. To construct the three-dimensional map we use the Andoyer–Deprit variables (L,G,H,l,g,h). The problems described above require two additional integrals of motion; therefore, it is necessary to use three-dimensional maps, and such maps are not necessarily possess an invariant measure (as against to Hamiltonian mechanics). Using the known formulas we make the transition from the variables $\boldsymbol{M}, \boldsymbol{\gamma}$ to the Andoyer–Deprit variables

$$M_1 = \sqrt{G^2 - L^2} \sin l, \quad M_2 = \sqrt{G^2 - L^2} \cos l, \quad M_3 = L,$$

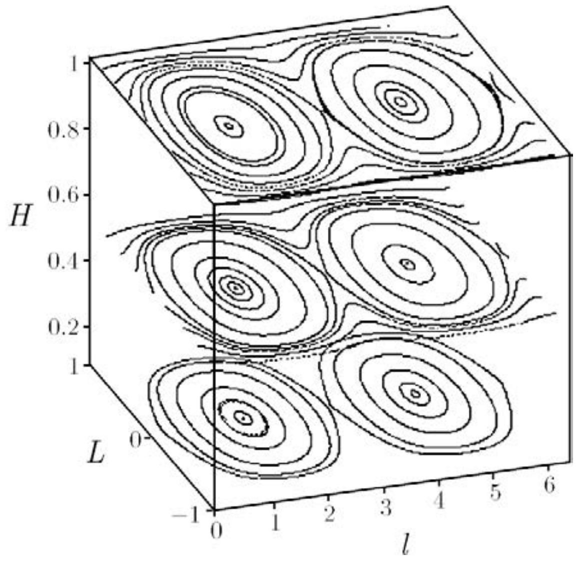
$$\gamma_1 = \left(\frac{H}{G}\sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G}\sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \sin l + \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \cos l$$

$$\gamma_2 = \left(\frac{H}{G}\sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G}\sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \cos l - \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \sin l$$

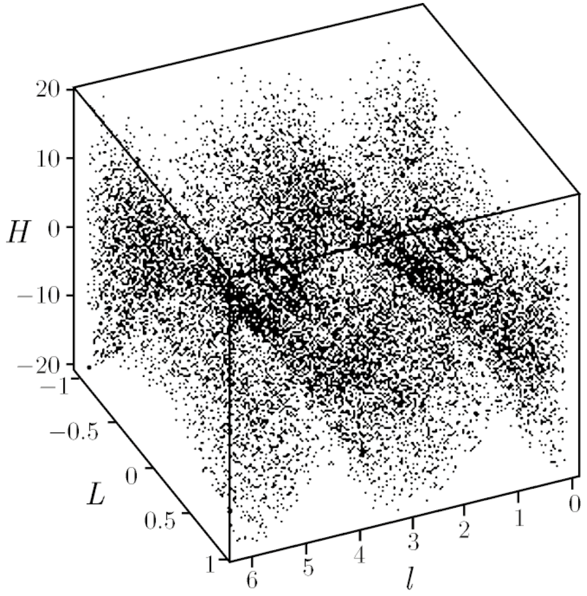
$$\gamma_3 = \left(\frac{H}{G}\right)\left(\frac{L}{G}\right) - \sqrt{1 - \left(\frac{L}{G}\right)^2}\sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g$$

in which we can express the energy $E = E(L, G, H, l, g)$.

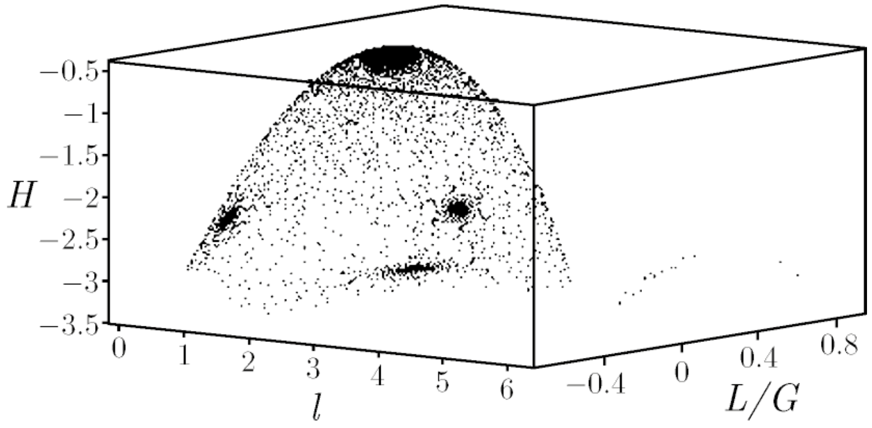
We fix the level of energy $E = E_0$, then choose the intersecting plane, for example, as $g = g_0 = \text{const}$, and obtain the three-dimensional map. induced by sequential intersections of the phase trajectory with the chosen intersecting plane.



The three-dimensional map for the case of Chaplygin ball. The figure shows very clearly that all trajectories are situated on joint level surfaces of two integrals.



Some trajectories in the problem of rolling of ellipsoid with the spherical tensor of inertia on plane. A random layer (which is obtained from one trajectory) in this case is not situated on any surface.



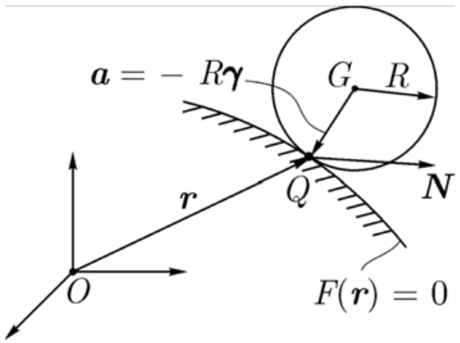
One of trajectories in the problem of rolling of unbalanced ball on plane. The figure shows clearly that all points are situated on some surface; the condensations of points correspond to asymptotic approximations of the trajectory to periodic solutions. The trajectory goes out from the top and approaches to the three points in lower part of surface

Hierarchy of nonholonomic dynamics

Table 1. Rolling of body on plane

tensor of inertia	dynamically nonsymmetric case $I_1 \neq I_2 \neq I_3 \neq I_1$			axial dynamical symmetry $I_1 = I_2$, $U = U(\gamma_3)$			total dynamical symmetry $I_1 = I_2 = I_3 = \mu$
surface of body	ball		ellipsoid	an arbitrary body of revolution	round disk with sharp edge	unbalanced ball	arbitrary
geometrical and dynamical restrictions	the center of mass coincides with the geometrical center	the center of mass does not coincide with the geometrical center	the axes of dynamical and geometrical ellipsoid coincide $\mathbf{I} = \mu\mathbf{E} + m\mathbf{B}$	the geometrical and dynamical axes coincide and contain the center of mass			—
measure	$(1 - D(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma}))^{-1/2}$	unknown	$(\mu + m\boldsymbol{r}^2)^{-1/2}$	$(I_1 I_3 + m(\boldsymbol{r}, \mathbf{I}\boldsymbol{r}))^{-1/2}$	const	$(I_1 I_3 + m(\boldsymbol{r}, \mathbf{I}\boldsymbol{r}))^{-1/2}$	$(\mu + m\boldsymbol{r}^2)^{3/2}$ (for variables $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$)
additional integrals	$\boldsymbol{M}^2 = \mathrm{const}$ $(\boldsymbol{M}, \boldsymbol{\gamma}) = \mathrm{const}$ (two integrals)	$\boldsymbol{M}^2 - m\boldsymbol{r}^2(\boldsymbol{M}, \boldsymbol{\omega}) = \mathrm{const}$ (one integral)	none of integrals are found	two integrals are obtained from the solution of system of two linear equations (2.13)	two integrals are obtained from the solution of hypergeometric equation (2.17)	$\omega_3/\rho = \mathrm{const}$ $(\boldsymbol{M}, \boldsymbol{r}) = \mathrm{const}$	none of integrals are found
integrable addition of gyrostat	possible (A. P. Markeev 1986)	it seems to be impossible	the measure is preserved	S. A. Chaplygin (1897)	S. A. Chaplygin (1897)	at $I_1 = I_2 = I_3$ the gyrostat was added by D. K. Bobylev, at $I_1 = I_2 \neq I_3$ by A. S. Kuleshov (2000)	the measure is preserved
Hamiltonian form	the system is Hamiltonian after the change of time (A. V. Borisov, I. S. Mamaev, 2001)	it seems that the system is not Hamiltonian	unknown	the reduced system is Hamiltonian after the change of time, defined by the reducing multiplier (A. V. Borisov, I. S. Mamaev, 2001)			unknown
authors	S. A. Chaplygin (1903)	A. V. Borisov, I. S. Mamaev (2001)	V. A. Yaroschuk (1995)	S. A. Chaplygin (1897)	S. A. Chaplygin (1897), P. Appell, D. Korteweg (1898)	E. J. Routh (1884), S. A. Chaplygin (1897)	V. A. Yaroschuk (1992.)
generalizations and remarks	the integrable addition of Brun field is possible (V. V. Kozlov, 1985) the Hamiltonian form is preserved for arbitrary fields with the loss of one integral	the Brun field can not be added (preserving the integral)	—	—	—	—	—

; Borisov A. V., Mamaev I. S., Kilin A. A., The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics, Regular and Chaotic Dynamics, 2002, vol. 7, no. 2, pp. 201-219



$$\dot{M} = D\dot{\gamma} \times (\omega \times \gamma) + M_F, \quad \dot{r} + R\dot{\gamma} = \omega \times R\gamma$$

$$M = \mu\omega + D\gamma \times (\omega \times \gamma), \quad \gamma = \frac{\nabla F(r)}{|\nabla F(r)|}$$

Table 1. The rolling of a ball on a surface

surface type	cylindrical surface	surface of the second order		surface of revolution			
		ellipsoid, hyperboloid, paraboloid	cone of the second order	arbitrary surface	ellipsoid, hyperboloid	paraboloid, cone, cylinder	sphere
measure	measure exists	$\rho = (\gamma, \mathbf{B}\gamma)^{-2}$	$\rho = \sqrt{(\mathbf{B}^{-1}r_c, \mathbf{B}^{-1}r_c)}$	$\rho = (f(\gamma_3))^3\Big(f(\gamma_3) - \frac{1-\gamma_3^2}{\gamma_3}f'(\gamma_3)\Big)$			
additional integrals	system is integrable by quadratures	$\frac{(\gamma\times M, \mathbf{B}^{-1}(\gamma\times M))}{(\gamma, \mathbf{B}\gamma)} = \text{const}$ (one integral)	$((M\times\mathbf{B}^{-1}r_c), \mathbf{B}^{-1}(M\times\mathbf{B}^{-1}r_c)) = \text{const}$ (one integral)	two linear integrals, defined by the system of linear equations	there exist two linear integrals that can be expressed in terms of elementary functions		
Hamiltonianity	nothing is known about the Hamiltonianity of these systems			upon change of time (prescribed by the reducing multiplier) the reduced system becomes Hamiltonian			
authors	A.V.Borisov, I.S.Mamaev, A.A.Kilin (2001)			E.Routh (1884)	A.V.Borisov, I.S.Mamaev, A.A.Kilin (2001)	E.Routh (1884)	
generalizations and remarks	integrable addition of the gravity field along a cylinder generatrix is possible			A.V.Borisov, I.S.Mamaev, and A.A.Kilin have shown the integrability in terms of elementary functions of the case of the ball rolling on the ellipsoid of revolution; have found an invariant measure for an arbitrary surface of revolution; for a paraboloid, a cone, and a cylinder they have shown the integrability of the case, when the ball is rolling on a surface rotating around its axis of symmetry			the problems of the rolling of a ball on an unconstrained and rotating sphere are also solved. System also allows integrable additions of potentials.

Borisov A. V., Mamaev I. S., Kilin A. A., The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics, Regular and Chaotic Dynamics, 2002, vol. 7, no. 2, pp. 201-219

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\omega} - m\boldsymbol{r} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}} + \lambda_o \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}.$$

The equations of motion in the local variables are represented in the form of the Chaplygin system

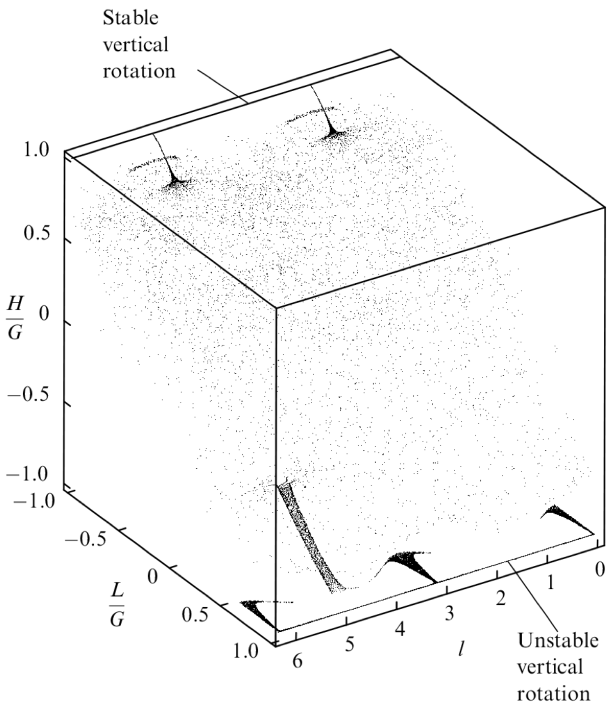
$$\left(\frac{\partial L}{\partial \dot{u}}\right)^{\cdot} - \frac{\partial L}{\partial u} = \dot{v} S, \quad \left(\frac{\partial L}{\partial \dot{v}}\right)^{\cdot} - \frac{\partial L}{\partial v} = -\dot{u} S$$

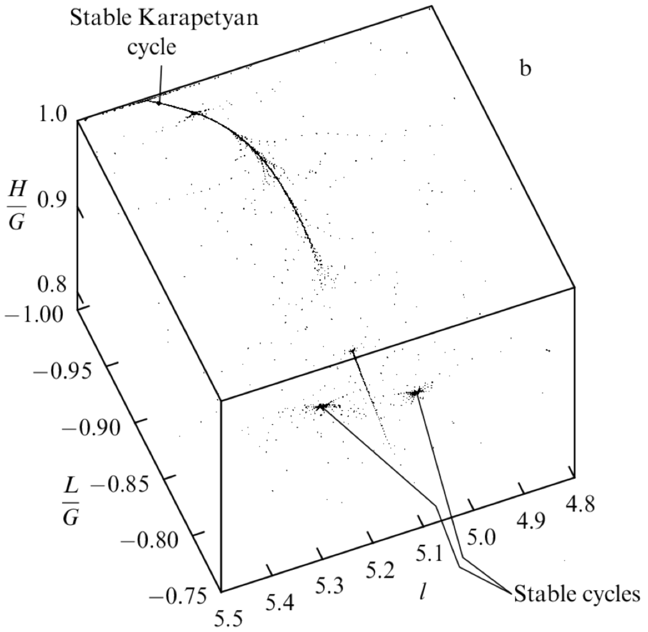
Table 1. The hierarchy of dynamics of a body rolling without spinning on a plane

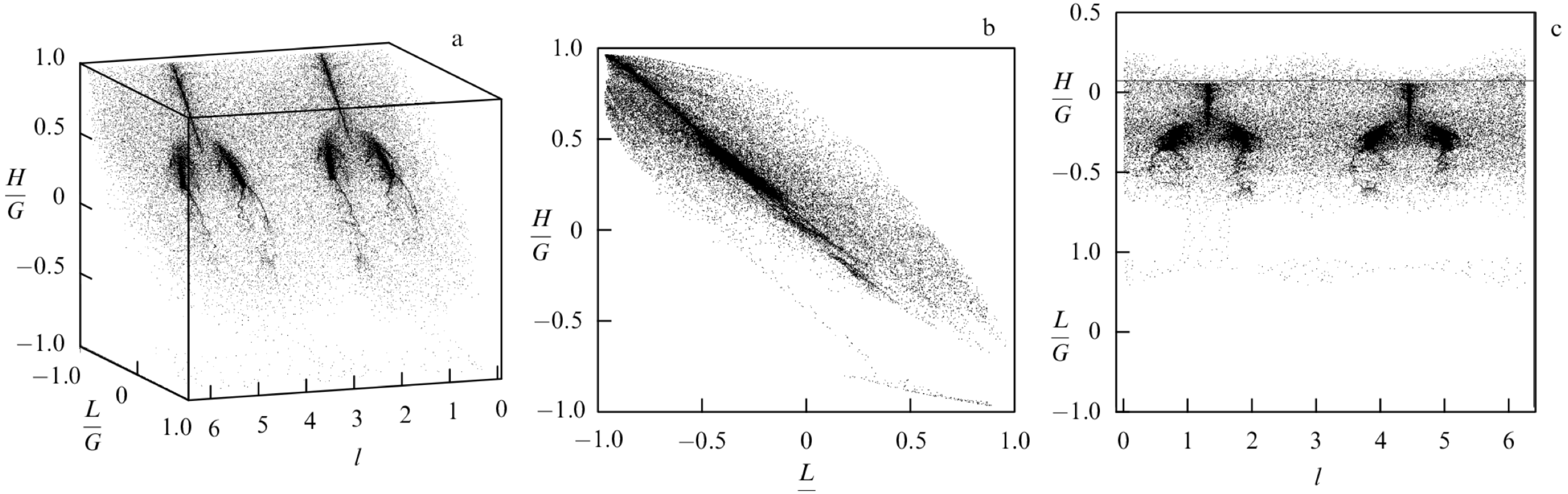
tensor of inertia	dynamically asymmetric case $I_1 \neq I_2 \neq I_3 \neq I_1$				axial dynamical symmetry $I_1 = I_2$, $U = U(\gamma_3)$	complete dymanical symmetry $I_1 = I_2 = I_3 = \mu$
surface of the body	ball		ellipsoid		body of revolution	arbitrary
geometrical and dynamical restrictions	the center of mass coincides with the geometrical center	the center of mass does not coincides with the geometrical center	geometrical and dynamical axes coincide $\mathbf{I}_k = m \det \mathbf{B}(\alpha + a_k^{-2}) \times (\beta + a_k^2)^{-1}$ $k = 1, 2, 3$		geometrical and dynamical axes coincide and contain the center of mass	—
			$\mathbf{I} \neq m\mathbf{B} + \mu\mathbf{E}$	$\mathbf{I} = m\mathbf{B} + \mu\mathbf{E}$		
measure	$(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma})^{-1/2}$ Borisov, Mamaev, 2005	does not exist	$\big(\chi_b + (\alpha\beta - 1)\det \mathbf{B}\big)^{1/2} \times \big((\boldsymbol{r}, \boldsymbol{\gamma})^2(\Delta_b + \beta \boldsymbol{r}^2) - \det \mathbf{B}\big)(\boldsymbol{r}, \boldsymbol{\gamma})^{-2} f^{-1}(\beta)$		$g_1(\gamma_3)^{1/2} g_2(\gamma_3)$ Borisov, Mamaev, 2008	$(\mu + m\boldsymbol{r}^2)(\mu + m(\boldsymbol{r}, \boldsymbol{\gamma})^2)^{1/2}$
additional integral	$(\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\gamma})$ Borisov, Mamaev, 2005	$(\widetilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\gamma}, \widetilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\gamma}) - 2mR(\boldsymbol{\gamma}, \boldsymbol{a})(\widetilde{\mathbf{I}}\boldsymbol{\omega}, \boldsymbol{\gamma})$ Borisov, Mamaev, 2008	does not exist in the general case[1]	$\dfrac{\det \widetilde{\mathbf{I}}}{(\mu + m\boldsymbol{r}^2)}(\boldsymbol{\omega}, \mathbf{B}^{-1}\boldsymbol{\omega})$	$g_1^{1/2}(\gamma_3)\omega_3$ Borisov, Mamaev, 2008	does not exist in the general case[1]
integrable addition of a gyrostat	possible; Borisov, Mamaev, 2005	unknown	unknown but the measure is preserved		possible — along the axis of dynamical symmetry	unknown
generalizations and remarks	integrable addition of the Bruns field is possible; $-\frac{\varepsilon}{4}(\mathbf{J}\boldsymbol{\gamma}, \boldsymbol{\gamma})$ Borisov, Mamaev, 2005					

Borisov A. V., Mamaev I. S., Bizyaev I. A., The Hierarchy of Dynamics of a Rigid Body Rolling without Slipping and Spinning on a Plane and a Sphere, Regular and Chaotic Dynamics, 2013, vol. 18, no. 3, pp. 277-328

Strange Attractors in Rattleback Dynamics

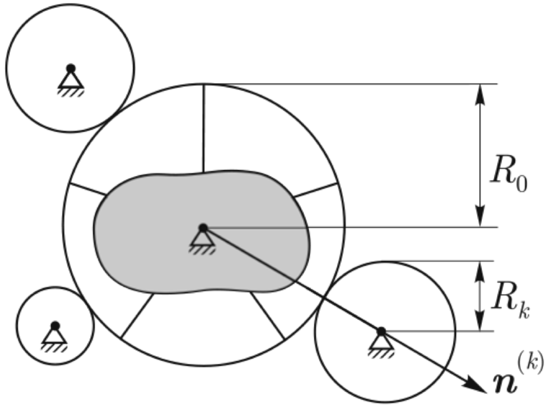






Borisov A. V., Mamaev I. S., Strange Attractors in Rattleback Dynamics, Physics-Uspekhi, 2003, vol. 46, no. 4, pp. 393-403

Borisov A. V., Kilin A. A., Mamaev I. S., Generalized Chaplygin's Transformation and Explicit Integration of a System with a Spherical Support, Regular and Chaotic Dynamics, 2012, vol. 17, no. 2, pp. 170-19

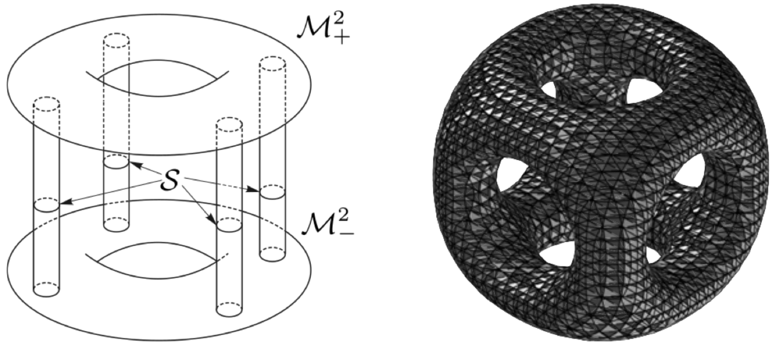


$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}$$

$$\boldsymbol{M} = (\mathbf{J} - \mathcal{D}_\alpha \boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \mathcal{D}_\beta \boldsymbol{\beta} \otimes \boldsymbol{\beta})\boldsymbol{\omega}$$

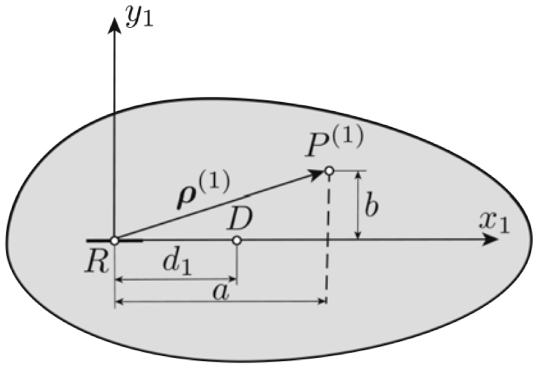
$$\mathbf{J} = \mathbf{I} + \sum_k \frac{\mu_k}{r_k^2}\mathbf{E} - \mathcal{D}_\gamma^{(0)}\mathbf{E}, \quad \mathcal{D}_\alpha = \mathcal{D}_\alpha^{(0)} - \mathcal{D}_\gamma^{(0)}, \quad \mathcal{D}_\beta = \mathcal{D}_\beta^{(0)} - \mathcal{D}_\gamma^{(0)}$$

Borisov A. V., Kilin A. A., Mamaev I. S., Invariant Submanifolds of Genus 5 and a Cantor Staircase in the Nonholonomic Model of a Snakeboard, International Journal of Bifurcation and Chaos, 2019, vol. 29, no. 3, 1930008, 19 pp.



(a) (b)

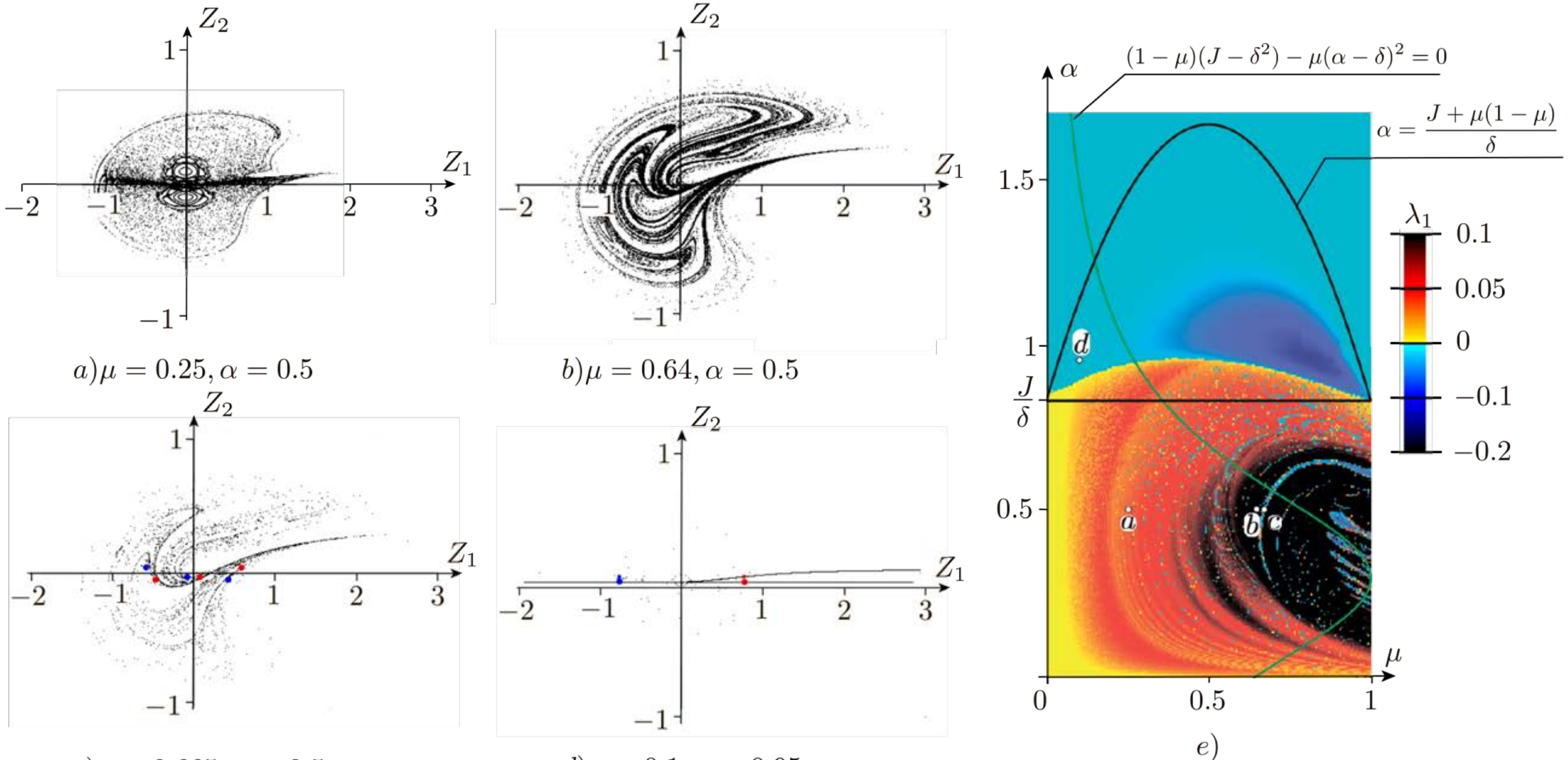
Bizyaev I. A., Borisov A. V., Mamaev I. S., The Chaplygin Sleigh with Parametric Excitation: Chaotic Dynamics and Nonholonomic Acceleration, Regular and Chaotic Dynamics, 2017, vol. 22, no. 8, pp. 955–975;



$$\frac{dZ_1}{d\tau} = \frac{(Z_2-\alpha\mu\cos\tau)\big(\delta(Z_2-\alpha\mu\cos\tau)+\mu\cos\tau\big(J+\mu(1-\mu)\sin^2\tau\big)\big)}{\big(J+\mu(1-\mu)\sin^2\tau\big)^2},$$

$$\frac{dZ_2}{d\tau} = -\frac{\delta(Z_2-\alpha\mu\cos\tau)Z_1}{J+\mu(1-\mu)\sin^2\tau},$$

$$\alpha = \frac{a}{b}, \quad \delta = \frac{c_1}{b}, \quad \mu = \frac{m_p^{(1)}}{m}, \quad J = \frac{I_s+m_p^{(1)}a^2}{mb^2}.$$



$a)\ \mu = 0.25, \alpha = 0.5$ $b)\ \mu = 0.64, \alpha = 0.5$

$c)\ \mu = 0.665, \alpha = 0.5$ $d)\ \mu = 0.1, \alpha = 0.95$ $e)$

We explore the dynamics of a mechanical multicomponent system with a nonholonomic constraint. The system consists of a platform, which slides on a horizontal plane like the Chaplygin sleigh. On this platform, n material points move according to a given law.
The most interesting problem in the dynamics of the nonautonomous Chaplygin sleigh is that of its speed-up (acceleration).
In this case, the problem reduces to investigating a reduced system of two first-order equations with periodic coefficients

$$\frac{dv}{d\tau} = f_2(\tau)u^2 + f_1(\tau)u + f_0(\tau), \quad \frac{du}{d\tau} = -uv + g(\tau)$$

where the coefficients are periodic functions of time τ with the same period.
In order to formulate the main result, we will need to define some average values for system. We recall that the average of a periodic function $h(\tau) = h(\tau + T)$ is given by

$$\langle h \rangle = \frac{1}{T}\int_0^T h(\tau)d\tau,$$

we define its periodic antiderivative with zero average $\tilde{h}(\tau)$ by the conditions. Set $c_1 = \langle f_0 \rangle, \; c_2 = \langle f_1 g \rangle, \; c_3 = \langle f_2 g^2 - f_1 g' - g f_1 \tilde{f}_0 \rangle$

Theorem. *Let* $c_k, k = 1, 2, 3$, *be the first nonzero coefficient. Assume that* $c_k > 0$. *Then for any* $a > 0$ *and a sufficiently small* $\varepsilon > 0$ *for any initial condition* $u(0), v(0)$ *in the region*

$$D = \{(u, v) : v \geqslant \varepsilon^{-1}, \; |u| \leqslant a\},$$

we have $v(\tau) \to +\infty$ *and* $u(\tau) \to 0$ *as* $\tau \to +\infty$ *in the form*

$$v(\tau) = (kc_k\tau)^{\frac{1}{k}} + o(\tau^{\frac{1}{k}}), \; u(\tau) = g(\tau)(kc_k\tau)^{-\frac{1}{k}} + o(\tau^{-\frac{1}{k}}).$$

Bizyaev I. A., Bolotin S. V., Mamaev I. S., Normal forms and averaging in an acceleration problem in nonholonomic mechanics, Chaos, 2021, vol. 31, 013132, 16 pp.

Topology and stability of integrable systems

Suppose that we have a Hamiltonian system with two degrees of freedom, which in canonical variables $\boldsymbol{q} = (q_1, q_2)$, $\boldsymbol{p} = (p_1, p_2)$ is given in the form

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}$$

where $H(\boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian. We assume that the system has an additional first integral $F(\boldsymbol{q}, \boldsymbol{p})$ and, consequently, is integrable. Henceforth we denote a point on a manifold and phase variables by x, in particular, $x = (\boldsymbol{p}, \boldsymbol{q})$ in this case. We recall the Liouville–Arnol'd theorem, which describes the behaviour of such systems in the general position situation.

Theorem. *Suppose that on a symplectic manifold* $\mathcal{M}^4$ *we are given a pair of functions H(x), F(x) in involution, that is,* $\{H, F\} = 0$
Then equations can be integrated by quadratures. Let $\mathcal{M}_{h,f} = \{x \in \mathcal{M}^4 \mid H = h, F = f\}$ *be a common level of the first integrals.*
If the functions H and F are independent on $\mathcal{M}_{h,f}$*, then*
1) $\mathcal{M}_{h,f}$ *is a smooth manifold that is invariant under the phase flow of the system;*
2) every compact connected component of the surface $\mathcal{M}_{h,f}$ *is diffeomorphic to a two-dimensional torus;*
3) in a neighbourhood of a compact connected component of $\mathcal{M}_{h,f}$ *action-angle variables* $\boldsymbol{I} = (I_1, I_2)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ *can be chosen such that* $H(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{I})$ *and the system (1) is represented in the form*

$$\dot{\mathbf{I}} = \frac{\partial H}{\partial \varphi} = 0, \quad \dot{\varphi} = \frac{\partial H}{\partial \mathbf{I}} = \omega(\mathbf{I})$$

Our goal: *to find periodic solutions (closed trajectories) of the system (1) and investigate their (orbital) stability.*

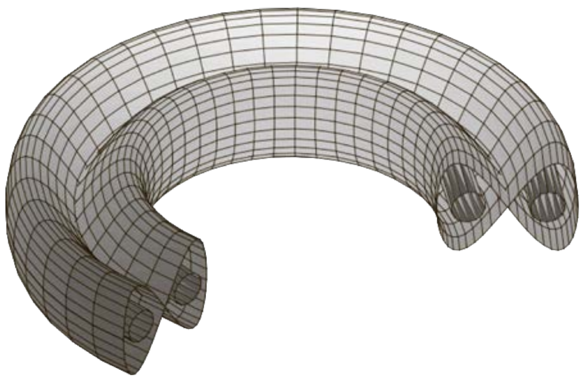
Bolsinov A. V., Borisov A. V., Mamaev I. S., Topology and stability of integrable systems, Russian Mathematical Surveys, 2010, vol. 65, no. 2, pp. 259–318

We denote by $\Phi = (H, F) : \mathcal{M}^4 \to R^2$ the integral map of an integrable system: $\boldsymbol{x} \to \Phi(\boldsymbol{x}) = (H(\boldsymbol{x}), F(\boldsymbol{x})) \in R^2$
In some papers this map is also called the energy–momentum map, or simply the momentum map. We define a number of objects that play important roles in the study of periodic solutions:
1) the region of possible motion (RPM) $\Phi(\mathcal{M})$ is the full image of the phase space on the plane of first integrals (h, f) (to each point $(h, f) \in \Phi(\mathcal{M})$ there corresponds the integral manifold $\mathcal{M}_{h,f} = \{x \in \mathcal{M}^4 \mid H = h, F = f\}$ of the system, which, generally speaking, can contain several connected components);
2) the set S of critical points of the integral map, $S = \{\mathbf{x} \in \mathcal{M} \mid \operatorname{rank} d\Phi(\mathbf{x}) < 2\}$
3) the bifurcation set Σ of the first integrals — the image $\Sigma = \Phi(S)$ of the set of critical points, that is, the values of the first integrals corresponding to the critical points of the integral map.

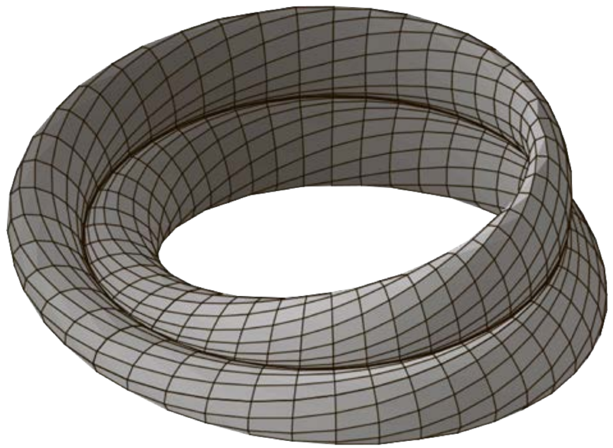
Definition. The bifurcation diagram of an integrable system is defined to be the region of possible motion $\Phi(\mathcal{M})$ depicted on the plane of first integrals (h, f) together with the image Σ of the critical set and the indication of the images Σ_0 and Σ_1.

Theorem. *Consider a one-parameter family of closed critical trajectories. Suppose that this family is isolated in the sense that there are no other critical points of the integral map in a neighbourhood of this family. Suppose further that this family is mapped onto some individual branch of the bifurcation diagram Σ which can be given as the graph of some smooth function $F = f(H)$.*
Suppose that at least one trajectory in the family is non-degenerate. Then the following hold:
1) almost all the trajectories of the family are non-degenerate and have one and the same type (either elliptic or hyperbolic);
2) if one of the trajectories of the family has elliptic type, then all the trajectories of the family (both degenerate and non-degenerate) are stable;
3) on the other hand, if at least one of the trajectories has hyperbolic type, then all the trajectories of the family (both degenerate and non-degenerate) are unstable.

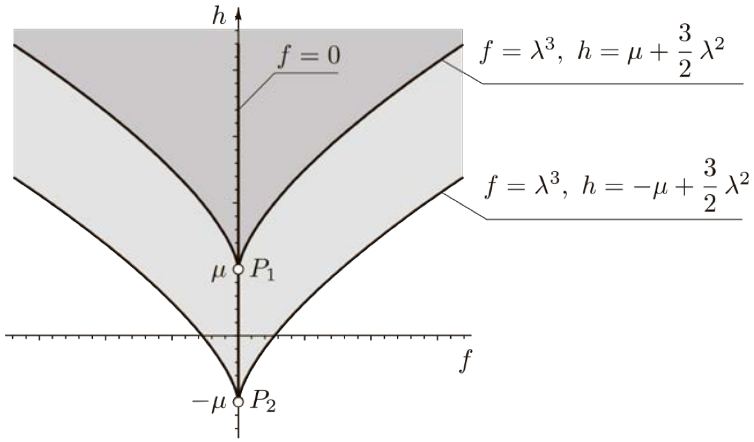
Proposition. *If some family of periodic trajectories satisfies the hypotheses of Theorem and corresponds to a branch of the bifurcation diagram that lies on the boundary of the RPM, then all the trajectories in this family are stable.*



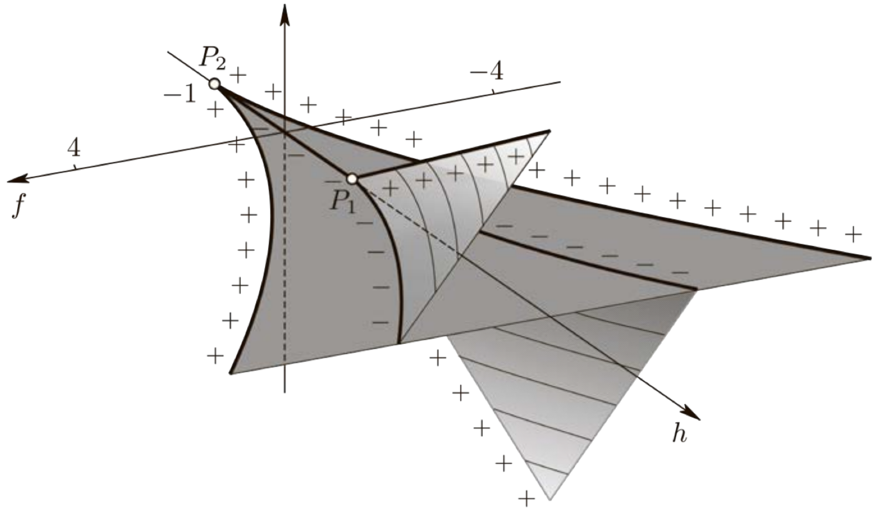
Liouville tori and a singular manifold adjoining an unstable periodic solution.



Singular leaf containing a hyperbolic trajectory with nonorientable separatrix diagram.



The bifurcation diagram of the Goryachev–Chaplygin case. The darker colour marks the domain in which to each point there corresponds a pair of invariant tori.



The bifurcation complex of a Goryachev–Chaplygin top. The symbol + indicates stable critical periodic solutions, and the symbol − indicates unstable ones.

The Gaffet system is a relatively new interesting class of integrable systems of astrophysical nature, which was found by Gaffet and since then has been investigated in detail analytically in his numerous papers. All our methods of analysis are applicable to this reduced system. In fact, in this case they have no alternative, since the explicit solution obtained by Gaffet has an extremely complicated form, from which it is impossible to extract any information about the dynamics of the system. Our approach enables one to qualitatively describe the dynamics of the system and to investigate its stability. This system describes the evolution of relative sizes of an expanding gaseous ellipsoid filled with a monatomic ideal gas. The Hamiltonian in this case has the form

$$H = \frac{1}{2}\boldsymbol{M}^2 + \frac{3}{2}\frac{a}{(\gamma_1\gamma_2\gamma_3)^{2/3}} + \frac{c^2}{(\gamma_1-\gamma_2)^2} + \frac{c^2}{(\gamma_1+\gamma_2)^2}$$

where the quantities γ_i are expressed in terms of the principle semi-axes A_i of the ellipsoid by the formulae $\gamma_i = A_i/\sqrt{\Sigma A_i^2}$. Thus, the system is defined in the first quadrant of the sphere: $\{\boldsymbol{\gamma}^2 = 1 \mid \gamma_i > 0, i = 1,2,3\}$, and for $c \neq 0$ it is necessary to exclude the diagonal $\gamma_1 = \gamma_2$. Everywhere inside the domain of definition the energy level $H = h$ is a compact three-dimensional manifold.
The system admits the integral of degree six

$$F_6 = (F_3 + F_c)^2 + 4\Phi\left(3a + G\frac{\gamma_1^2}{\gamma_3^2}\right)\left(3a + G\frac{\gamma_2^2}{\gamma_3^2}\right)$$

$$F_3 = M_1M_2M_3 - 3a(\gamma_1\gamma_2\gamma_3)^{1/3}\sum\frac{M_i}{\gamma_i}, \quad F_c = \frac{4c^2\gamma_1\gamma_2\gamma_3^2}{(\gamma_1^2-\gamma_2^2)^2}M_3,$$

$$\Phi = 4c^2(\gamma_1\gamma_2\gamma_3)^{2/3}\frac{\gamma_3^2}{(\gamma_1^2-\gamma_2^2)^2}, \quad G = (\gamma_1\gamma_2\gamma_3)^{2/3}\frac{M_1M_2}{\gamma_1\gamma_2} + \Phi - 3a.$$

Furthermore, it is also necessary to take into account that the possible values of θ_0 must satisfy the requirement $p_\psi^2(\theta_0) \geqslant 0$. Calculating the corresponding values of the first integrals, we find two curves on the bifurcation diagram in the form

$$\sigma_+ : (f_+(\theta_0), h_+(\theta_0)), \quad \theta_0 \in [\theta_*, \theta_1] \qquad \sigma_- : (f_-(\theta_0), h_-(\theta_0)), \quad \theta_0 \in \left[\theta_*, \frac{\pi}{2}\right)$$

$$h_\pm = \frac{\tilde{a}^2}{2} \frac{1 \pm \sqrt{4 - 3(1 - 3\cos^2\theta_0)^2}}{(\sin^2\theta_0 \cos\theta_0)^{2/3} \cos^2\theta_0 (3\cos^2\theta_0 - 2)} \qquad f_\pm = \frac{p_+}{\eta} \frac{(1 - 3\cos^2\theta_0)(1 - 2\cos^2\theta_0) \pm \sqrt{4 - 3(1 - \cos^2\theta_0)}}{(\sin^2\theta_0 \cos\theta_0)^{2/3}}$$

so that the bifurcation set $\sum_1$ is $\sigma_+ \cup \sigma_-$, where $\theta_* = \arccos(1/\sqrt{3})$ and $\theta_1 = \arccos(\sqrt{3} + 2\sqrt{3}/2)$ (the expression under the radical sign in formula (29) vanishes at this point). The corresponding bifurcation diagram is given in Fig. 17, and the typical form of the corresponding periodic solutions in Fig. 18.

From the physical viewpoint, a spherically symmetric expansion of the gaseous cloud corresponds to the point P_0, and an expansion of the domain such that its relative sizes change periodically corresponds to solutions on the branches of the diagram.

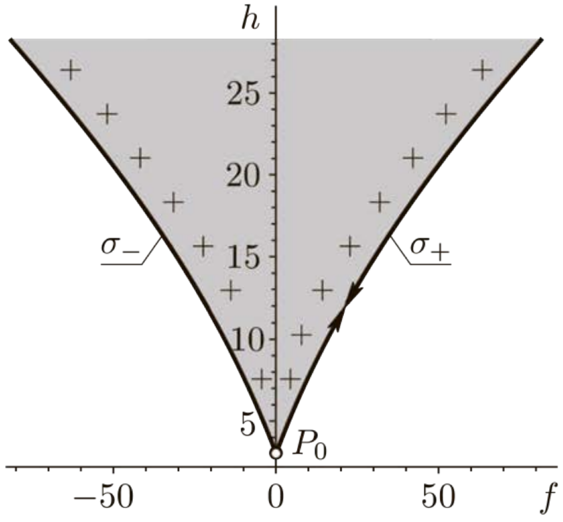


Figure 17.

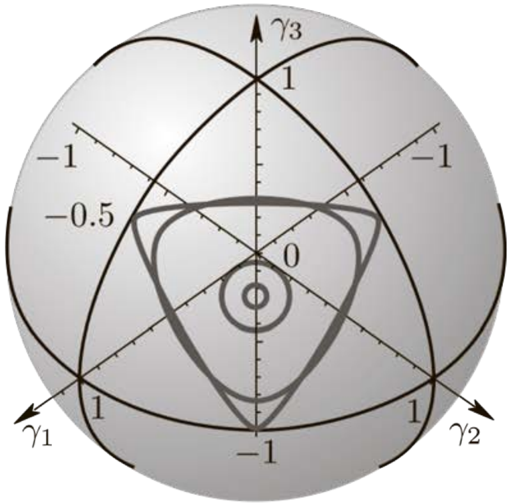


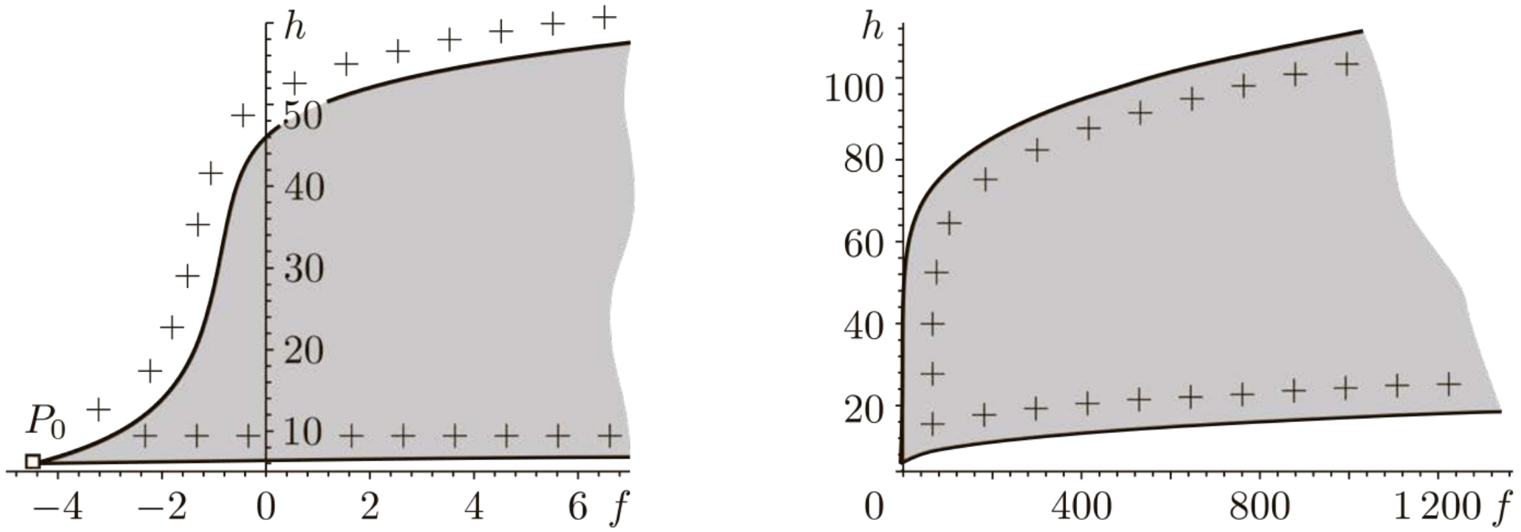
Figure 18.

The case $c \neq 0$ (integral of degree six). We again use the canonical variables; here the potential of the system has a singularity (becomes infinite) on the meridian $\varphi = \pi/2$ (that is, $\gamma_1 = \gamma_2$). Because the system is invariant under mirror reflections with respect to the meridian $\varphi = \pi/2$, it is sufficient in this case to consider the domain $\theta \in (0, \pi/2), \varphi \in (0, \pi/2)$.

We perform the change of variables

$$R = \cos^2\theta, \quad S = \left(\sin\psi\cos\theta\sin^2\theta\right)^{1/3},$$

which reduces the subsequent equations to an algebraic form, and then the region of possible motion of the system is bounded by the line segment $S = 0, R \in (0, 1)$ and the curve $R(1 - R)^2 - S^3 = 0$ intersecting the straight line $S = 0$ in the points $R = 0$ and $R = 1$.



The Bifurcation Analysis and the Conley Index in Mechanics

Consider a system whose Hamiltonian $H(\boldsymbol{x},\alpha)$ is a smooth function on $\mathbb{R}^n$ depending on a parameter $\alpha \in [-1,1]$. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be its nondegenerate singular point at $\alpha = -1$. What happens to this point under a change of the parameter? It is well known that under a small variation of the parameter it remains nondegenerate and its Morse index does not change (the point itself can, of course, slightly change its location). Moreover, as long as it remains nondegenerate, its index remains the same. This easily follows from the continuity argument and the implicit function theorem. The following natural question arises:

can the Morse index of the point change during a passage through degeneration, and if it can, what does this change imply?

Theorem. Let $H(\boldsymbol{x},\alpha)$ be a smooth function on $\mathbb{R}^n$ smoothly depending on a parameter $\alpha \in [-1,1]$. and let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be an isolated singular point of $H(\boldsymbol{x},\alpha)$ for each α . Let $\boldsymbol{x}$ be nondegenerate at $\alpha \neq 0$ and let its Morse index change as α passes through zero. Then in an arbitrarily small neighborhood of $\boldsymbol{x}_0$ there exist other singular points for some values of α.

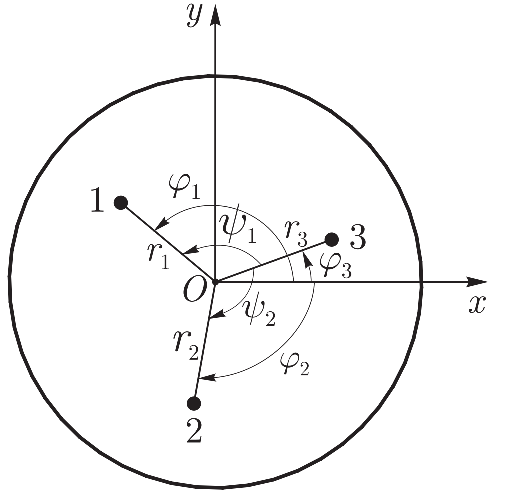
Many problems in mechanics often require to find and analyze the equilibria of Hamiltonian systems depending on some parameter.
In fact, this reduces to analysis of critical points of the Hamiltonian $H(\boldsymbol{x},\alpha)$, which is a smooth (analytical) function of phase variables $\boldsymbol{x}_0 \in \mathbb{R}^n$ and the parameter α . One of the most common problems is to describe the relative equilibria of a Hamiltonian system possessing a cyclic integral (see the following section for more details). In this case one needs to find critical points of the Hamiltonian reduced by the action of the integral, where α is the value of the cyclic integral. As a rule, for almost all values of α the critical points turn out to be nondegenerate and form one-parameter families $\boldsymbol{x}^{(i)}(\alpha), i = 1,\ldots,m.$ Note that on the plane of values of the first integrals $\mathbb{R}^2 = \{(\alpha, H)\}$ one can construct in a natural way the bifurcation curves σ_i corresponding to the singular points. These curves σ_i are given as graphs of the functions

$$h_i(\alpha) = H(\boldsymbol{x}^{(i)}(\alpha),\alpha).$$

After calculating the Morse indices of the Hamiltonian for the corresponding families of singular points $\boldsymbol{x}^{(i)}(\alpha)$, we can place them on the corresponding curves σ_i and track their change under variation of α . This diagram is called the *bifurcation diagram of the system*.

Bolsinov A. V., Borisov A. V., Mamaev I. S., The Bifurcation Analysis and the Conley Index in Mechanics, Regular and Chaotic Dynamics, 2012, vol. 17, no. 5, pp. 457-478

As an example, we consider a system describing the dynamics of three equal point vortices in a circular domain on a plane. Let us choose the origin of a fixed coordinate system O to coincide with the center of a circle, assume the vortex intensities to be equal to 1 and let their position be given by polar coordinates r_k, φ_k . Then the equations of motion are expressed in Hamiltonian form



$$\dot{r}_k = \{r_k, H\} = \frac{1}{r_k}\frac{\partial H}{\partial \varphi_k}, \quad \dot{\varphi}_k = \{\varphi_k, H\} = -\frac{1}{r_k}\frac{\partial H}{\partial r_k}, \quad k = 1, 2, 3,$$

$$H = -\frac{1}{4\pi}\sum_{k<j}^{3} \ln \frac{r_k^2 + r_j^2 - 2r_k r_j \cos(\varphi_k - \varphi_j)}{R^4 + r_k^2 r_j^2 - 2R^2 r_k r_j \cos(\varphi_k - \varphi_j)} + \frac{1}{4\pi}\sum_{k=1}^{3} \ln(R^2 - r_k^2)$$

The equations of motion for vortices admit an additional first integral of motion – the *moment of vorticity*, which in this case can be written as

$$I = \frac{1}{2}\sum_{k=1}^{3} r_k^2$$

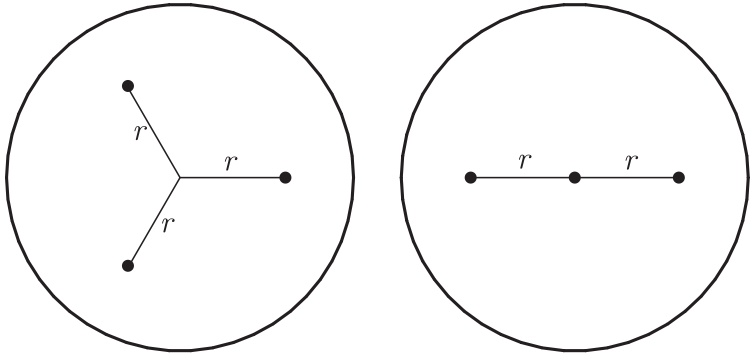
To find relative equilibria, we carry out a reduction by symmetry; to do so, we pass to the new variables $\rho_k, \psi_k, k = 1, 2, 3, I = \rho_3$ using the formulae

$$\psi_1 = \varphi_1 - \varphi_3, \quad \psi_2 = \varphi_2 - \varphi_3, \quad \psi_3 = \varphi_3, \quad \rho_1 = r_1^2/2, \quad \rho_2 = r_2^2/2, \quad \rho_3 = I = (r_1^2 + r_2^2 + r_3^2)/2,$$

where $\psi_k \in (-\pi, \pi), \; k = 1, 2, 3$, are the angle variables.

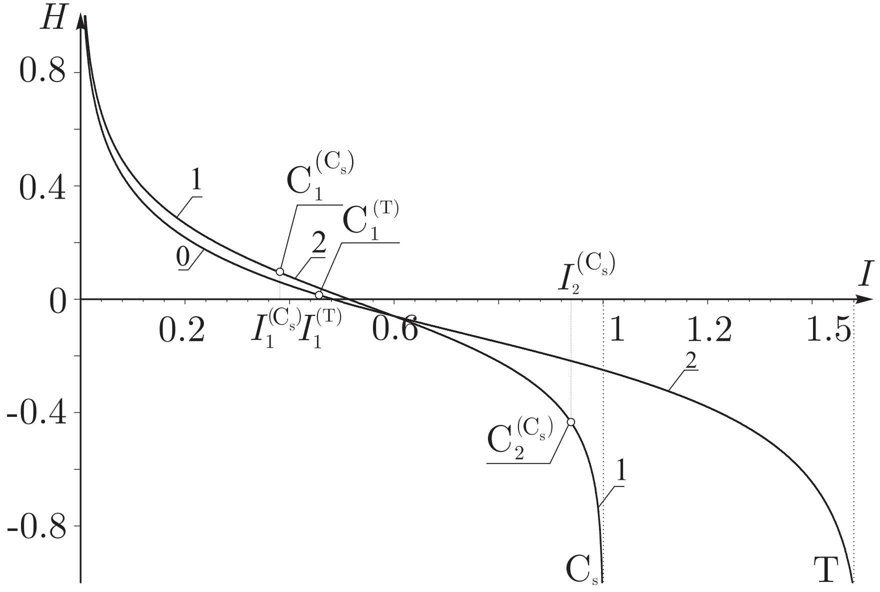
Two stationary configurations of three vortices in a circular domain are well known:

- *equilateral triangle (Thomson's configuration);*
- *symmetric collinear configuration.*



(a) Thomson configuration — T

(b) collinear symmetric configuration — C_s

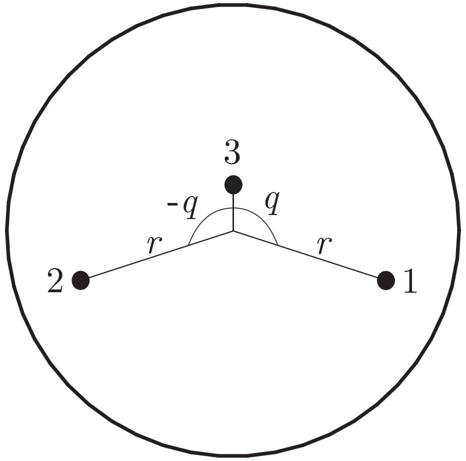


A natural question arises: do these configurations exhaust all possible relative equilibria in this system ?

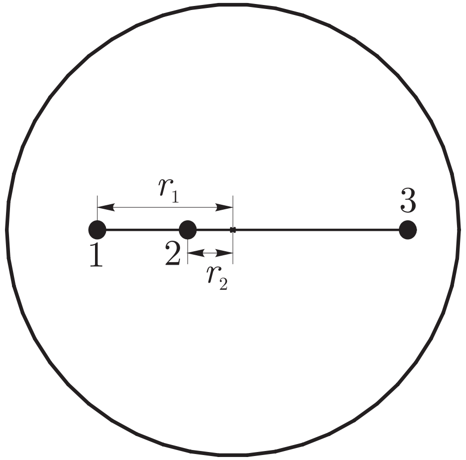
From Figure, where the bifurcation curves T and C_s for Thomson's and collinear configurations are shown, it can be seen that
1) the curves T and C_s intersect transversally,
2) on the bifurcation curves T and C_s there are isolated points at which there occurs a change of the index of the critical points of the Hamiltonian which correspond to these curves.
Using the results of the previous section, we can draw the following conclusions:
1) in the phase space, the critical points corresponding to different families T and C_s, are isolated from each other for all values of the parameter I (since for nonisolated families the bifurcation curves at the point of merging or intersection are tangent to each other);
2) by Theorem of an isolated family, at the points of change of the index there must arise new critical points corresponding to hitherto unknown configurations.

We show that in this case the second scenario takes place, i.e. new stationary configurations are born under the change of the index.

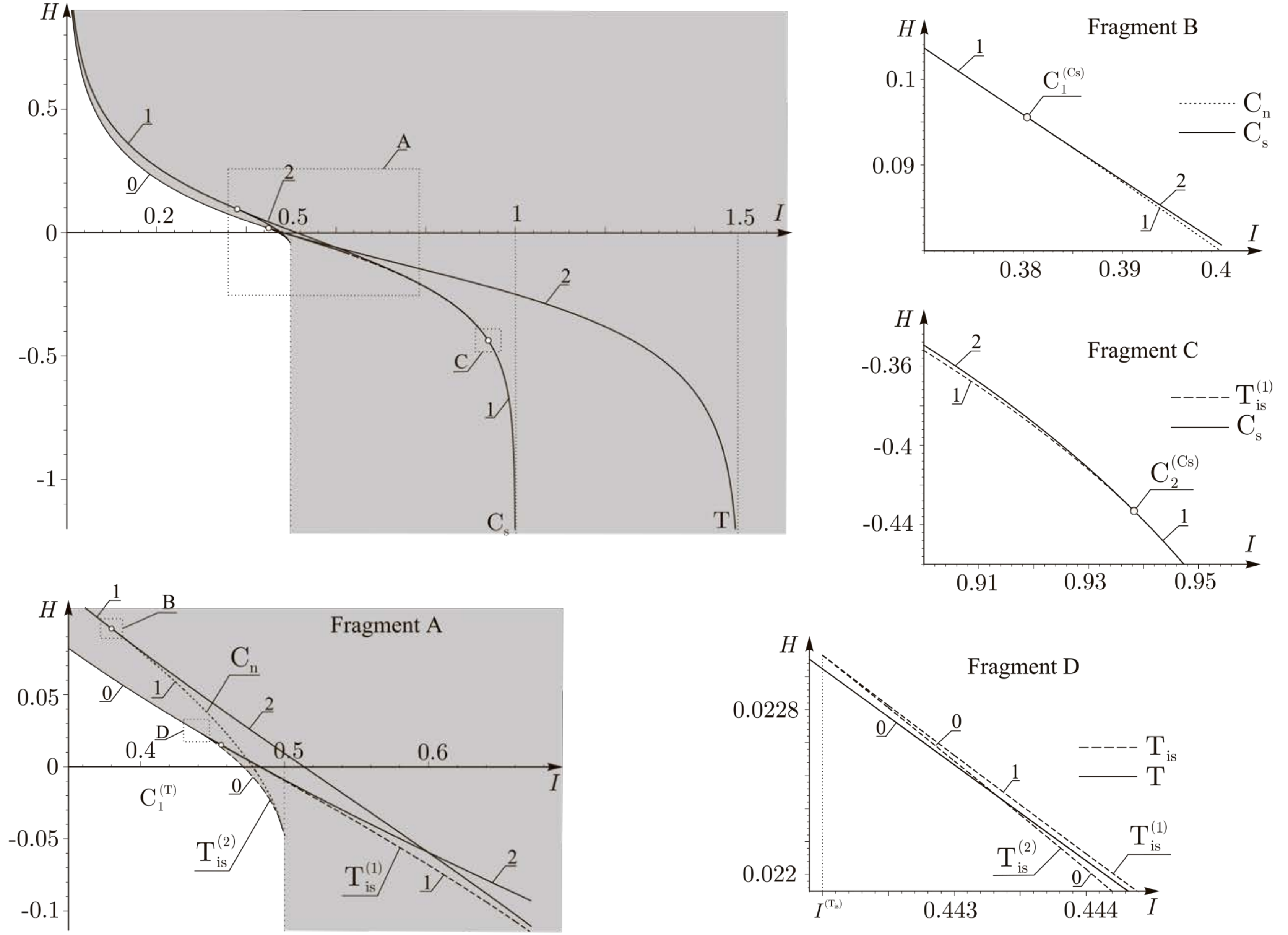
It is clear that as the parameters change, the sought-for new configurations should tend to Thomson's or collinear symmetric configurations in such a way that the corresponding bifurcation curves on the plane I, H merge with the curves T and C_s at the points of change of the index. In addition, it is natural to expect that for new configurations the symmetry will decrease but not disappear completely. Possible configurations which satisfy these requirements are isosceles and collinear asymmetric configurations.



(a) isosceles configuration — T_{is}



(b) collinear assymmetric configuration — C_n


H
I
0.5
0
-0.5
-1
0.2
0.5
1
1.5
A
C
C_s
T
Fragment B
0.1
0.09
0.38
0.39
0.4
C_1^(Cs)
C_n
C_s
Fragment C
-0.36
-0.4
-0.44
0.91
0.93
0.95
T_is^(1)
C_s
C_2^(Cs)
Fragment A
B
C_n
D
0.05
-0.05
-0.1
0.4
0.5
0.6
C_1^(T)
T_is^(2)
T_is^(1)
Fragment D
0.0228
0.022
0.443
0.444
I^(T_is)
T_is
T
T_is^(2)
T_is^(1)