Transverse foliations in the Euler problem of two fixed centers

Geometry, Dynamics and Mechanics Seminar

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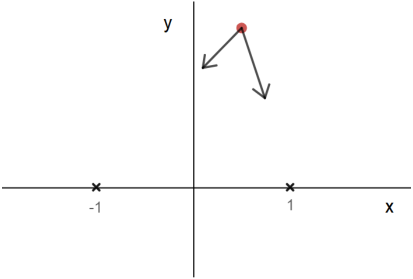
The Euler problem of two centers in the plane



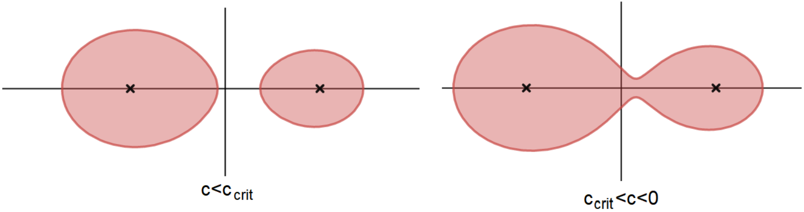
- In canonical coordinates:

$$H_\mu = \frac{p_x^2 + p_y^2}{2} - \frac{\mu}{\sqrt{(x+1)^2 + y^2}} - \frac{1-\mu}{\sqrt{(x-1)^2 + y^2}}$$

- mass ratio: $0 < \mu < 1$.
- Euler (1760): integral of motion

$$G = \frac{1}{2}(xp_y - yp_x)^2 + \frac{p_x^2}{2} + x\frac{\mu}{\sqrt{(x+1)^2 + y^2}} - x\frac{1-\mu}{\sqrt{(x-1)^2 + y^2}}$$

Hill's regions



$$H_\mu = c \Rightarrow V(x,y) \le c$$

$$c_{\mathrm{crit}} = -\frac{1}{2} - \sqrt{\mu - \mu^2}.$$

- $c < c_{\mathrm{crit}}$: trajectories stay close to a center.
- $c_{\mathrm{crit}} < c < 0$: transit trajectories.
- $c > 0$: the typical trajectory is unbounded.

Some related models in Celestial Mechanics

- circular planar restricted three-body problem

$$H_\mu = \frac{p_x^2 + p_y^2}{2} + (x-\mu)p_x - yp_x - \frac{\mu}{\sqrt{(x-1)^2 + y^2}} - \frac{1-\mu}{\sqrt{x^2+y^2}}$$

- Hill's lunar problem

$$H = \frac{p_x^2 + p_y^2}{2} + yp_x - xp_y - \frac{1}{\sqrt{x^2+y^2}} - x^2 + \frac{y^2}{2}.$$

- Rotating Kepler problem

$$H = \frac{p_x^2 + p_y^2}{2} + yp_x - xp_y - \frac{1}{\sqrt{x^2+y^2}}.$$

- Harmonic oscillator

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2+y^2}{2}.$$

- Hénon-Heiles system

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2+y^2}{2} + x^2 y - \frac{y^3}{3}.$$

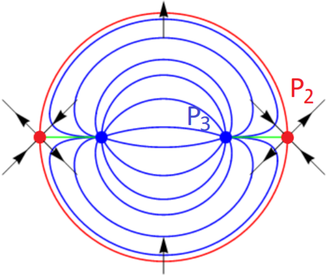
Reeb flows

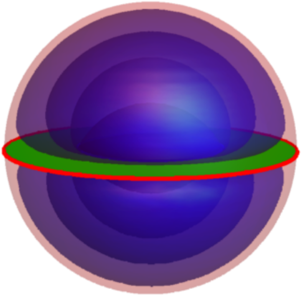
- M- compact, connected 3-manifold.
- λ - contact form on M:
$$\lambda \wedge d\lambda \neq 0.$$
- $\xi = \ker \lambda \subset TM$ contact structure.
- Reeb vector field R
$$d\lambda(R, \cdot) = 0 \quad \text{and} \quad \lambda(R) = 1.$$
- The Reeb flow parametrizes Hamiltonian flows on contact-type energy levels.

Transverse foliations for Reeb flows

A transverse foliations consists of a singular foliation of M adapted to the Reeb flow:

- A link $L \subset M$ formed by finitely many closed orbits.
- a smooth foliation of $M \setminus L$ by properly embedded surfaces transverse to the flow. The closure of each leaf is a compact surface with boundary contained in L.





Why transverse foliations?

- some order inside chaos.
- two-dimensional reduction.
- existence and multiplicity of periodic orbits.
- symbolic dynamics.
- the dynamics of transit trajectories.
- homoclinics to the Lyapunoff orbit.

Pseudo-holomorphic curves

- Let $J : \xi \to \xi$ be a $d\lambda$-compatible complex structure.
- Consider the $\mathbb{R}$-invariant almost complex structure $\widetilde{J}$ on the symplectization $\mathbb{R} \times M$ so that:

$$\widetilde{J}|_{\xi} = J \quad \text{and } \widetilde{J} \cdot \partial_a = R,$$

  where ∂_a is the unit vector in the $\mathbb{R}$-direction.
- $\tilde{u} = (a, u) : (\dot{\Sigma}, i) \to (\mathbb{R} \times M, \widetilde{J})$ is pseudo-holomorphic:

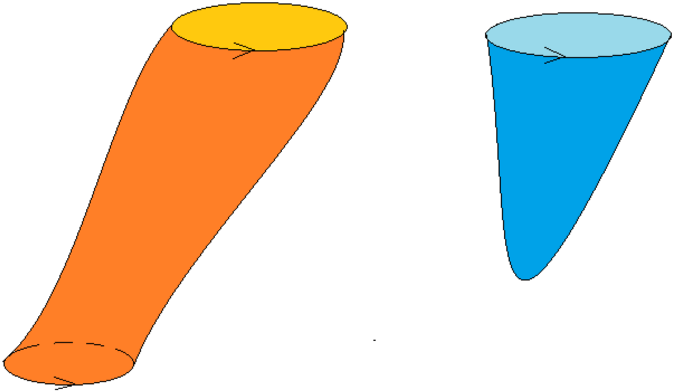
$$d\tilde{u} \circ i = \widetilde{J}(u) \circ d\tilde{u}.$$

- Finite Hofer's energy

$$0 < E(\tilde{u}) := \sup_{\varphi} \int_{\dot{\Sigma}} \tilde{u}^* d(\varphi(a)\lambda) < +\infty,$$

  where $\varphi : \mathbb{R} \to [0, 1]$ satisfies $\varphi' \geq 0$.

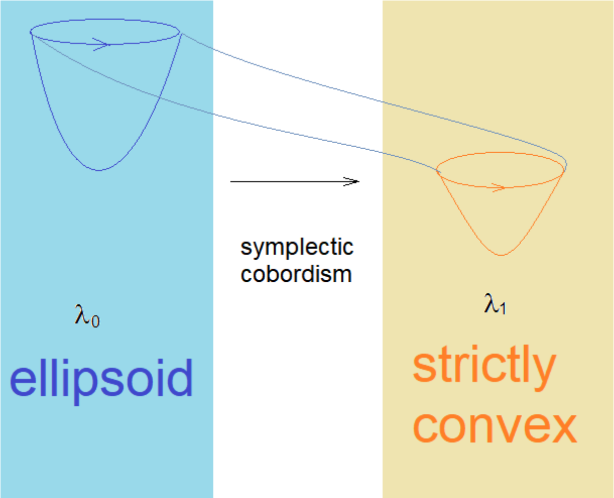
Periodic orbits at the ends of a finite energy curve



Why pseudo-holomorphic curves?

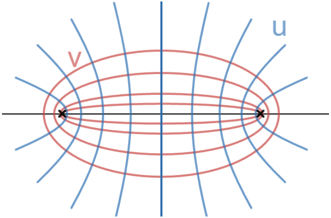
- pseudo-holomorphic curves can be found almost explicitly for many integrable Reeb flows.
- one can deform the pseudo-holomorphic curves to find periodic orbits, global surfaces of section and transverse foliations for Reeb flows far from integrable ones.

Symplectic cobordism



Hofer-Wysocki-Zehnder (Ann. of Math. 1998)

Back to the Euler problem: elliptic coordinates



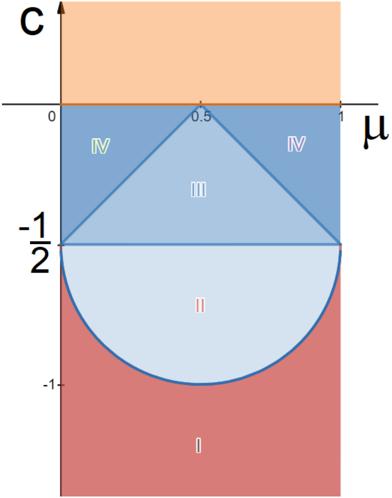
- New canonical coordinates: (u, v, p_u, p_v):

$$x = \cosh u \cos v$$
$$y = \sinh u \sin v$$
$$p_x dx + p_y dy = p_u du + p_v dv$$

- Regularized Hamiltonian:

$$K_{\mu,c} = (\cosh^2 u - \cos^2 v)(H_\mu - c)$$
$$= \frac{p_v^2}{2} + (2\mu - 1)\cos v + c\cos^2 v + \frac{p_u^2}{2} - \cosh u - c\cosh^2 u$$

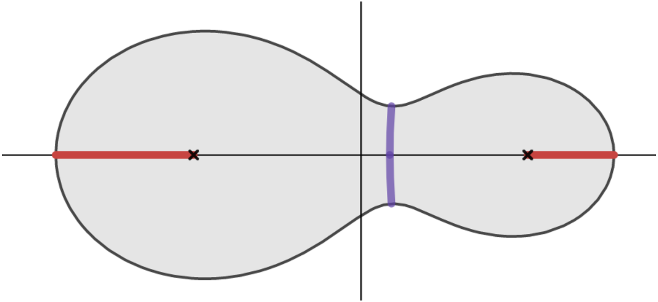
Energy x mass ratio



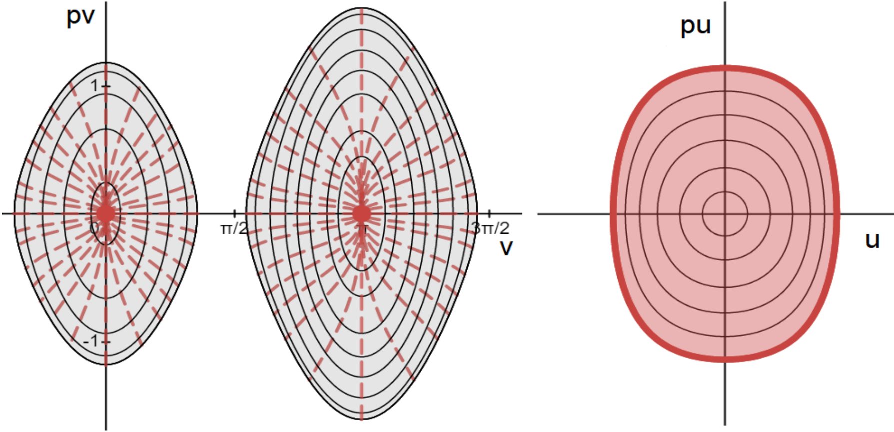
- II, III and IV - $c_{\text{crit}} < c < 0$ (high energy).
- I -$c < c_{\text{crit}} = -\frac{1}{2} - \sqrt{\mu - \mu^2}$ (low energy).

Main theorem

Theorem(de Paulo, Hryniewicz, Kim) For every $0 < \mu < 1$ and $c_{\mathrm{crit}} < c < 0$, the regularized energy surface $K_{\mu,c}(0)$ admits a transverse foliation, which is the projection of a finite energy foliation. The binding orbits are the Lyapunoff orbit and the exterior collision orbits.

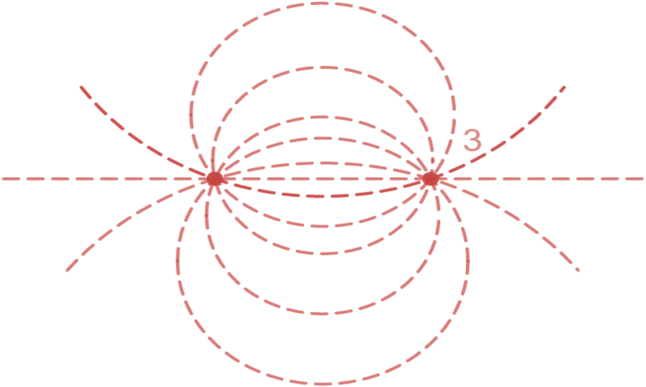


Low energy I: $c < c_{\text{crit}}(\mu)$



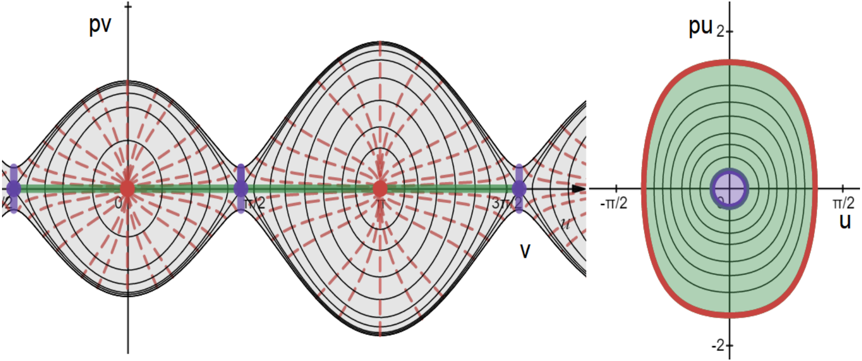
- 2 components diffeomorphic to S^3.
- disk-like global surface of section

Low energy I: $c < c_{\text{crit}}(\mu)$



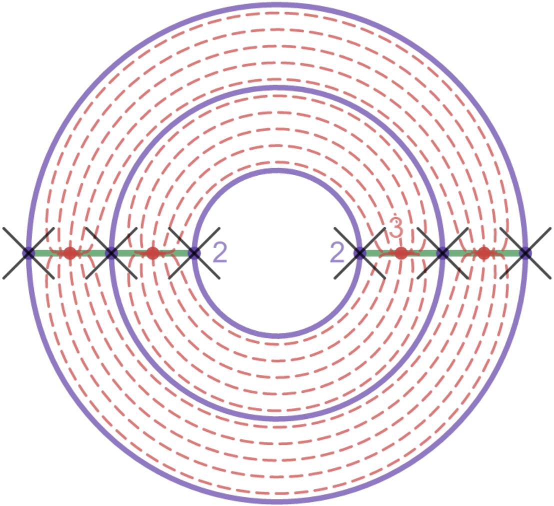
- Open book decomposition.
- Each page is a disk-like global surface of section.

High energy *II* : $c_{\mathrm{crit}} < c < -\frac{1}{2}$



- 1 component diffeomorphic to $S^1 \times S^2$.

High energy *II* : $c_{\mathrm{crit}} < c < -\frac{1}{2}$



- Weakly convex foliation: 2 pairs of rigid cylinders, 2 pairs of rigid planes and 2 families of planes.

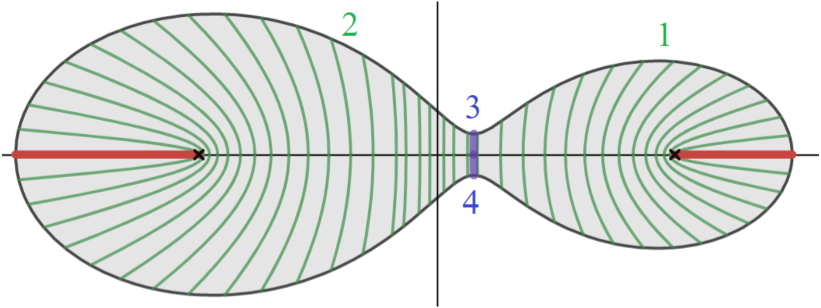
Pseudo-holomorphic

- some planes and cylinders can be made projections of pseudo-holomorphic curves.
- take action-angle coordinates in the (u, p_u) plane.
- choose a suitable decoupled contact form on $K^{-1}_{\mu,c}(0)$.
- use the complex structure $J : \xi \to \xi$ induced by the quaternions.
- consider solutions of the form

  $$(u, p_u, v, p_v) = (r(s)\cos 2\pi t, r(s)\sin 2\pi t, 0, g(s)),$$

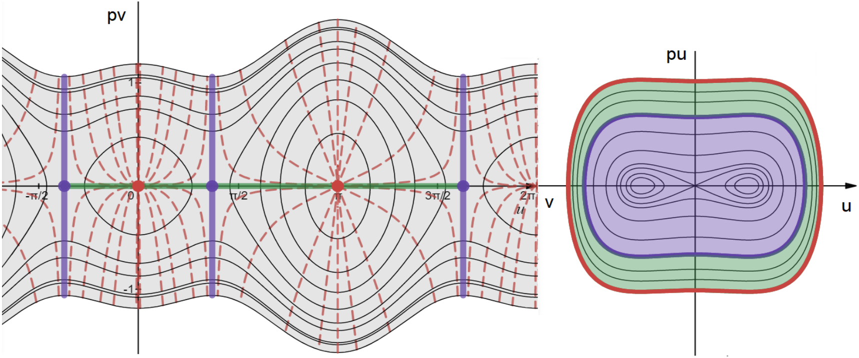
  $\forall (s, t) \in \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$.
- solve first order ODEs for $r(s)$ and $f(s)$ to obtain a pseudo-holomorphic plane.

Symbolic dynamics



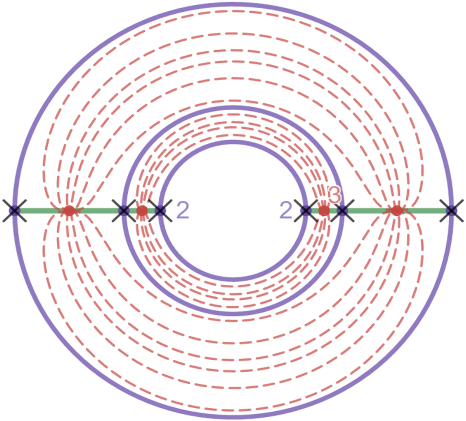
- Satellite: 1111111... or 2222222...
- Lemniscate: 34343434343434.....

High energy *III* : $-\frac{1}{2} < c < c_2(\mu)$



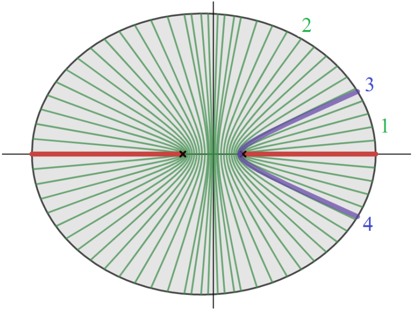
- 1 component diffeomorphic to $S^1 \times S^2$.

High energy III : $-\frac{1}{2} < c < c_2(\mu)$



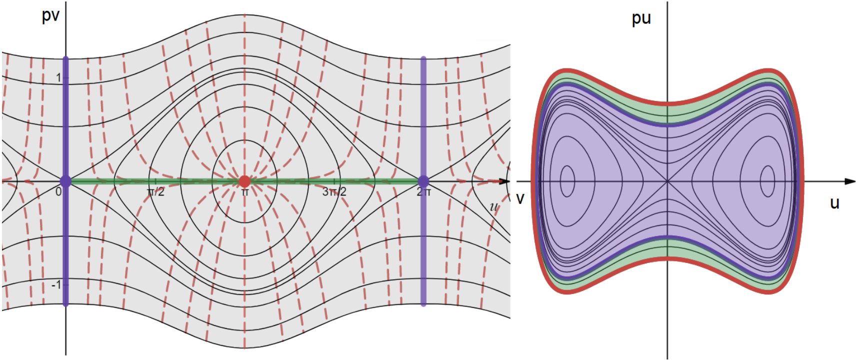
- Weakly convex foliation: 2 pairs of rigid cylinders, 2 pairs of rigid planes and 2 families of planes.

Symbolic dynamics



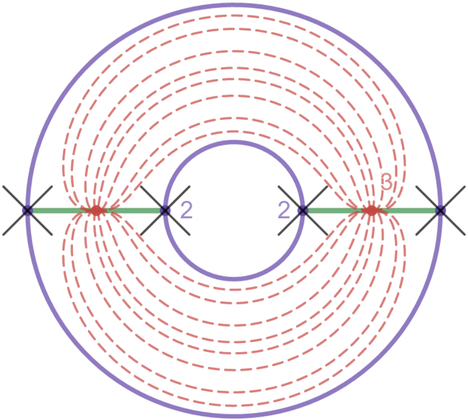
- Satellite: 1111111... or 2222222...
- Lemniscate: 34343434343434.....
- Planet: 343434343434....

High energy IV : $c_2(\mu) < c < 0$



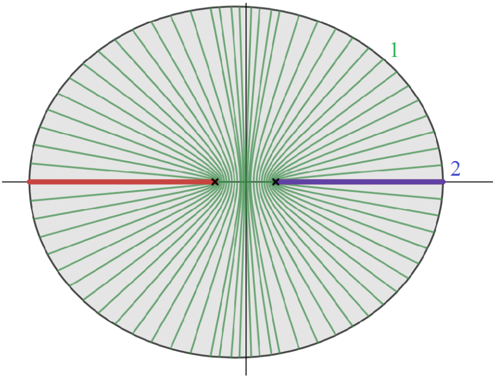
- 1 component diffeomorphic to $S^1 \times S^2$.

High energy IV : $c_2(\mu) < c < 0$



- Weakly convex foliation: 1 pair of rigid cylinders, 1 pair of rigid planes and 1 family of planes.

Symbolic dynamics



- Satellite': 1111111...
- Lemniscate: 22222222.....
- Planet: 22222222....

Hénon-Heiles system