Non existence of small amplitude breathers for Klein-Gordon equations

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Breathers and the Klein-Gordon equation

- Klein-Gordon equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} - \tilde{u} + f(\tilde{u}), \qquad f(0) = f'(0) = 0, \qquad x \in \mathbb{R}$$

- Breathers: Periodic in t localized in x solutions $\tilde{u}(x, t)$.
- The linearized Klein-Gordon equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} - \tilde{u}, \qquad x \in \mathbb{R}$$

  has linear decay as $t \to \infty$
- The existence of Breathers shows a big non-linear effect: Breathers are an "obstacle" to non-linear decay
- Breathers for the Sine-Gordon equation $\tilde{u}_{tt} = \tilde{u}_{xx} - \sin \tilde{u}$:

$$\tilde{u}(x, t) = 4 \arctan \left(\frac{m}{\omega} \frac{\sin(\omega t)}{\cosh(mx)} \right), \quad m, \omega > 0, \; m^2 + \omega^2 = 1.$$

  They are $\frac{2\pi}{\omega}$-periodic in time and $\lim_{x \to \pm\infty} \tilde{u}(x, t) = 0$.
- What about other nonlinearities?
- Families of breathers should be unlikely to happen.

Non-existence of breathers for the Klein-Gordon eq.

Perturbative result: Perturbed Sine-Gordon equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} - \sin\tilde{u} + \varepsilon\Delta(\tilde{u}), \quad \varepsilon \ll 1, \quad \Delta \text{ analytic}$$

Persistence of the family of breathers implies $\Delta(\tilde{u})$ is a trivial perturbation.

- Denzler, J.: Nonpersistence of breather families for the perturbed Sine-Gordon equation. Comm. Math. Phys. (1993).
- Birnir, B., McKean, H. P., Weinstein, A.: The rigidity of sine-Gordon breathers. Comm. Pure Appl. Math. (1994).

Global result:

- Kowalczyk, M., Martel, Y., Muñoz, C.: Nonexistence of small, odd breathers for a class of nonlinear wave equations. Lett. Math. Phys. (2017)
  Nonexistence of odd (in x) breathers for any odd f.
  but The breathers of the Sine-Gordon equation are even!
- Soffer, A., Weinstein, M. I.: Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. Invent. Math. (1999)
  Non-existence of breathers if one adds a potential (under some hypotheses).

Small amplitude breathers for the Klein Gordon Equation

There are some results about the existence (or not) of small amplitude breathers $\|\tilde{u}\|_{L^\infty} \ll 1$

- H. Segur, M.D. Kruskal.: Nonexistence of small-amplitude breather solutions in ϕ^4 theory. Phys. Rev. Lett. (1987) They give formal arguments using divergent formal series to show that the ϕ^4-model:

  $$\partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} + \tilde{u} - \frac{1}{3}\tilde{u}^3 = 0,$$

  has no breathers of frequency ω, $0 < 1 - \omega \ll 1$, and amplitude $\tilde{u} = \mathcal{O}(\sqrt{1-\omega^2})$

- M.D. Groves, G. Schneider: Modulating pulse solutions for a class of nonlinear wave equations. Comm. Math. Phys. (2001) Rigourous analysis on breathers $\tilde{u} = \mathcal{O}(\varepsilon)$ with exponentially small errors.

Small amplitude generalized breathers for the Klein Gordon Equation

N. Lu: Small generalized breathers with exponentially small tails for Klein-Gordon equations. J. Differential Equations (2014)

- Proves the existence of breathers with exponentially small tails.

- For any $\varepsilon \ll 1$ there exist solutions $\tilde{u}(t,x)$ such that are $\frac{2\pi}{\omega}$ – periodic, $\omega = \sqrt{1-\varepsilon^2}$ and

$$|\tilde{u}(t,x)| \lesssim \varepsilon \quad \text{and} \quad \limsup_{x\to\pm\infty} |\tilde{u}(t,x)| \lesssim e^{-c/\varepsilon}, \qquad c > 0.$$

All these results approach the problem using the spatial dynamics where breathers can be seen as homoclinic orbits.

Breathers as homoclinic orbits

$$\tilde{u}_{tt} = \tilde{u}_{xx} - \tilde{u} + f(\tilde{u}), \qquad f(\tilde{u}) = \mathcal{O}(\tilde{u}^2)$$

- Dynamical system with x as time: phase space is space of $2\pi/\omega$-periodic functions in t for some $\omega > 0$.
- As Breathers satisfy $\tilde{u}(x, t) \to 0$, as $x \to \pm\infty$,
  Breathers $\equiv$ Homoclinic orbits to the critical point $\tilde{u} = 0$.
- We will see that $\tilde{u} = 0$ has finite dimensional stable and unstable eigenspaces: therefore it has finite dimensional stable and unstable manifolds and should be unlikely that those manifolds intersect.
- But this is hard to prove in general...

Small amplitude breathers for the odd Klein Gordon eq.

- In this talk we will look for small amplitude breathers $\|\tilde{u}\|_{L^\infty} \ll 1$?
- Equivalent to small homoclinics to $\tilde{u} = 0$.
- They should appear at bifurcations of the critical point $\tilde{u} = 0$.
- Simplest setting: Odd Klein gordon equation

$$\partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} + \tilde{u} - \frac{1}{3}\tilde{u}^3 - g(\tilde{u}) = 0, \quad g(\tilde{u}) = \mathcal{O}(\tilde{u}^5), \text{ odd}$$

Main result: Non existence of small (single-bump) breathers

MAIN RESULT: There exists $C_{\rm in} \in \mathbb{R}^2$, which depends on $g(\cdot)$ analytically, such that if $C_{\rm in} \neq 0$, then for any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there DOES NOT EXIST any solution $\tilde{u}(x,t)$ such that $\tilde{u}$ is $\frac{2\pi}{\omega}$-periodic in t for some $\omega > 0$, σ-single-bump, and satisfies

$$\|\tilde{u}(x,\cdot)\|_{H^1_t\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x \tilde{u}(x,\cdot)\|_{L^2_t\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \quad x \to \pm\infty$$

and

$$\sup_{x\in\mathbb{R}} \|\tilde{u}(x,\cdot)\|_{\ell_1} < \min\{1, \rho^* \sqrt{\omega}\}.$$

Main results

$$\partial_{tt}\tilde{u} - \partial_{xx}\tilde{u} + \tilde{u} - \frac{1}{3}\tilde{u}^3 - g(\tilde{u}) = 0, \quad g(\tilde{u}) = \mathcal{O}(\tilde{u}^5), \text{ odd and analytic}$$

The spatial properties of solutions which are $\frac{2\pi}{\omega}$-periodic in t would heavily depend on ω.
We shall divide $\omega \in \mathbb{R}^+$ into two primary cases:

$$I_k(\varepsilon_0) = \left[\frac{1}{k}(1+\frac{\varepsilon_0^2}{k})^{-\frac{1}{2}}, \frac{1}{k}\right) \; k \in \mathbb{N},$$

and

$$J_k(\varepsilon_0) = \left[\frac{1}{k+1}, \frac{1}{k}(1+\frac{\varepsilon_0^2}{k})^{-\frac{1}{2}}\right), \; k \in \mathbb{N} \cup \{0\},$$

where $0 < \varepsilon_0 \leq 1/2$.
Note that $J_0(\varepsilon_0) = [1, \infty)$ and $(0, \infty) = (\cup_{k\geq 1} I_k) \bigcup (\cup_{k\geq 0} J_k)$.
The main difficulty will be to study I_k.

Main results: Leading order of the exponentially small splitting

There exists $C_{\text{in}} \in \mathbb{R}^2$ such that, $\forall k \in \mathbb{N}$, if $\omega \in I_k$, $\omega = \frac{1}{k}(1 + \frac{\varepsilon^2}{k})^{-\frac{1}{2}}$,
$\exists\, u^{\pm}(t,x)$ solutions which are $2\pi/\omega-$periodic in t, such that
$\|u^{\pm}(x,\cdot)\|_{\ell_1} = \mathcal{O}(\varepsilon\sqrt{k}\omega)$ and:

$$\|u^{\pm}(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u^{\pm}(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \quad x \to \pm\infty$$

and

$$\left|\left(\frac{(-\partial_t^2 - k^{-2})^{\frac{1}{2}}}{\omega}(u^+ - u^-) + i\partial_x(u^+ - u^-)\right)(0) - 4\sqrt{2}C_{\text{in}} e^{-\frac{\sqrt{2k}\pi}{\varepsilon}} \sin 3k\omega t\right|$$

$$\leq \mathcal{O}\Big(\frac{e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{|\log\frac{k}{\varepsilon}|}\Big)$$

- The solutions $u^{\pm}$ belong to the stable and unstable manifolds of the steady solution $u = 0$.

Main results: Leading order of the exponentially small splitting

As $u^{\pm}(x,t) = \sum u_n^{\pm}(x)e^{in\omega t}$ the last inequality reads, in components:

$$|-(9k^2+\frac{k^2}{\omega^2})(u_{3k}^{+}-u_{3k}^{-})(0)+i(\frac{u_{3k}^{+}}{dx}-\frac{u_{3k}^{-}}{dx})(0)-4\sqrt{2}C_{\rm in}e^{-\frac{\sqrt{2}k\pi}{\varepsilon}}| \leq \mathcal{O}(\frac{e^{-\frac{\sqrt{2}k\pi}{\varepsilon}}}{|\log\frac{k}{\varepsilon}|})$$

- At the other Fourier modes the difference $u_n^{+} - u_n^{-} = \mathcal{O}(\frac{e^{-\frac{\sqrt{2}k\pi}{\varepsilon}}}{|\log\frac{k}{\varepsilon}|})$ is smaller.
- We can show that the solutions $u^{\pm}$ are the only possible small breathers because all the other solutions in the stable and unstable manifold become "big" at some value of x.
- If $C_{\rm in} \neq 0$ the solutions $\tilde{u}^{\pm}$ can not intersect and therefore small breathers can not exist

Small amplitude breathers for the odd Klein Gordon eq.

$$\partial_{tt}\tilde{u} - \partial_{xx}\tilde{u} + \tilde{u} - \frac{1}{3}\tilde{u}^3 - g(\tilde{u}) = 0, \quad g(\tilde{u}) = \mathcal{O}(\tilde{u}^5), \text{ odd}$$

- Fix periodicity in t to be $2\pi/\omega$. Change time to $\tau = \omega t$. Call $u(\tau, x) = \tilde{u}(\frac{\tau}{\omega}, x)$, then $u(\tau, x)$ has to be 2π periodic in τ and satisfies:
$$\omega^2 \partial_{\tau\tau} u - \partial_{xx} u + u - \frac{1}{3} u^3 - g(u) = 0,$$
which is an equation depending on a parameter ω.
- Linearization around $u = 0$:
$$\omega^2 \partial_{\tau\tau} u - \partial_{xx} u + u = 0$$
write $u(\tau, x) = \sum_{n\in\mathbb{Z}} u_n(x) \mathrm{e}^{in\tau}$, and:
$$-n^2\omega^2 u_n - u_n'' + u_n = 0$$
- Eigenvalues: $\pm\sqrt{1 - n^2\omega^2}$, $n \geq 1$.
- Eigenfunctions: $u_n^{\pm}(x) = \mathrm{e}^{\pm x\sqrt{1-n^2\omega^2}}$

Small amplitude breathers for the odd Klein Gordon eq.

- Eigenvalues: $\pm\sqrt{1-n^2\omega^2}$, $n \geq 1$.
- $\omega > 1$: $u = 0$ is an elliptic critical point. No homoclinic orbits to $u = 0$ are possible. SMALL BREATHERS DO NOT EXIST
- First bifurcation value: $\omega \approx 1$.
  For $\omega \in I_1$, $\omega = (1+\varepsilon^2)^{-1/2}$, two real (small) eigenvalues: $\pm\frac{\varepsilon}{1+\varepsilon^2}$ and the rest are elliptic.
  The critical point $u = 0$ has one-dimensional stable and unstable manifolds; they can give rise to small homoclinic orbits to $u = 0$. SMALL BREATHERS CAN EXIST
- For $\omega \in J_1$, the hyperbolic eigenvalues are $\mathcal{O}(1)$ and the associated stable and unstable manifolds become "big". SMALL BREATHERS CANNOT EXIST
- At $\omega \approx \frac{1}{k}$, bifurcations where 2 elliptic eigenvalues become hyperbolic (real). At these values the stable and unstable manifolds increase a dimension and, for $\omega \in I_k$ there are $k-1$ strong hyperbolic directions and two "weak" hyperbolic directions. These weak directions can create a small homoclinic loop. SMALL BREATHERS CAN EXIST

Small amplitude breathers for the odd Klein Gordon eq.

$$\omega^2\partial_{\tau\tau}u - \partial_{xx}u + u - \frac{1}{3}u^3 - g(u) = 0, \quad g(u) = \mathcal{O}(u^5), \text{ odd}$$

$\omega \in I_k$. Scaling:

$$u = \varepsilon\sqrt{k}\omega v \text{ and } y = \varepsilon\sqrt{k}\omega x.$$

A small breather is now $v = \mathcal{O}(1)$!! $v(y,\tau)$ satisfies

$$\partial_y^2 v - \frac{1}{\varepsilon^2 k}\partial_\tau^2 v - \frac{1}{\varepsilon^2 k\omega^2}v + \frac{1}{3}v^3 + \frac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3} g\left(\varepsilon\sqrt{k}\omega v\right) = 0,$$

which is a Hamiltonian partial differential equation with respect to

$$\mathcal{H}(v, \partial_y v) = \int_{\mathbb{T}} \left(\frac{(\partial_y v)^2}{2} + \frac{(\partial_\tau v)^2}{2\varepsilon^2 k} - \frac{v^2}{2\varepsilon^2 k\omega^2} + \frac{v^4}{12} + \frac{G(\varepsilon\sqrt{k}\omega v)}{\varepsilon^4 k^2\omega^4} \right) d\tau,$$

where G is an analytic function such that $G'(z) = g(z)$.

Equation for the Fourier coefficients

- Odd (in τ) breathers: $v(y,\tau) = \sum_{n\geq 1} v_n(y)\sin(n\tau)$, not odd in y!
- Equations for v_n form an infinite dimensional Hamiltonian system:

$$\begin{cases} \ddot{v}_n = \frac{\lambda_n^2}{\varepsilon^2} v_n - \Pi_n \left[\frac{v^3}{3} + \frac{1}{\varepsilon^3 k^{\frac{3}{2}} \omega^3} g(\varepsilon\sqrt{k}\omega v) \right], \ n < k, \\ \ddot{v}_k = v_k - \Pi_k \left[\frac{v^3}{3} + \frac{1}{\varepsilon^3 k^{\frac{3}{2}} \omega^3} g(\varepsilon\sqrt{k}\omega v) \right], \\ \ddot{v}_n = -\frac{\lambda_n^2}{\varepsilon^2} v_n - \Pi_n \left[\frac{v^3}{3} + \frac{1}{\varepsilon^3 k^{\frac{3}{2}} \omega^3} g(\varepsilon\sqrt{k}\omega v) \right], \ n > k. \end{cases}$$

with $\cdot = d/dy$ and $\lambda_n = \sqrt{\frac{1}{k}\left|n^2 - \frac{1}{\omega^2}\right|} \geq \frac{1}{2}$, for $n \neq k$ ($\frac{1}{\omega^2} = k(k+\varepsilon^2)$).

The first k modes correspond to hyperbolic (stable and unstable) directions and the rest are elliptic.

The origin has k-dim. stable and unstable invariant manifolds $W^s(0)$ and $W^u(0)$ characterized as solutions v^s and v^u satisfying the asymptotic conditions

$$\lim_{y\to+\infty} v_n^s(y) = \lim_{y\to-\infty} v_n^u(y) = 0,$$

The first bifurcation: $\omega \approx 1$

The most difficult case is the first bifurcation $\omega \approx 1$. Let's take $\omega \in I_1$, $\omega^2 = \frac{1}{1+\varepsilon^2} = 1 - \frac{\varepsilon^2}{1+\varepsilon^2}$, $|\varepsilon| \ll 1$.
The system becomes:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1 \left[\dfrac{v^3}{3} + \dfrac{1}{(\varepsilon\omega)^3} g(\varepsilon\omega v) \right], \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \Pi_n \left[\dfrac{v^3}{3} + \dfrac{1}{(\varepsilon\omega)^3} g(\varepsilon\omega v) \right], \; n > 1, \quad \text{odd} \end{cases}$$

$$\lambda_n = \sqrt{\left| n^2 - \frac{1}{\omega^2} \right|} = \sqrt{|n^2 - 1 - \varepsilon^2|} \geq \tfrac{1}{2}, \text{ for } n \neq 1.$$

A normally elliptic slow manifold

- Equivalently, we have a singularly perturbed problem:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1 \left[\dfrac{v^3}{3} + \dfrac{1}{(\varepsilon\omega)^3} g(\varepsilon\omega v) \right], \\ \varepsilon^2 \ddot{v}_n = -\lambda_n^2 v_n - \Pi_n \left[\varepsilon^2 \dfrac{v^3}{3} + \frac{1}{\varepsilon\omega^3} g(\varepsilon\omega v) \right], \ n \geq 3. \end{cases}$$

- Taking the singular limit $\varepsilon \to 0$, the critical manifold is the (normally elliptic) plane

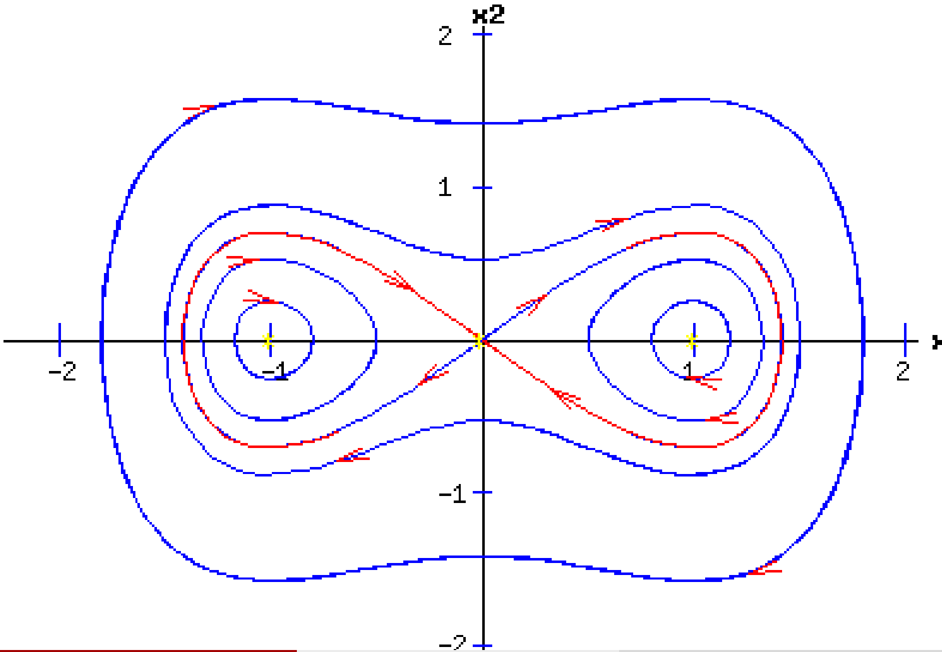
$$\mathcal{M} = \{v_n = 0, \ \dot{v}_n = 0, \ n \geq 3\}.$$

and the dynamics in $\mathcal{M}$ is given by the Duffing equation: $\ddot{v}_1 = v_1 - \dfrac{v_1^3}{4}$.

- Limit equation has a homoclinic orbit to $v_1 = \dot{v}_1 = 0$.

$$v_1^h(y, \tau) = \frac{2\sqrt{2}}{\cosh(y)}, \quad v_n^h = 0, \ n \geq 3.$$

- In the limit problem, the invariant manifolds $W^s(0)$ and $W^u(0)$ coincide.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$\varepsilon \neq 0$: Homoclinic breakdown

- Does the homoclinic orbit persist for the full problem?
- It is a singular perturbation problem:

$$\begin{array}{r} \text{Fast rotation versus} \\ \text{weak hyperbolicity} \end{array} \longrightarrow \text{Exponentially small phenomena.}$$

- Hard to measure the distance between the one dimensional perturbed invariant manifolds $W^{\mathrm{st}}(0)$ and $W^{\mathrm{uns}}(0)$.
- Classical perturbative methods (Melnikov Theory) cannot be applied.

$\varepsilon > 0$: Formal series expansions

- Look for parameterizations $v_n^-(y,\varepsilon)$, $v_n^+(y,\varepsilon)$, $n \geq 1$ of $W^-(0)$ and $W^+(0)$ solutions of:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1 \left[\dfrac{v^3}{3} + \dfrac{1}{(\varepsilon\omega)^3} g(\varepsilon\omega v) \right], \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \Pi_n \left[\dfrac{v^3}{3} + \dfrac{1}{(\varepsilon\omega)^3} g(\varepsilon\omega v) \right], \ n > 1. \end{cases}$$

  satisfying:

$$\lim_{y \to -\infty} v_n^-(y,\varepsilon) = 0, \quad \lim_{y \to +\infty} v_n^+(y,\varepsilon) = 0$$

- As this is a (infinite dymensional) system of non-linear differential equations we look for $v_n^\pm(y,\varepsilon)$ as formal power series of ε:

$$v_n^\pm(y,\varepsilon) = v_{n,0}(y) + \varepsilon v_{n,1}^\pm(y) + \varepsilon^2 v_{n,2}^\pm(y) + \dots$$

Formal series expansions

$$v_n^{\pm}(y,\varepsilon) = v_{n,0}(y) + \varepsilon v_{n,1}^{\pm}(y) + \varepsilon^2 v_{n,2}^{\pm}(y) + \dots$$

- One can check:

$$v_{n,k}^{-}(y) = v_{n,k}^{+}(y) \;\; \forall k \in \mathbb{N}$$

- Thus: their difference is beyond all orders:

$$v_n^{-}(y,\varepsilon) - v_n^{+}(y,\varepsilon) = \mathcal{O}(\varepsilon^k) \;\; \forall k \in \mathbb{N}.$$

- We have two possibilities:
  1. The power series in ε are convergent and the manifolds coincide.
  2. The power series in ε are divergent and the difference between manifolds is flat with respect to ε.

  Typically, we expect the second case to happen.

Kruskal and Segur work

H. Segur, M.D. Kruskal. Nonexistence of small-amplitude breather solutions in ϕ^4 theory. Phys. Rev. Lett. 58 (1987)

- They give formal arguments which indicate that the series is not convergent and that there is breakdown.
- Questions:
  - How to make rigorous the formal arguments to show breathers breakdown.
  - Do small amplitude breathers with exponentially small (with respect to the amplitude) tails exist?

Breathers with tails

N. Lu, Small generalized breathers with exponentially small tails for Klein-Gordon equations. J. Differential Equations 256 (2014)

- Proves the existence of breathers with exponentially small tails.
- I. e. for any $\varepsilon \ll 1$ there exist solutions $u(t,x)$ such that are $\frac{2\pi}{\omega}$ – periodic, $\omega = \sqrt{1-\varepsilon^2}$ and

$$|u(t,x)| \lesssim \varepsilon \quad \text{and} \quad \limsup_{x\to\pm\infty} |u(t,x)| \lesssim e^{-c/\varepsilon}, \qquad c > 0.$$

- Equivalently: Homoclinic orbits to $W^c(0)$ whose ω/α-limits are confined $\mathcal{O}(e^{-c/\varepsilon})$ – close to $u = 0$.
- Also $W^+(0)$ and $W^-(0)$ are $\mathcal{O}(e^{-c/\varepsilon})$ – close
- Proven through Normal form + Invariant manifolds techniques.
- M.D. Groves, G. Schneider, Modulating pulse solutions for a class of nonlinear wave equations. Comm. Math. Phys. 219 (2001) Related results

Main result

- Take a section transversal to the solutions

$$\Sigma = \{(v, \partial_y v);\ \mathcal{H}(v, \partial_y v) = 0 \quad \text{and} \quad \Pi_1 [\partial_y v] = 0\}$$

$\mathcal{H}$ is the Hamiltonian.

Theorem (Guardia-Gomide-S.-Zeng)

Call $P^{\pm}$ the first intersection points of $W^{\pm}$ with Σ. Then, there exists a constant $C_{\rm in} \in \mathbb{C}$ such that, for $\varepsilon \ll 1$, the distance $d(\varepsilon) = P^- - P^+$ satisfies

$$\Pi_3[d(\varepsilon)] = \frac{2}{\varepsilon} e^{-\frac{\pi\lambda_3}{2\varepsilon}} \left(C_{\rm in} + \mathcal{O}\left(\frac{1}{\log(\varepsilon)}\right)\right)$$

where $\lambda_3 = 2\sqrt{2}$ and

$$\Pi_n[d(\varepsilon)] = \frac{2}{\varepsilon} e^{-\frac{\pi\lambda_3}{2\varepsilon}} \mathcal{O}\left(\frac{1}{\log(\varepsilon)}\right) \qquad n > 1.$$

Implication on breathers (if $C_{\rm in} \neq 0$)

- If $C_{\rm in} \neq 0$, then the invariant manifolds $W^-(0)$ and $W^+(0)$ do not intersect the first time they reach Σ.
- It rules out the existence of homoclinics continuation of those of the singular limit problem.
- Even if $C_{\rm in} \neq 0$, $W^-(0), W^+(0)$ may still coincide after more rounds. This would give multi-bump breathers.
- It implies non-existence of one bump breathers with period $2\pi\sqrt{1+\varepsilon^2}$

Working more... (if $C_{\mathrm{in}} \neq 0$)

Analyzing the dynamics of the central-stable and central-unstable manifolds close to the non-coincident stable and unstable invariant manifolds:

- One has transverse intersections of the center-stable and center-unstable manifolds of $\tilde{u} = 0$.
- We obtain a lower bound for the size of the tails of (one bump) breathers in Nan Lu's work.

Comments on C_{in}

- C_{in} is a Stokes constant and is hard to compute.
- It cannot be computed perturbatively.
- It is obtained through an equation independent of ε: the inner eq.
- It depends on the whole Fourier series of v and all Taylor expansion of f.
- For a “typical” f, $C_{\mathrm{in}} \neq 0$.
- For a given equation, say cubic Klein-Gordon, prove $C_{\mathrm{in}} \neq 0$ is much harder:
    - Borel summation techniques, Resurgence Theory.
    - Computer assisted proofs?

Non-vanishing C_{in}

Considers models like

$$\partial_z^2\phi - \partial_\tau^2\phi - \phi + \sin\phi + \mu\Delta(\phi) = 0, \qquad \Delta(z) = \mathcal{O}_5(z)$$

one can show that:

- $C_{\mathrm{in}}(\mu)$ is analytic in μ.
- $C_{\mathrm{in}}(0) = 0$ and for g satisfying a non-degeneracy condition $C'_{\mathrm{in}}(0) = 0$.
- Thus
  - For $\mu \ll 1$: $C_{\mathrm{in}}(\mu) \neq 0$ (There cannot exist small amplitude breathers).
  - Analyticity: $C_{\mathrm{in}}(\mu) \neq 0$ for almost all μ.

Some ideas about the proof

- Exponentially small splitting of separatrices: We follow the ideas by Lazutkin for the homoclinic breakdown for the Standard Map (also Kruskal and Segur).

- Mostly been applied to

  - 2 dimensional area preserving maps

  - Invariant manifolds of periodic orbits or invariant tori at resonances of nearly integrable Hamiltonian systems (Arnold diffusion)

  - Local bifurcations for Hamiltonian/Reversible/Volume preserving systems

  - Analytic unfoldings of the Hopf-Zero bifurcation.

Some ideas about the proof

The parameterizations of the manifolds are solutions of the equations:

$$\ddot{v}_1 = v_1 - \frac{v_1^3}{4} + \mathcal{O}(\varepsilon^2)$$
$$\ddot{v}_n = -\frac{\lambda_n^2}{\varepsilon^2} v_n + \mathcal{O}(\varepsilon^2), \; n \geq 3.$$

- The perturbed invariant manifolds are ε^2–to the unperturbed ones, but we will see that their difference is exponentially small.
- We focus in $n \geq 3$ and write the second order equation as a system:

$$\dot{v}_n = w_n \quad \dot{w}_n = -\frac{\lambda_n^2}{\varepsilon^2} v_n + \mathcal{O}(\varepsilon^2).$$

- (v_n^-, w_n^-) and (v_n^+, w_n^+) are solutions of the same system, therefore the difference $(\Delta_n, \Xi_n) = (v_n^- - v_n^+, \partial_y v_n^- - \partial_y v_n^+)$ satisfies a linear system.

Some ideas about the proof

$$\dot{\Delta}_n = \Xi_n \quad \dot{\Xi}_n = (-\frac{\lambda_n^2}{\varepsilon^2} + \mathcal{O}(\varepsilon^2))\Delta_n.$$

Since the last term is much smaller than the oscillating one, to give a heuristic idea, let us assume that is zero. With the change of variables

$$\Gamma_n = \lambda_n \Delta_n + i\varepsilon \Xi_n \quad \Theta_n = \lambda_n \Delta_n - i\varepsilon \Xi_n \tag{1}$$

becomes

$$\dot{\Gamma}_n = -i\frac{\lambda_n}{\varepsilon}\Gamma_n \quad \dot{\Theta}_n = i\frac{\lambda_n}{\varepsilon}\Theta_n$$

The solutions of this system can be easily computed as

$$\Gamma_n(y) = e^{-i\frac{\lambda_n}{\varepsilon}(y-y_+)}\Gamma_n(y_+)$$
$$\Theta_n(y) = e^{i\frac{\lambda_n}{\varepsilon}(y-y_-)}\Theta_n(y_-)$$

for any points $y_\pm$ where Γ_n and Θ_n are known to exist.

remember:

$$\Gamma_n = \lambda_n(v_n^- - v_n^+) + i\varepsilon(\partial_y v_n^- - \partial_y v_n^+)$$
$$\Theta_n = \lambda_n(v_n^- - v_n^+) - i\varepsilon(\partial_y v_n^- - \partial_y v_n^+).$$

- If we are able to extend the stable and unstable manifolds to complex values $y_\pm = \pm i\sigma$, $\sigma > 0$. Then, one obtains exponentially small the estimates for $y \in \mathbb{R}$,

$$|\Gamma_n(y)| \leq e^{-\frac{\lambda_n \sigma}{\varepsilon}} |\Gamma_n(i\sigma)|$$
$$|\Theta_n(y)| \leq e^{-\frac{\lambda_n \sigma}{\varepsilon}} |\Theta_n(-i\sigma)|,$$

- The bounds depend on the size of the invariant manifolds at the complex points $y_\pm = \pm i\sigma$.
- As the unperturbed homoclinic $v_1^h(y,\tau) = \dfrac{2\sqrt{2}}{\cosh(y)}$ has poles $y = \pm i\pi/2$,one expects: $\sigma \simeq \pi/2$.
- To obtain asymptotics we need to find a good approximation of the manifolds near $\pm i\frac{\pi}{2}$, where the homoclinic blows up!

Lazutkin approach: analytic continuation to complex domains

- Unperturbed homoclinic:

$$v_h(y,\tau) = \frac{2\sqrt{2}}{\cosh(y)} \sin\tau.$$

has singularities at $y = \pm i\pi/2$.

- Parameterizations of the perturbed invariant manifolds

$$v^{\pm}(y,\tau) = v_h(y,\tau) + \xi^{\pm}(y,\tau).$$

- When $y \mp i\pi/2 \sim \varepsilon$, $W^-(0), W^+(0)$ are not well approximated by the unperturbed separatrix since

$$v_h(y,\tau) = \frac{2\sqrt{2}}{\cosh(y)} \sin\tau \sim \frac{1}{|y \mp i\pi/2|} \sim \frac{1}{\varepsilon}$$

whereas

$$\xi^{\pm}(y,\tau) \sim \frac{\varepsilon^2}{|y \mp i\pi/2|^3} \sim \frac{1}{\varepsilon}$$

The inner equation

- Singular change: $z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right)$ and $\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right)$.

- We obtained an scaled Klein-Gordon equation.

- Letting $\varepsilon \to 0$, we get a new equation for the first order: the inner equation
$$\partial_z^2\phi - \partial_\tau^2\phi - \phi + \frac{1}{3}\phi^3 + f(\phi) = 0$$

- The first order of the invariant manifolds close to the singularities must be solutions of this equation.

The inner equation

$$\partial_z^2 \phi - \partial_\tau^2 \phi - \phi + \frac{1}{3}\phi^3 + f(\phi) = 0$$

- One has to look for analytic solutions such that $\Re z \to \pm\infty$.
- Each asymptotic condition corresponds to the stable/unstable manifold.
- Both are asymptotic to the same series $\phi^{\text{uns,st}}(z,\tau) \sim \sum_{k\geq 0} \frac{a_k(\tau)}{z^{2k+1}}$ (in different sectorial domains $\subset \mathbb{C}$).
- Its difference satisfies

$$\phi^- - \phi^+ = e^{-i2\sqrt{2}z}\left(C_{\text{in}} \sin(3\tau) + \mathcal{O}\left(\frac{1}{z}\right)\right) \qquad \text{as } \Im z \to -\infty.$$

Last step

- Show that this difference is still the first order when one considers the true parameterizations of the invariant manifolds close to the singularities $i\pi/2$.

- Use analyticity to deduce that such the difference of the solutions of inner equations gives an asymptotic formula for real values of y.