Choreographies in the n-body problem and a conjecture of Marchal

Geometry, Dynamics and Mechanics Seminar

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Joint work with Eusebius Doedel, Carlos García Azpeitia, Jean-Philippe Lessard, and Jason Mireles-James.

Choreographies

The equations of motion for the n-body problem are

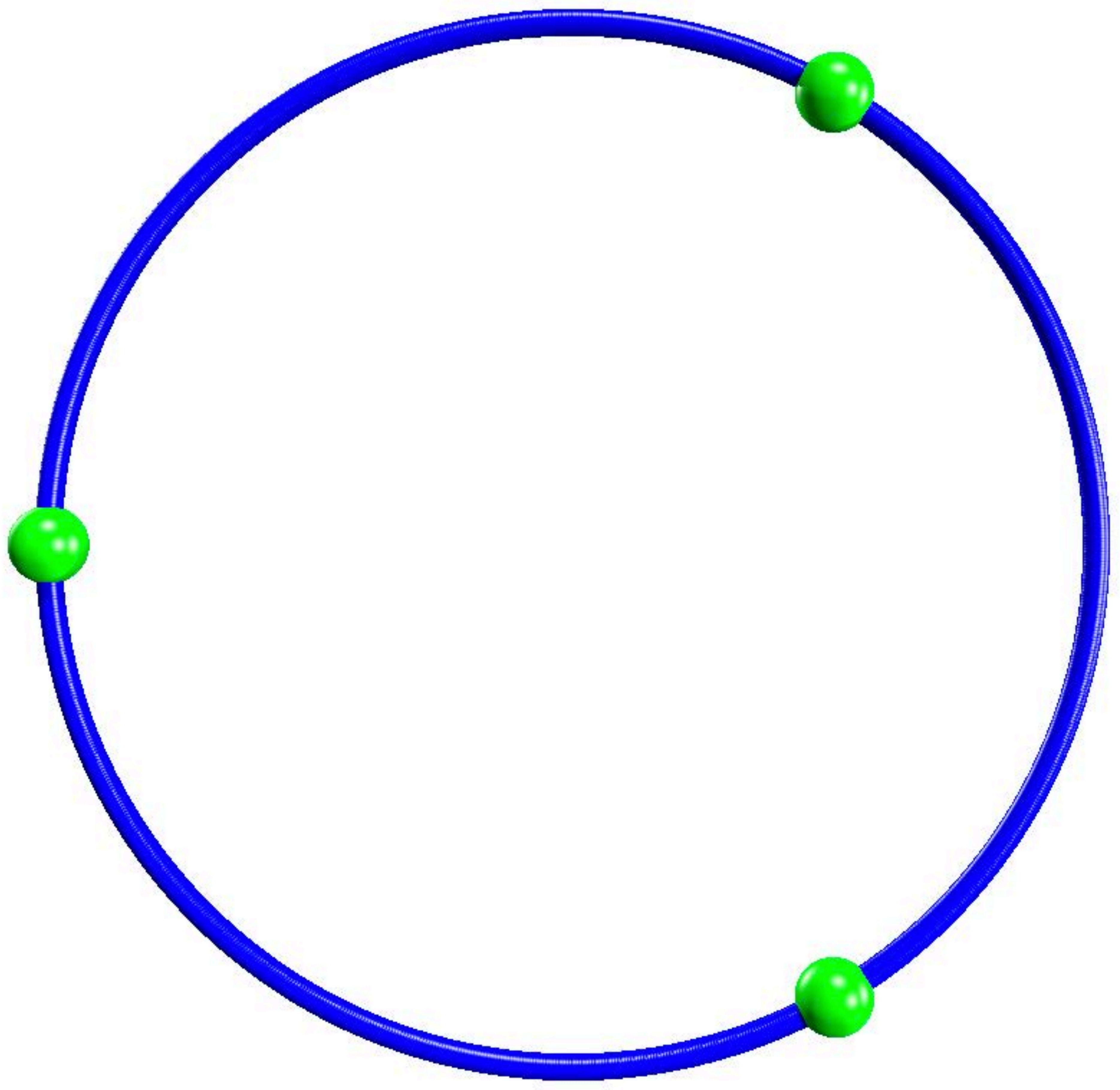
$$\ddot{x}_j = \sum_{i=1, i\neq j}^{n} \frac{m_j(x_i - x_j)}{\|x_i - x_j\|^3}$$

We look for 2π-periodic functions $q : \mathbb{R}^1 \to \mathbb{R}^3$ such that if

$$x_j(t) = q(t - j\zeta), j = 1,...,n \quad \zeta = \frac{2\pi}{n}$$

we find a solution to Newton's equations $\mathbb{Z}^n$ acts on the set of bodies and in $\mathbb{S}^1$ by shifting to the next body

Joseph-Louis Lagrange



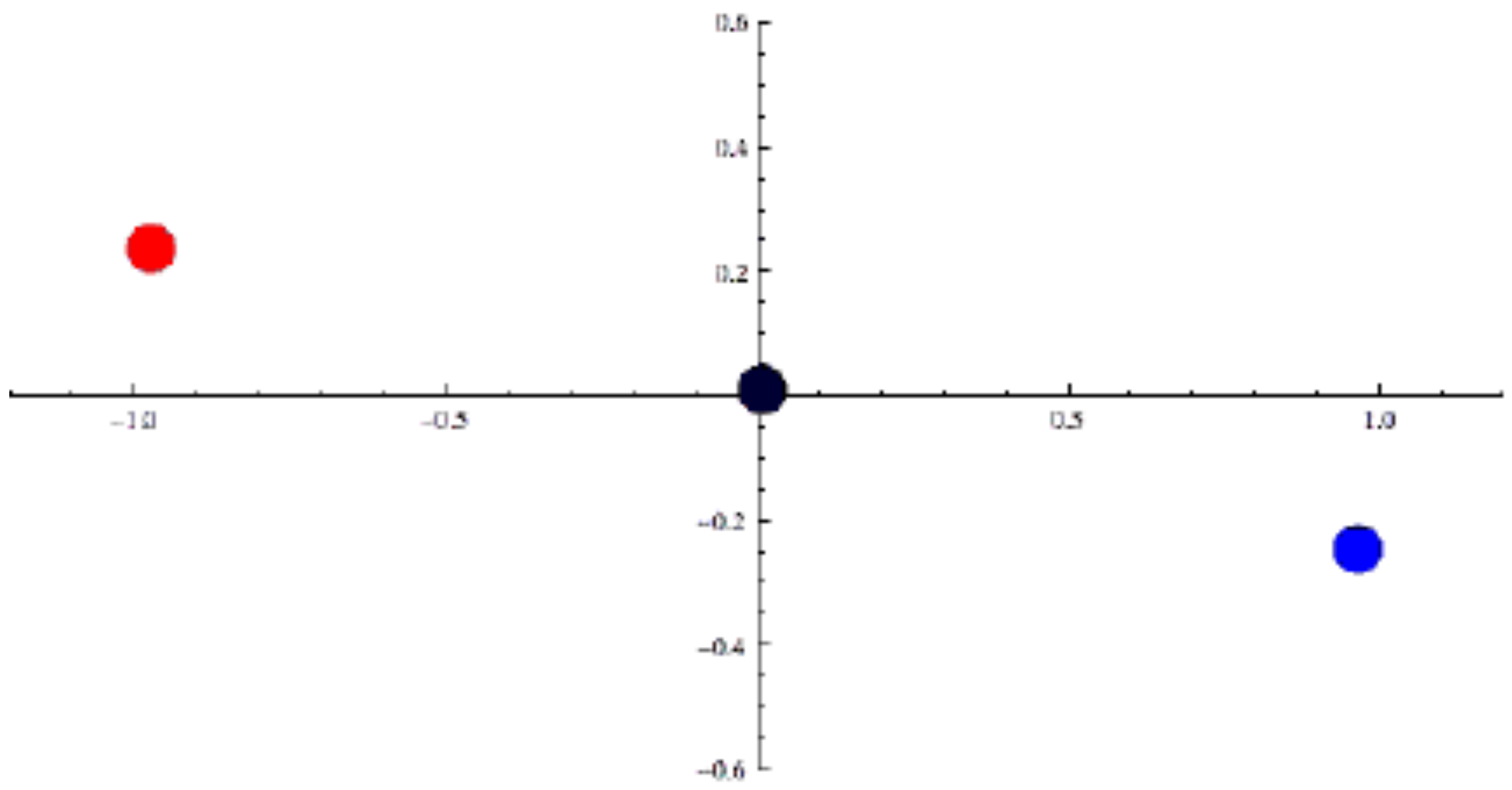
Choreographies

A choreography is an orbit where n masses follow the same path

- C. Moore in ’93, figure-eight orbit minimizing the action among symmetric paths
- Chenciner-Montgomery ’00 proved existence minimizing the action over paths that connect a colinear and an isosceles configuration
- C. Simó large number of different choreographies

Figure eight choreography

(http://www.scholarpedia.org/article/N-body_choreographies)



Variational Methods

Many important contributions started from the work of Chenciner and Montgomery ’00

An obstacle that one encounters is the existence of paths with collisions

- Terracini-Ferrario ’04 applied the principle of least action systematically over symmetric paths to avoid collisions, using ideas introduced by Marchal

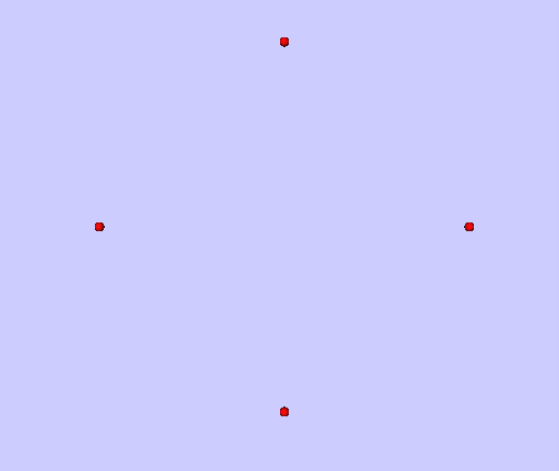
- Other contributions to variational approaches include (Barutello-Ferrario-Terracini ’08) (K.-C. Chen ’01) (Arioli-Barutello-Terracini ’03) (Ferrario ’06) (Ferrario-Portaluri ’08) (Terracini-Venturelli ’07)

- …

Continuation methods

Continuation methods usually give infinitely many choreographies

Chenciner-Féjoz ’09 observed that choreographies appear in dense sets along the vertical Lyapunov families that arise from n<7 rotating bodies in a regular polygon (Weinstein-Moser)



When the frequency varies along the vertical Lyapunov families, then an infinite number of choreographies exists

Equations of motion in a rotating frame

Ize-García Azpeitia '13, proved the global existence of bifurcating planar and vertical Lyapunov families using equivariant degree theory (Ize-Vignoli '03)

$$\ddot{w}_j + 2\sqrt{s_1}\; i\;\dot{w}_j = s_1 w_j - \sum_{i=1(i\neq j)}^{n} \frac{w_j - w_i}{\left\| (w_j, z_j) - (w_i, z_i) \right\|^3}\;, \ddot{z}_j = -\sum_{i=1(i\neq j)}^{n} \frac{z_j - z_i}{\left\| (w_j, z_j) - (w_i, z_i) \right\|^3},$$

where the $(w_j, z_j) \in \mathbb{C} \times \mathbb{R}$ are the positions of the bodies in space, and s_1 is defined by

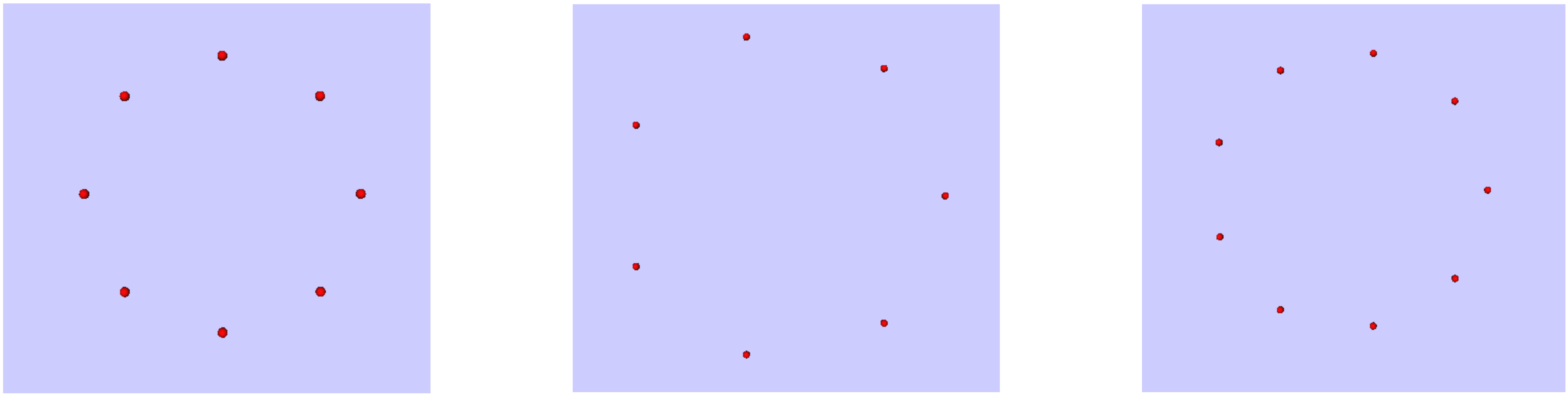
$$s_1 = \frac{1}{4}\sum_{j=1}^{n-1} \frac{1}{\sin(j\zeta/2)}\;, \qquad \zeta = \frac{2\pi}{n}\;.$$

Polygonal equilibria

The circular, polygonal relative equilibrium consists of the positions

$$w_j = e^{ij\zeta}, \qquad z_j = 0.$$

The frequency of the rotational frame is chosen to be $\sqrt{s_1}$



Some symmetries

These equations have symmetries inherited from Newton's equations in the inertial frame

1. rotations in the plane, $e^{\theta i}w_j$

2. *phase in time*

3. translation in $z_j + c$

Spatial branches

Theorem (Ize-García Azpeitia ’13)

For $n \geq 3$ and each k such that $1 \leq k \leq n/2$, the polygonal relative equilibrium has one global bifurcation of spatial periodic solutions, which start with frequency $\sqrt{s_k}$, have the symmetry

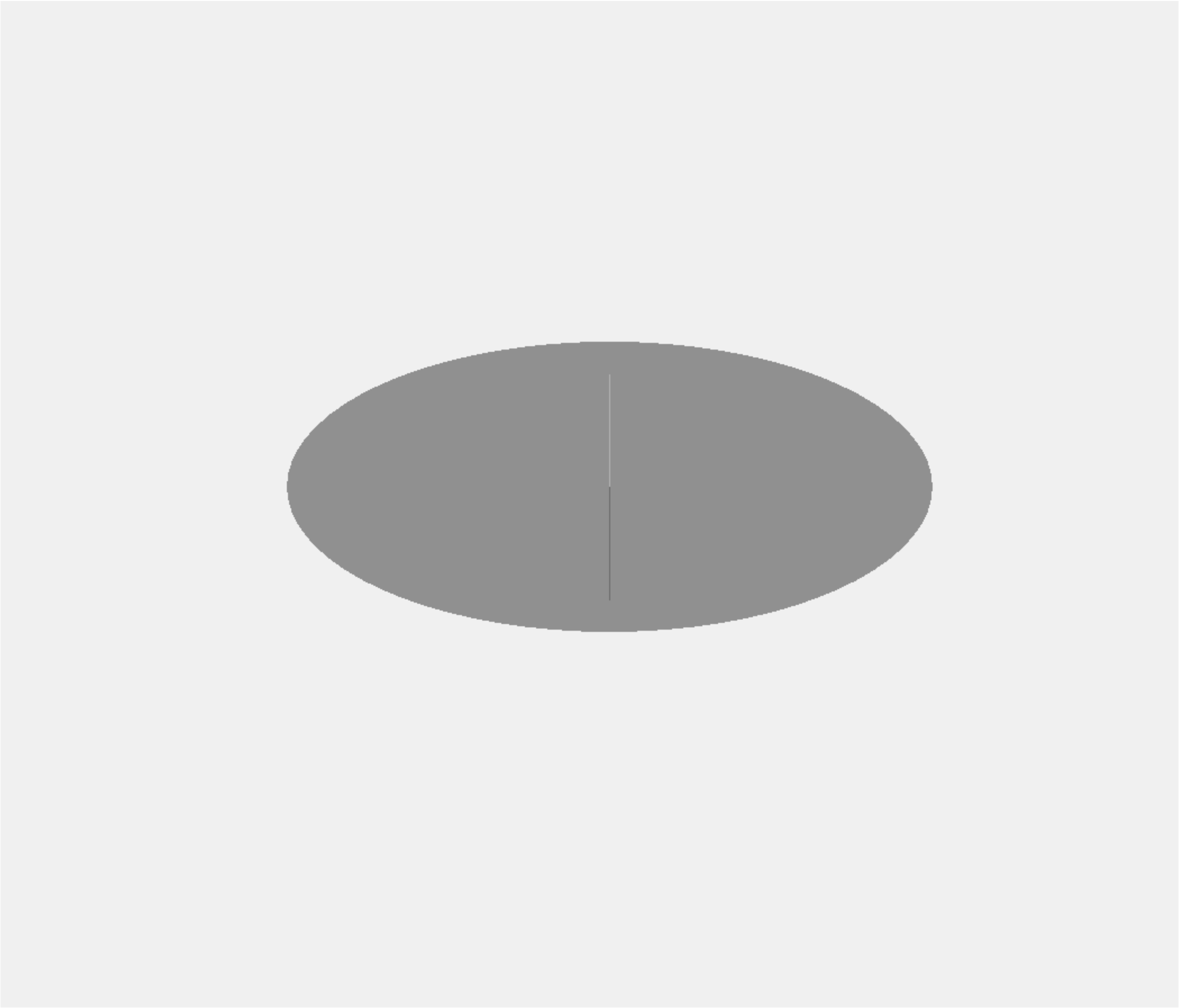
$$(w_j(t) = e^{ij\zeta} w_n(t + jk\zeta), \quad w_n(t) = \bar{w}_n(-t)),$$

as well as the symmetries,

$$z_j(t) = z_n(t + jk\zeta), \quad \text{and} \quad w_n(t) = w_n(t+\pi), \qquad z_n(t) = -z_n(t+\pi)$$

$$s_k = \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sin^2(kj\zeta/2)}{\sin^3(j\zeta/2)}, \quad \zeta = \frac{2\pi}{n}.$$

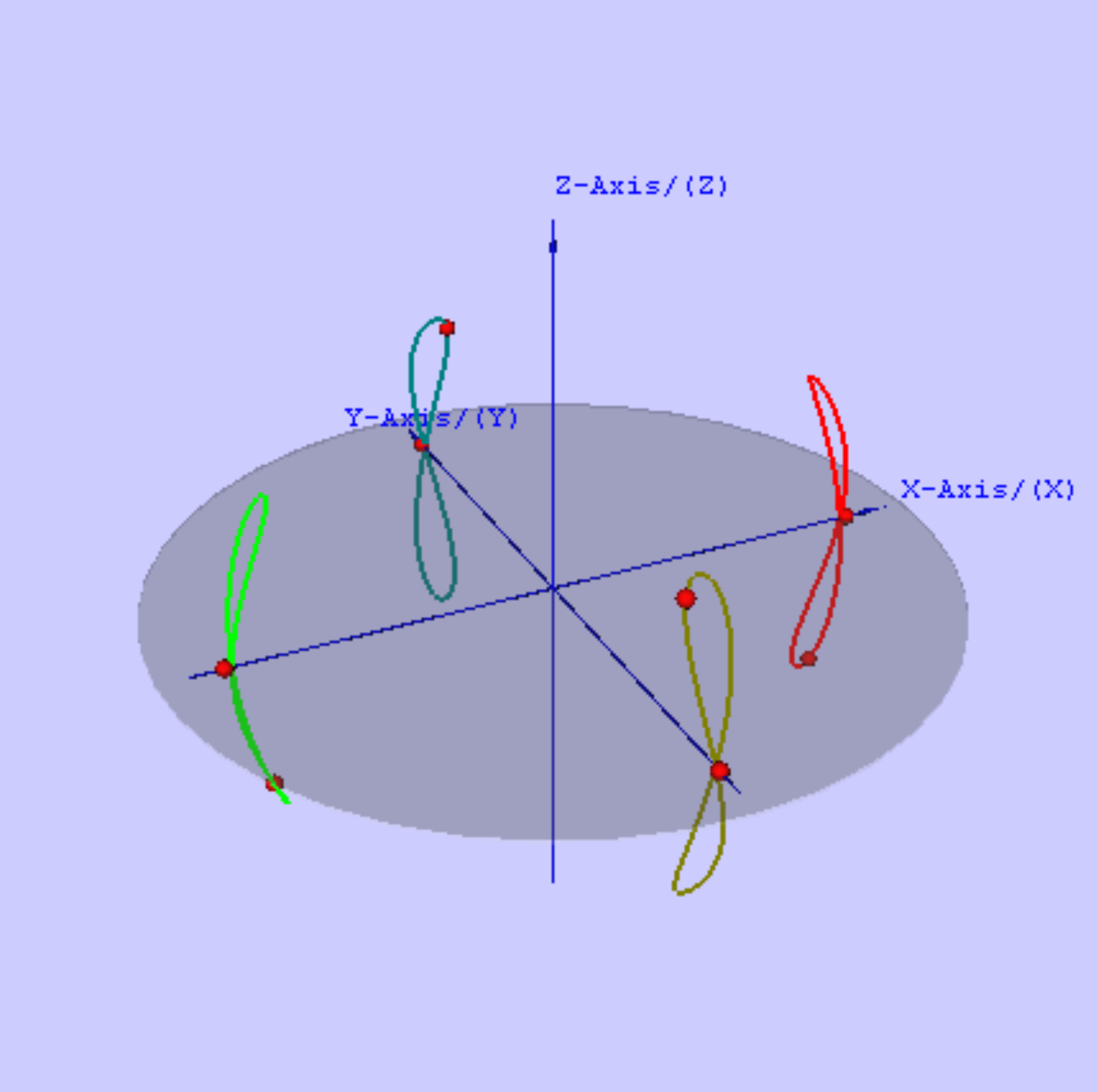
Spatial branches



Spatial branches

$$w_j(t) = e^{ij\zeta} w_n(t + jk\zeta), \qquad w_n(t) = \bar{w}_n(-t)$$

$$z_j(t) = z_n(t + jk\zeta), \quad \text{and} \quad w_n(t) = w_n(t + \pi), \qquad z_n(t) = -z_n(t + \pi)$$



$n = 4, \quad k = 2$

Symmetries along the branches

The periodic orbits in a branch satisfy the following (Ize-García Azpeitia '13)

$$w_j(t) = e^{ij\zeta} w_n(t + jk\zeta)$$

$$z_j(t) = z_n(t + jk\zeta)$$

We transform to the inertial frame

$w_j(t) = e^{ij\zeta} w_n(t + jk\zeta) \to q_j(t) = e^{-ij(2\pi)\Omega} q_n(t + jk\zeta)$

with $\Omega = \frac{1}{n}\left(k\frac{\sqrt{s_1}}{\nu} - 1\right)$

(…remember $x_j(t) = q(t - j\xi),\ j = 1, ..., n$ with $\xi = \frac{2\pi}{n}$)

Symmetries along the branches

We have that

$$q_j(t) = e^{-ij(2\pi)\Omega} q_n(t + jk\zeta)$$

We chose $\Omega = p/q$ and ℓ and m relatively prime integers such that

$$\frac{\ell}{m} = \frac{np + q}{kq}$$

then

$$q_j(t) = q_n(t + j\left(1_n k\zeta\right))$$

$(1_n = 1\, mod\, n)$ and q_n is a $2\pi m$ - periodic function

Symmetries along the branches

At the same time, the spatial component satisfies that

$$z_j(t) = z_n(t + jk\zeta) = z_n(t + j1_n k\zeta)$$

We finally conclude that

$$(q_j, z_j)(t) = (q_n, z_n)(t + j(k - (k\ell - m)\ell^*)\zeta)$$

which is the condition to have a Choreography and we can say more about the path

Resonant periodic orbits

We say that a planar or spatial Lyapunov orbit is $\ell : m$ *resonant* if its period and frequency are

$$T = \frac{2\pi}{\sqrt{s_1}} \left(\frac{\ell}{m} \right), \quad \nu = \sqrt{s_1} \frac{m}{\ell},$$

with

$$s_1 = \frac{1}{4} \sum_{j=1}^{n-1} \frac{1}{\sin(j\zeta/2)}, \qquad \zeta = \frac{2\pi}{n}$$

where ℓ and m are relatively prime, and such that

$$k\ell - m \in n\mathbb{Z}$$

Choreography Theorem

Theorem

In the inertial frame an $\ell : m$ *resonant Lyapunov orbit is a choreography,*

$$x_j(t) = x_n(t + j\tilde{k}\zeta),$$

where $x_j = (q_j, z_j)$, $\tilde{k} = k - (k\ell - m)\ell^*$ *with* ℓ^* *the* m*-modular inverse of* ℓ *.*

Choreography Theorem

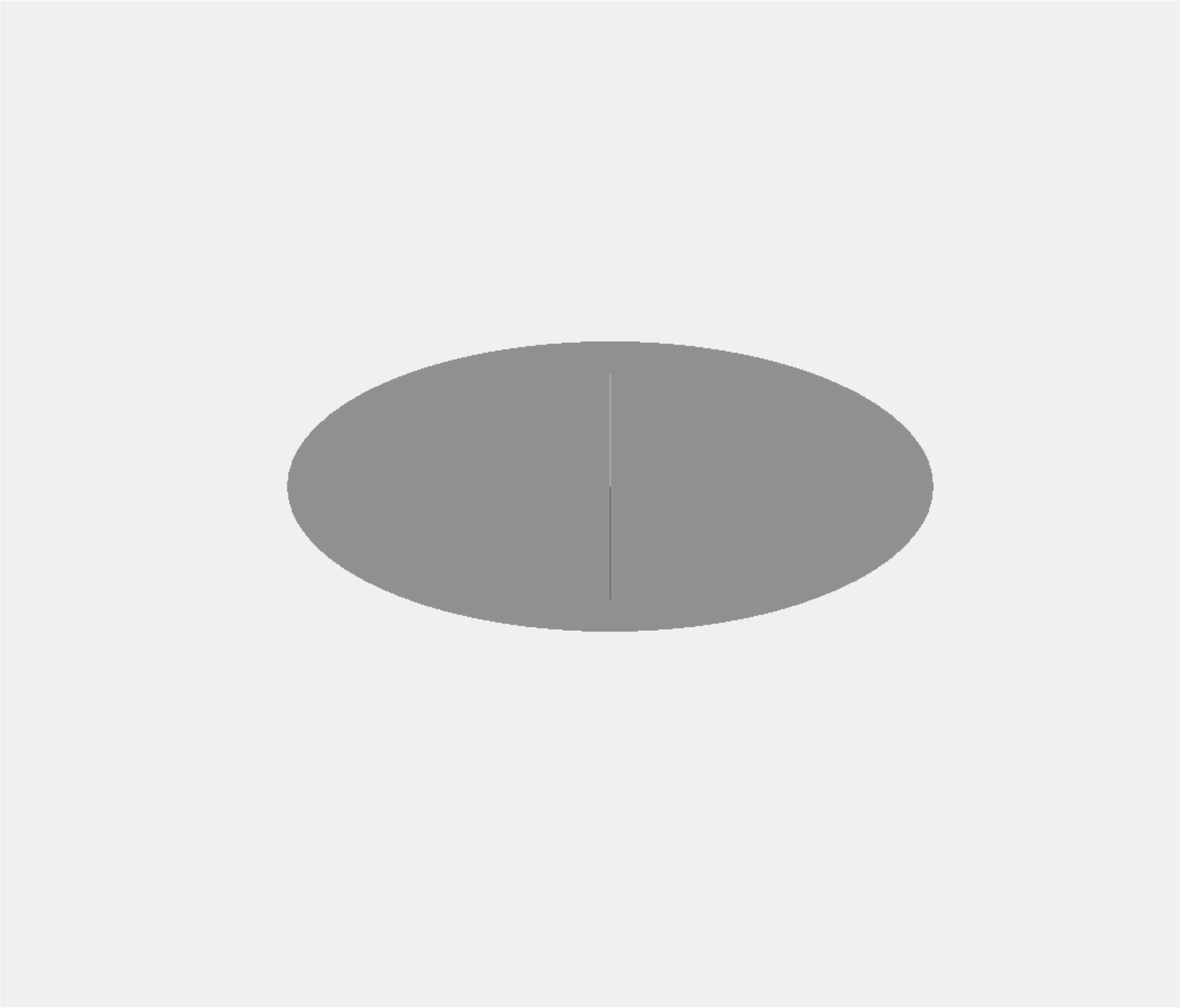
The projection on the xy-plane of the choreography

1) symmetric by rotations of the angle $2\pi/m$
2) winds around a center ℓ times

The period of the choreography is $m \times T_{\ell:m}$ where

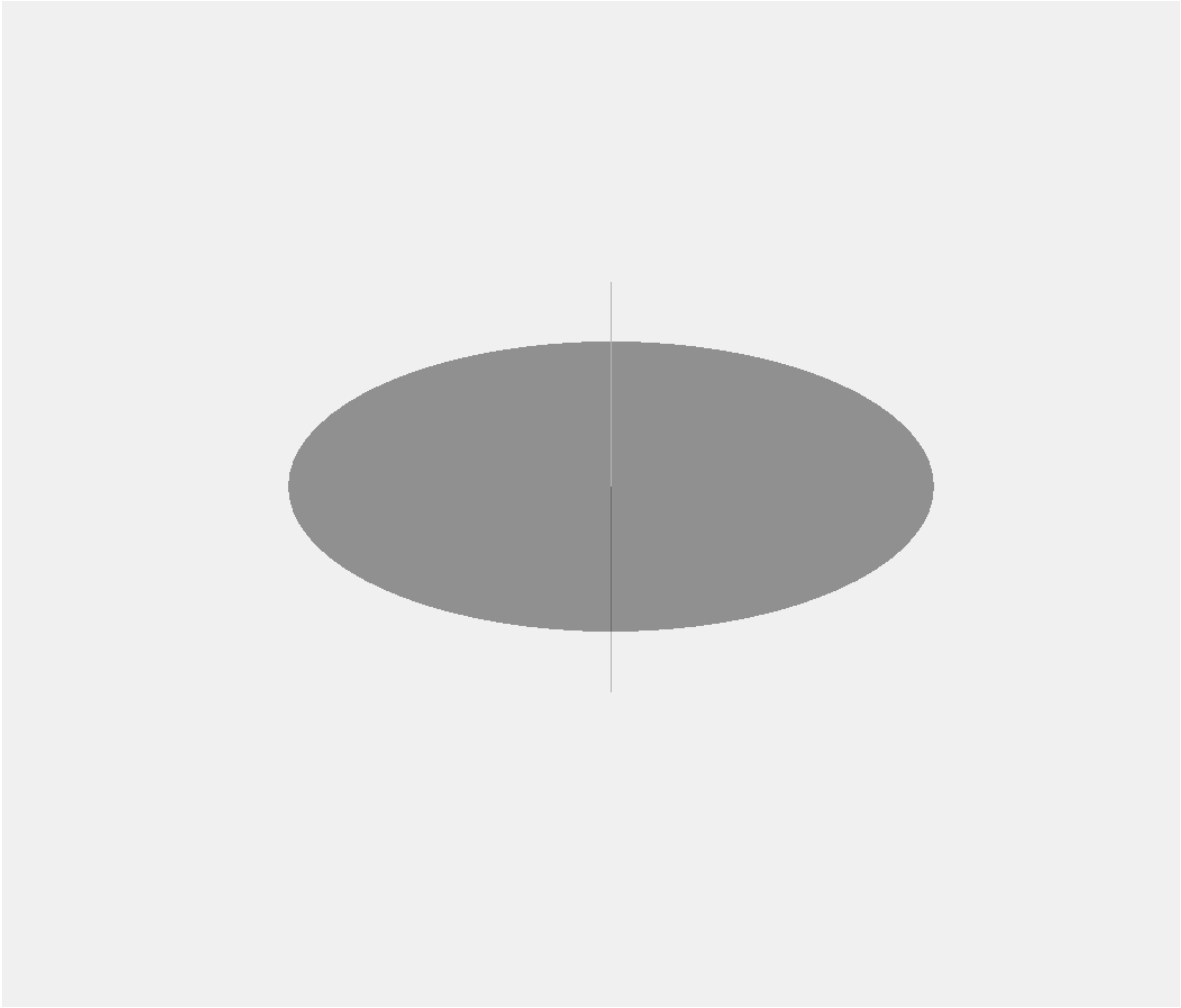
$$T_{\ell:m} = \frac{2pi}{\sqrt{s_1}}\frac{\ell}{m}$$

Spatial orbits and bifurcations

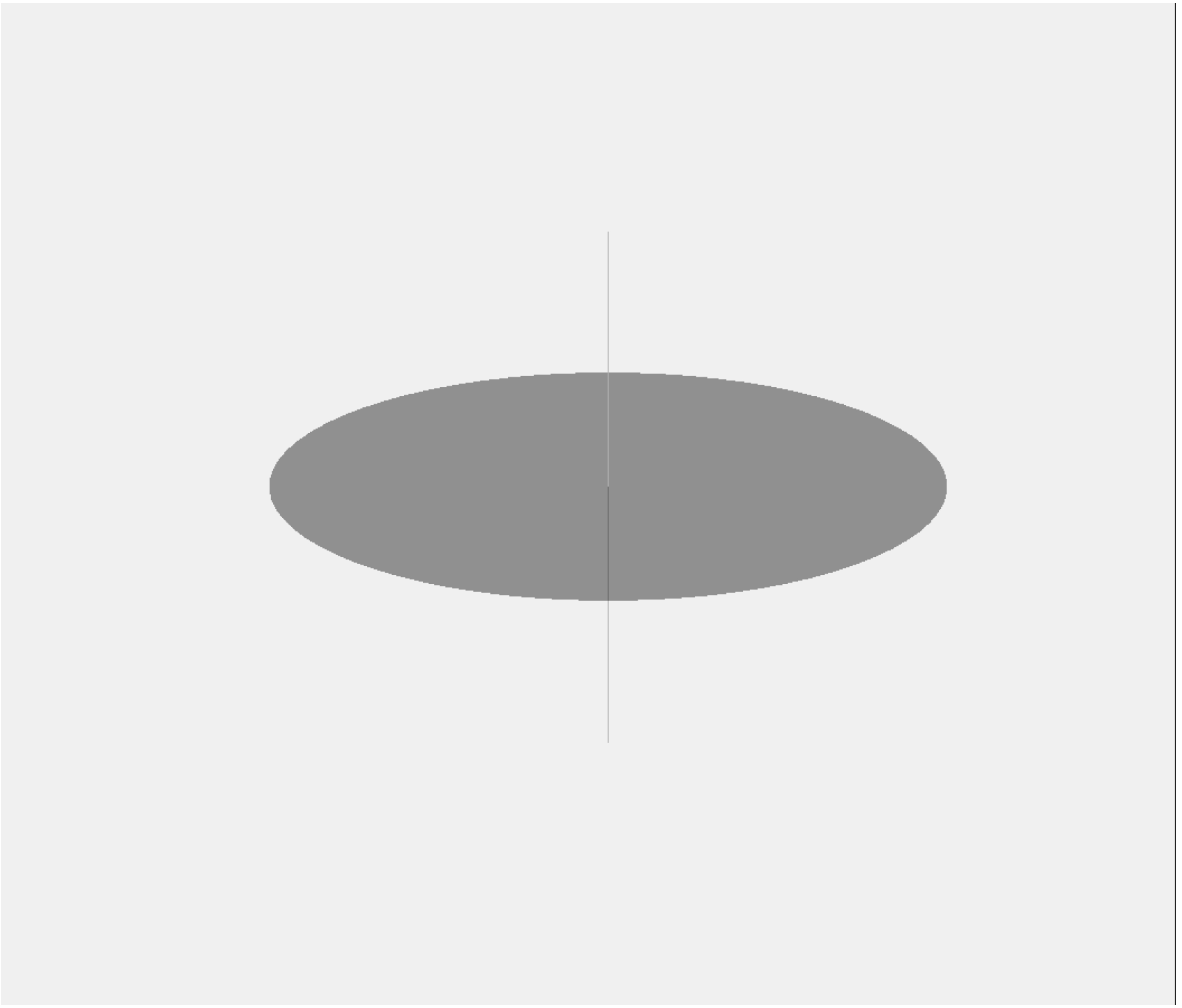


n = 4

Spatial orbits and bifurcations



n = 6



n = 8

DDE framework

$$\ddot{w}_j = -2\sqrt{s_1} i \dot{w}_j + s_1 w_j - \sum_{i=1 (i \neq j)}^{n} \frac{w_j - w_i}{\left\| (w_j, z_j) - (w_i, z_i) \right\|^3},$$

$$\ddot{z}_j = -\sum_{i=1 (i \neq j)}^{n} \frac{z_j - z_i}{\left\| (w_j, z_j) - (w_i, z_i) \right\|^3},$$

The solutions of Newton's equations with the symmetries $u_j(t) = e^{j\bar{J}\zeta} u_n(t + jk\zeta)$ are zeros of the map

$$\mathcal{G}(u_n, \omega) = \omega^2 \ddot{u}_n + 2\omega\sqrt{s_1}\bar{J}\dot{u}_n - s_1 \bar{I} u_n + \sum_{j=1}^{n-1} \frac{u_n - e^{j\bar{J}\zeta} u_n(t + jk\zeta)}{\left\| u_n - e^{j\bar{J}\zeta} u_n(t + jk\zeta) \right\|^3} : X \times \mathbb{R} \to Y$$

defined in spaces X and Y of analytic 2π-periodic functions, which we will specify later in Fourier components.

The equation $\mathcal{G}(u_n, \omega) = 0$ is a delay differential equation (DDE).

Symmetries

$$\mathcal{G}(u_n, \omega) = \omega^2 \ddot{u}_n + 2\omega\sqrt{s_1}\bar{J}\dot{u}_n - s_1\bar{I}u_n + \sum_{j=1}^{n-1} \frac{u_n - e^{j\bar{J}\zeta}u_n(t + jk\zeta)}{\left\| u_n - e^{j\bar{J}\zeta}u_n(t + jk\zeta) \right\|^3}$$

These equations have three symmetries inherited from Newton's equations so the solutions are invariant under

1. rotations in the plane, $e^{\theta i}w_j$
2. changes in the phase, $u(t + \varphi)$
3. translation in $z_j + c$

$$(\theta, \varphi, \tau)u(t) = e^{\bar{J}\theta}u(t + \varphi) + (0,0,\tau),$$

Augmented equations

The function $\mathscr{G}$ is invariant under the same group actions
$\mathscr{G}((\theta, \varphi, \tau)u_0) = (\theta, \varphi, \tau)\mathscr{G}(u_0) = 0$

We augment the equation with three Lagrange multipliers

$$\mathscr{G}(u, \omega) + \sum_{j=1}^{3} \lambda_j A_j(u) = 0.$$

where A_j are the three generator fields

1. $A_1 = diag(J,0)x_j$ rotations in the plane
2. $A_2 = \dot{u}$ changes in the phase
3. $A_3 = (0,0,1)$ translations in z

Poincaré Sections

The augmented equations $\mathscr{G}(u,\omega) + \sum_{j=1}^{3} \lambda_j A_j(u) = 0$ together with three equations (Poincaré sections)

1. $I_1(u) = \int_0^{2\pi} u(t) \cdot \bar{J}\tilde{u}'(t)\ dt = 0$

2. $I_2(u) = \int_0^{2\pi} \left(u(t) - \tilde{u}_n(t)\right) \cdot \tilde{u}_n'(t)\ dt = 0,$

4. $I_3(u) = \int_0^{2\pi} u_3(t)\ dt = 0$

correspond conservation of translations in z, rotations, and change of phase

Automatic Differentiation

We write the DDE as a higher dimensional equation with polynomial nonlinearities

$$\dot{u} = v$$

$$\dot{v} = \frac{1}{\omega^2}\left(-2\omega\sqrt{s_1}\bar{J}v + s_1\bar{I}u - \sum_{j=1}^{n-1} w_j^3\left(u(t) - e^{\,j\bar{J}\zeta}u(t+jk\zeta)\right)\right) + \lambda_1\bar{J}u + \lambda_2 v + \lambda_3 e_3$$

$$\dot{w}_j = -\,w_j^3\left\langle v(t) - e^{\,j\bar{J}\zeta}v(t+jk\zeta), u(t) - e^{\,j\bar{J}\zeta}u(t+jk\zeta)\right\rangle + \alpha_j w_j^3,$$

supplemented with the conditions

$$w_j(0) = \frac{1}{\left\| u(0) - e^{\,j\bar{J}\zeta}u(jk\zeta) \right\|}, \quad j = 1,\ldots,n-1$$

Fourier Map

Consider the Banach space

$$\ell_\nu^1 = \left\{ c = (c_\ell)_{\ell\in\mathbb{Z}} : \|c\|_\nu = \sum_{\ell\in\mathscr{Z}} |c_\ell| \nu^{|\ell|} < \infty \right\},$$

that is a Banach algebra under discrete convolution $(a * b)_k = \sum_{k_1+k_2=k} a_{k_1} b_{k_2}$, where $a, b \in \ell_\nu^1$.

Defining $Y = \mathbb{C}^3 \times \mathbb{C}^{n-1} \times (\tilde{\ell}_\nu^1)^3 \times (\tilde{\ell}_\nu^1)^3 \times (\tilde{\ell}_\nu^1)^{n-1}$ the Fourier map

$F : X \times \mathbb{R} \to Y$ is defined by

$$F(x,\omega) = \begin{pmatrix} \eta(u) \\ \gamma(u,w) \\ f(u,v) \\ g(\lambda,u,v,w,\omega) \\ h(\alpha,u,v,w) \end{pmatrix}.$$

A Posteriori Validation of the Fourier Map

Demonstrate that a Newton-like operator is a contraction in a closed ball centered at a numerical approximation $\bar{x}$.

Theorem (Radii Polynomial Approach)

$\bar{x} \in X$, $r > 0$, $F : X \to Y$ Fréchet differentiable on the ball $B_r(\bar{x})$. Let $A^\dagger \in B(X, Y)$ (approximation of $DF(\bar{x})$) and $A \in B(Y, X)$ (approximate inverse of $DF(\bar{x})$), $AF : X \to X$, with A injective.

Let $Y_0, Z_0, Z_1, Z_2 \geq 0$ be such that

$$\begin{aligned} \|AF(\bar{x})\|_X &\leq Y_0, \\ \|I - AA^\dagger\|_{B(X)} &\leq Z_0, \\ \|A[DF(\bar{x}) - A^\dagger]\|_{B(X)} &\leq Z_1, \\ \|A[DF(\bar{x} + b) - DF(\bar{x})]\|_{B(X)} &\leq Z_2 r, \quad \forall\ b \in B_r(0)\,. \end{aligned}$$

Define the radii polynomial $p(r) = Z_2 r^2 + (Z_1 + Z_0 - 1)r + Y_0$. If $0 < r_0 \leq r$ so that $p(r_0) < 0$, then $F(\tilde{x}) = 0$ with $\tilde{x} \in B_{r_0}(\bar{x})$

Several existence proofs

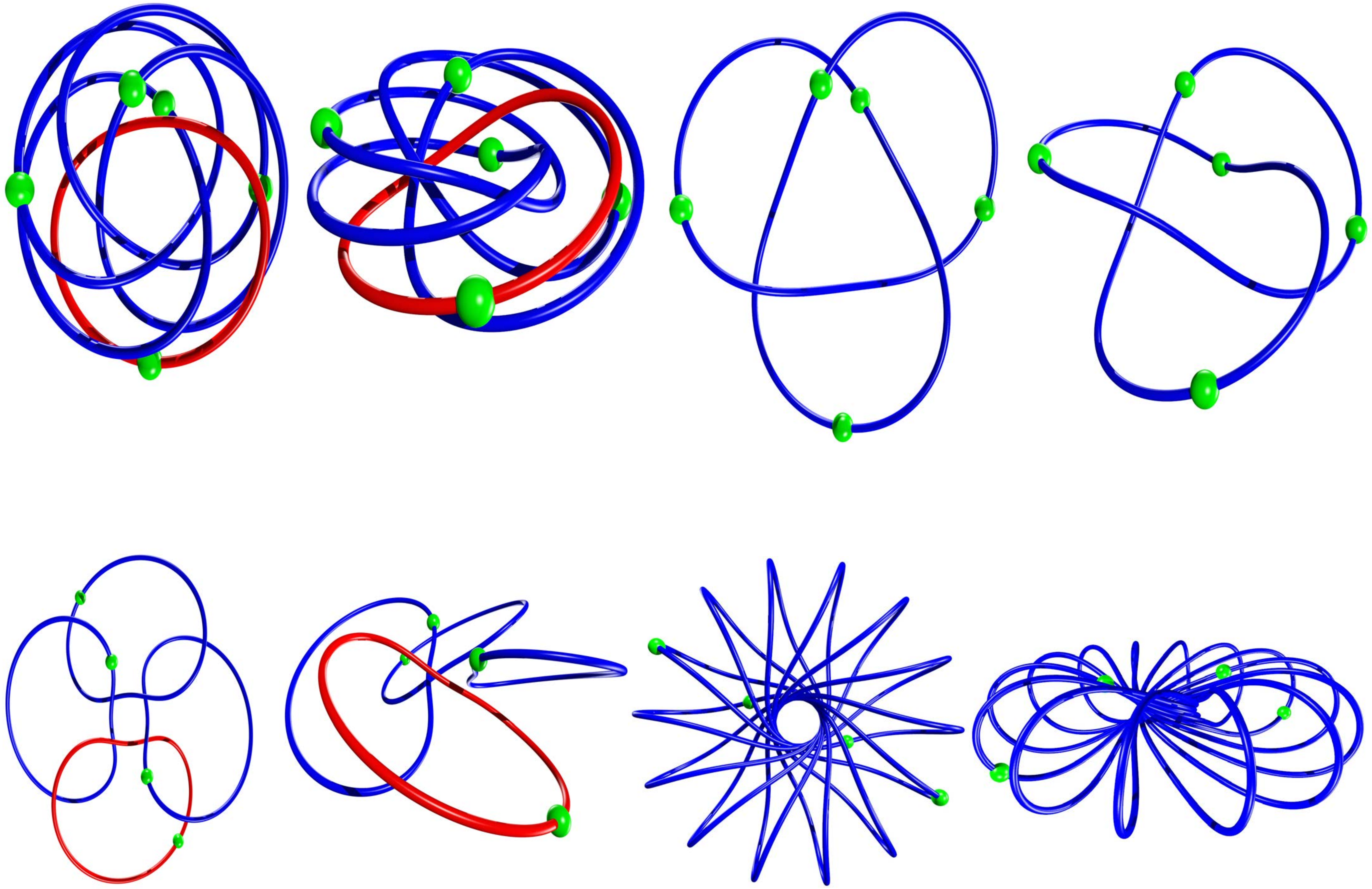
Several existence proofs for planar and spatial choreographies use a different setup

- Equations of motion in the phase space and CAPD (Kapela-Simó ’07), (Kapela-Simó '17), (Kapela-Zgliczynski '03)
- Functional analytic methods applied directly to periodic orbits of the Hamiltonian vector field (Arioli-Barutello-Terracini ’03)
- For spatial choreographies the dimension scales like $6n$

Our approach uses automatic differentiation to reduce to a polynomial nonlinearity adding an additional scalar equation for each body

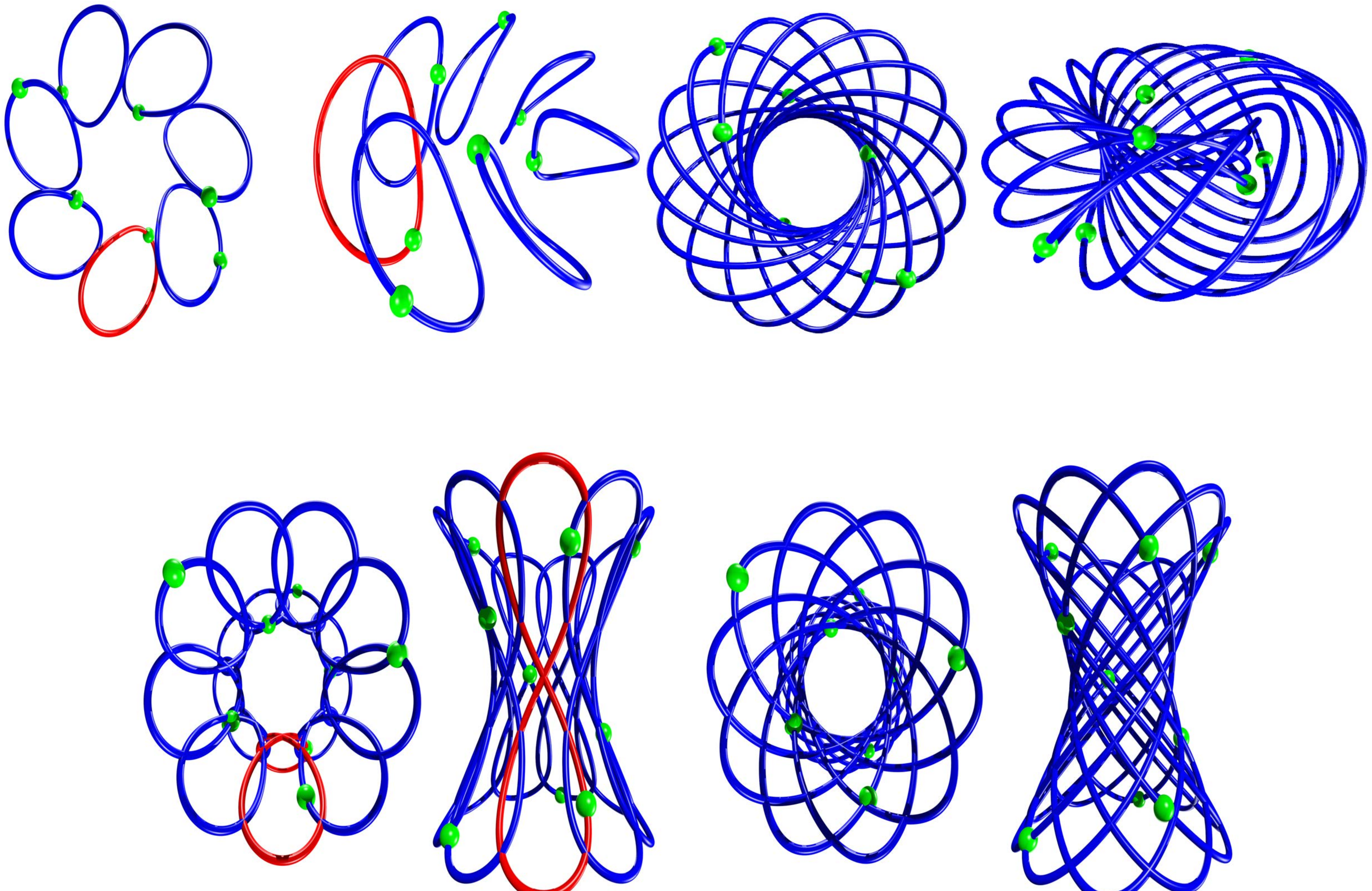
$$6 + (n - 1)$$

Computer Assisted Proofs



- R. C., C. García-Azpeitia, J.P. Lessard, and J.D. Mireles-James, **Torus knot choreographies in the n -body problem**, Nonlinearity, Volume 34, Number 1, (2021) 313-349

Computer Assisted Proofs



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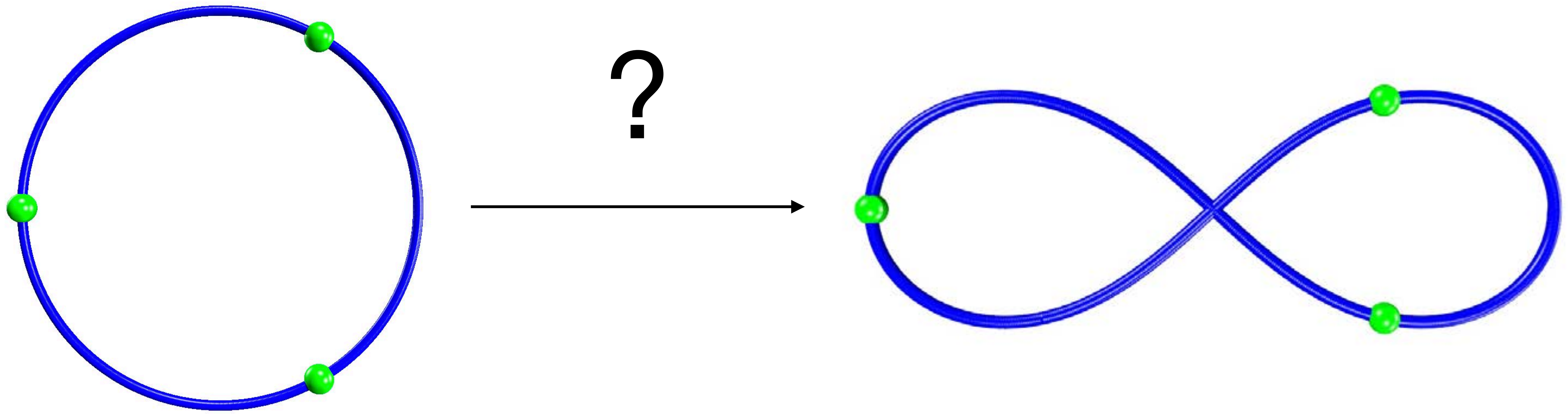
A conjecture of Marchal

The three body problem admits at least two choreography solutions



The symmetry group of the figure eight is a 12th-order subgroup of the equilateral triangle

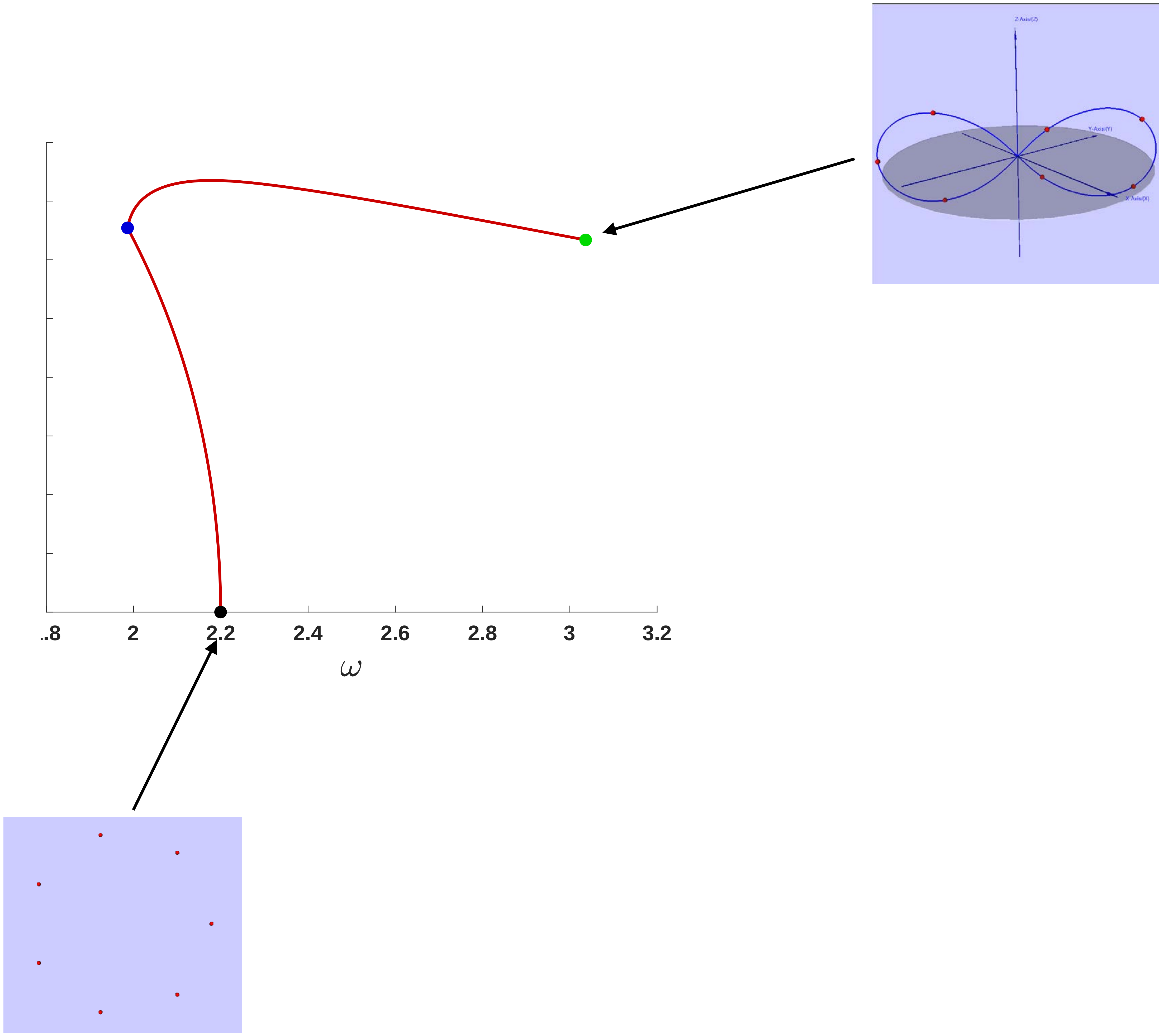
P12 is an out of plane family of periodic orbits coming out to the triangle

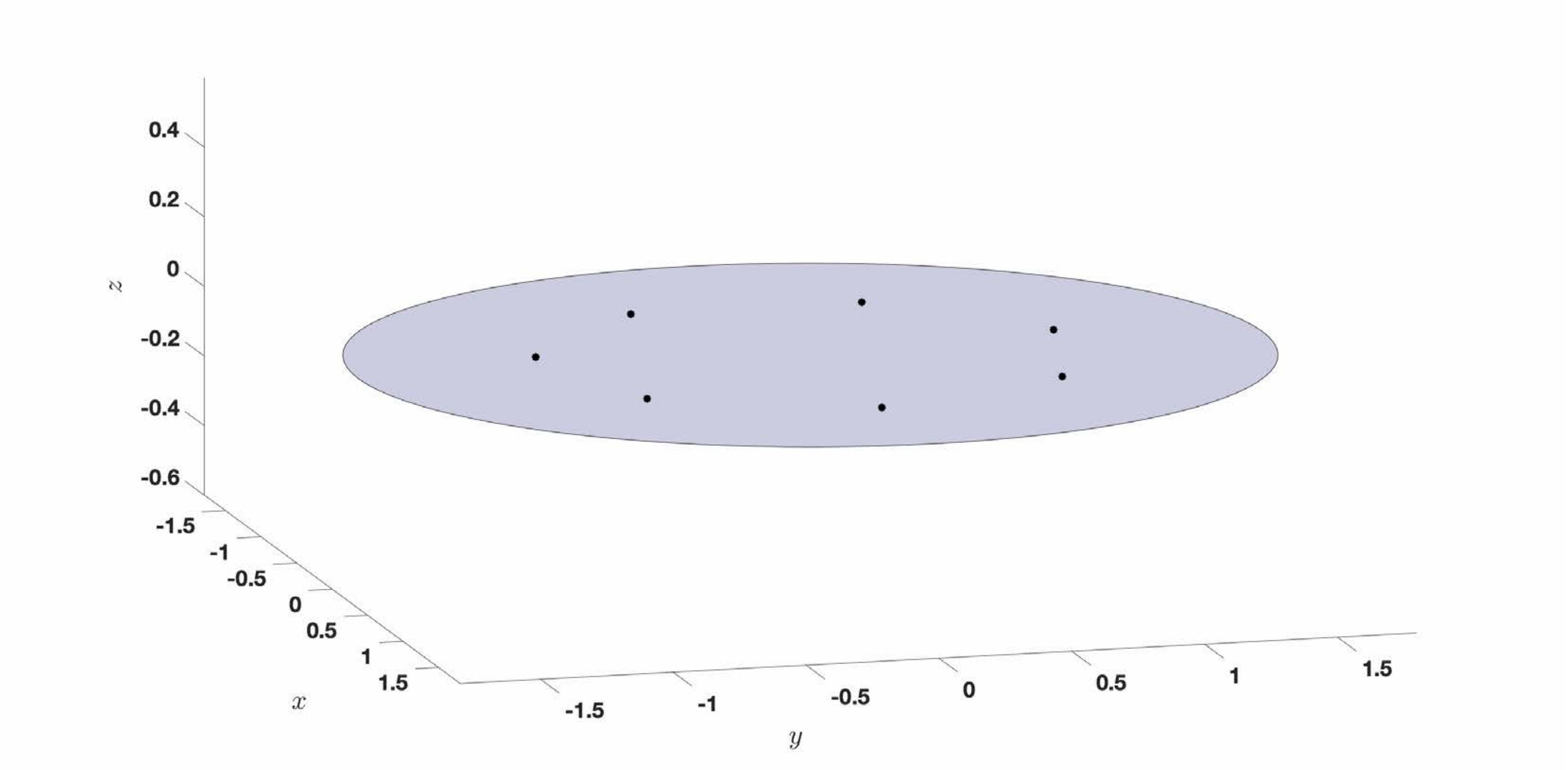
A conjecture of Marchal

“The three body equilateral triangle of Lagrange and the three body figure eight are in the same continuation class”

"In rotating coordinates the three body equilateral triangle configuration can be continued by the variation of the frequency to a periodic orbit of the rotating problem which, when converted back to inertial coordinates, is the figure eight of Moore, Chenciner, and Montgomery."

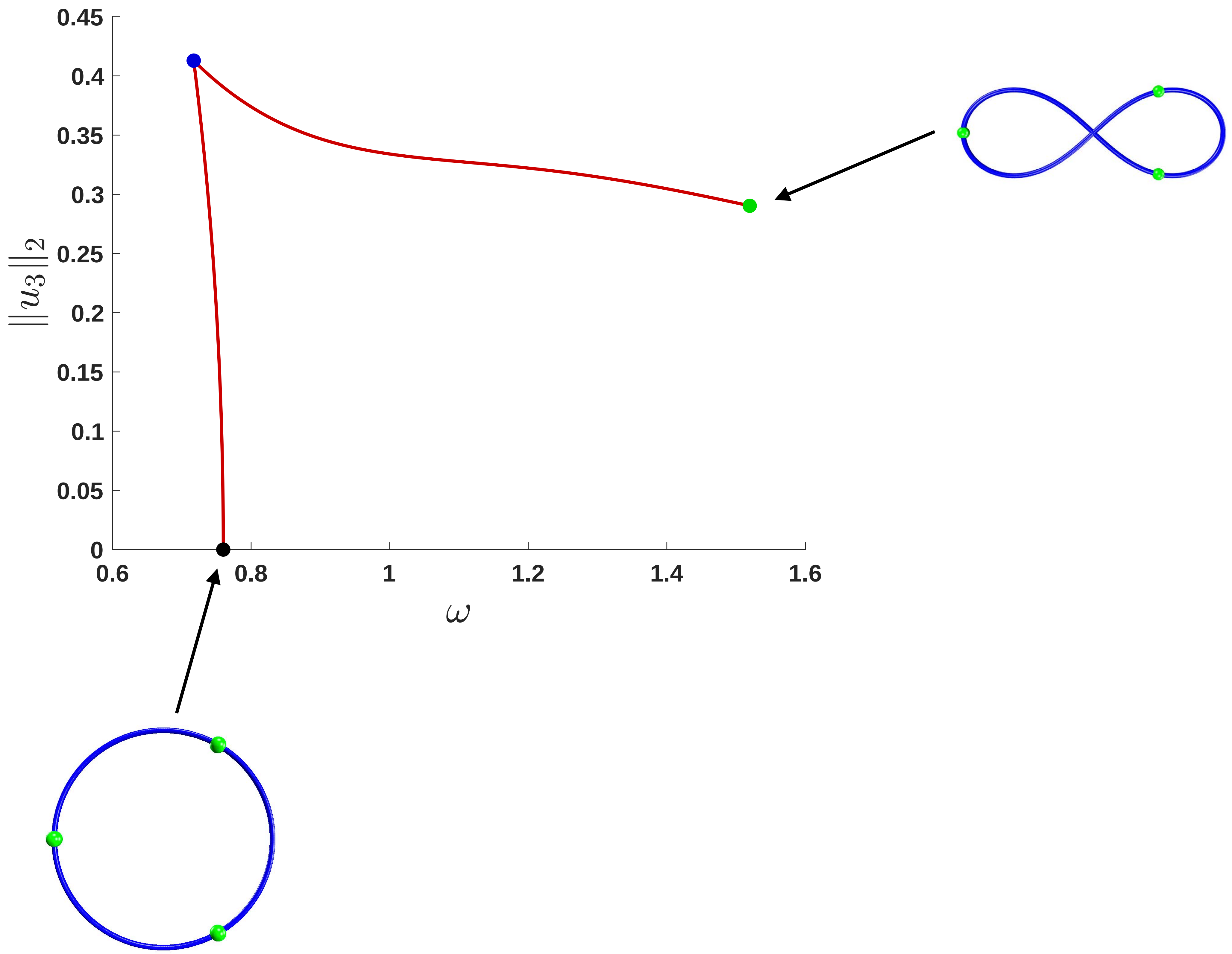
From the heptagon to the Figure eight

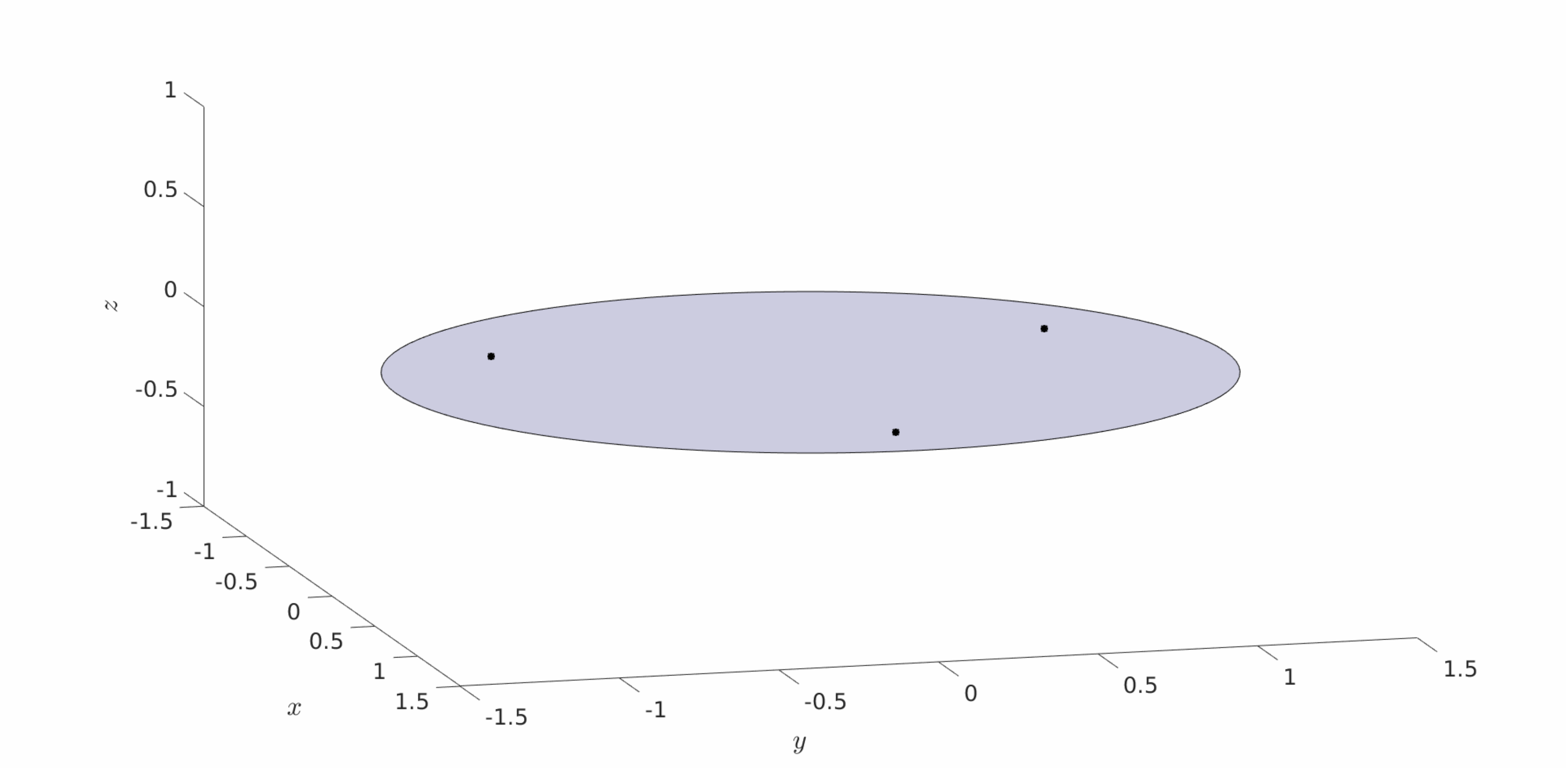




- R. C., E. Doedel, and C. García-Azpeitia, Symmetries and choreographies in families that bifurcate from the polygonal relative equilibrium of the n -body problem, Celestial Mech. Dynam. Astronom. 130 (2018), 130:48.

From the triangle to the eight

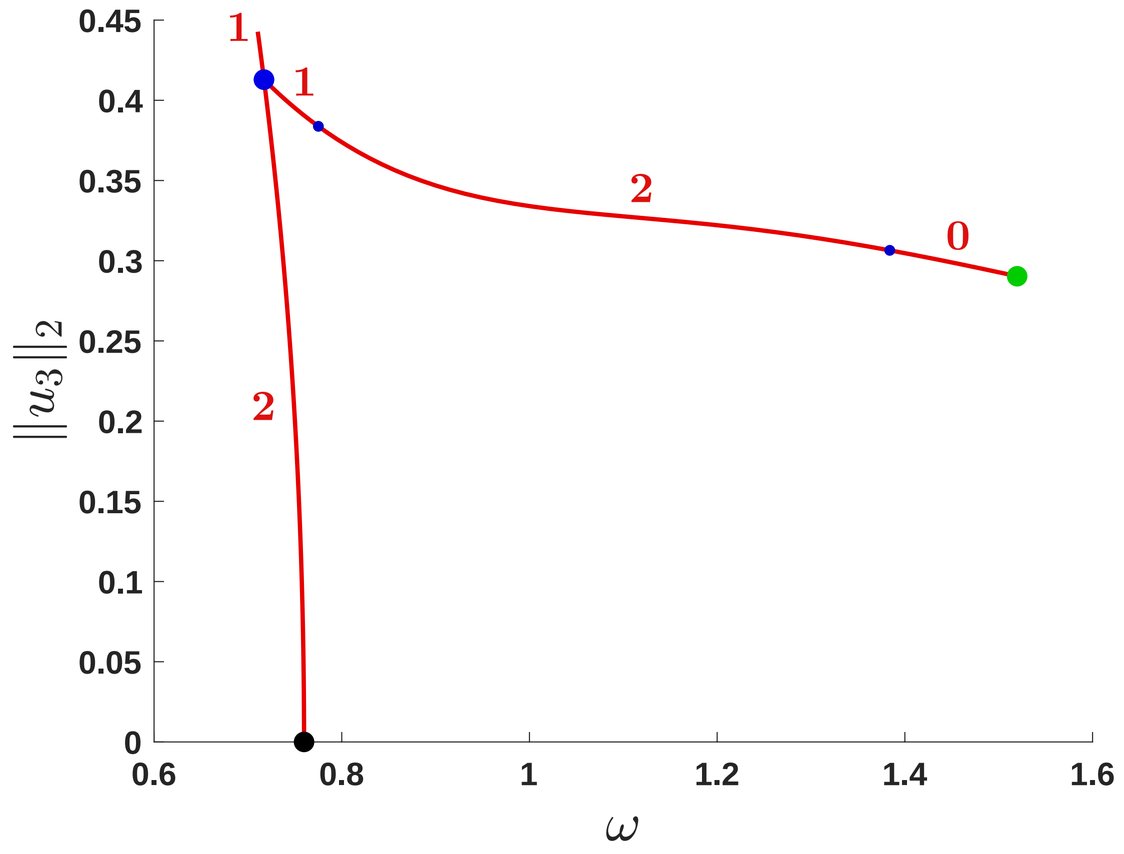




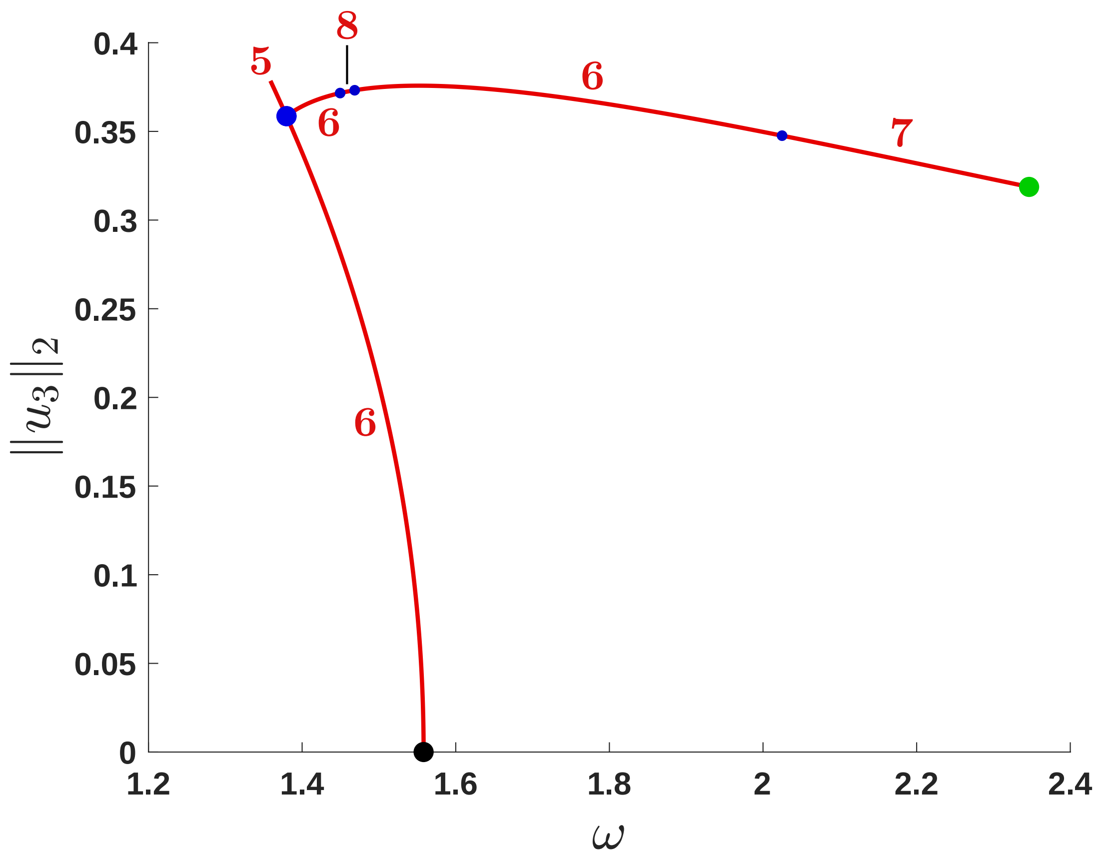
Evidence for Marchal´s conjecture.

- R. C., C. García-Azpeitia, J.P. Lessard, and J.D. Mireles-James, From the Lagrangre polygons to the figure eight I, Celestial Mech. Dynam. Astronom., volume 133, Article number: 10 (2021)

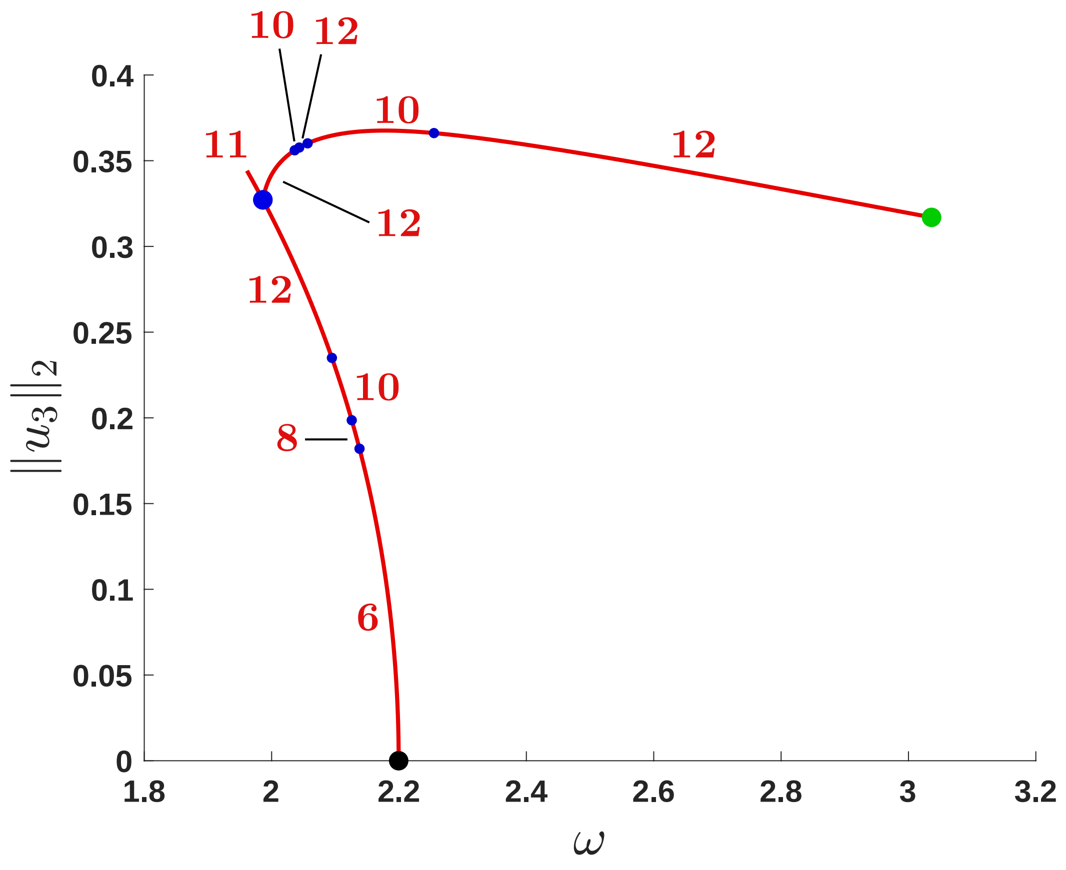
Many bifurcations (n = 3, 5, 7, 9)



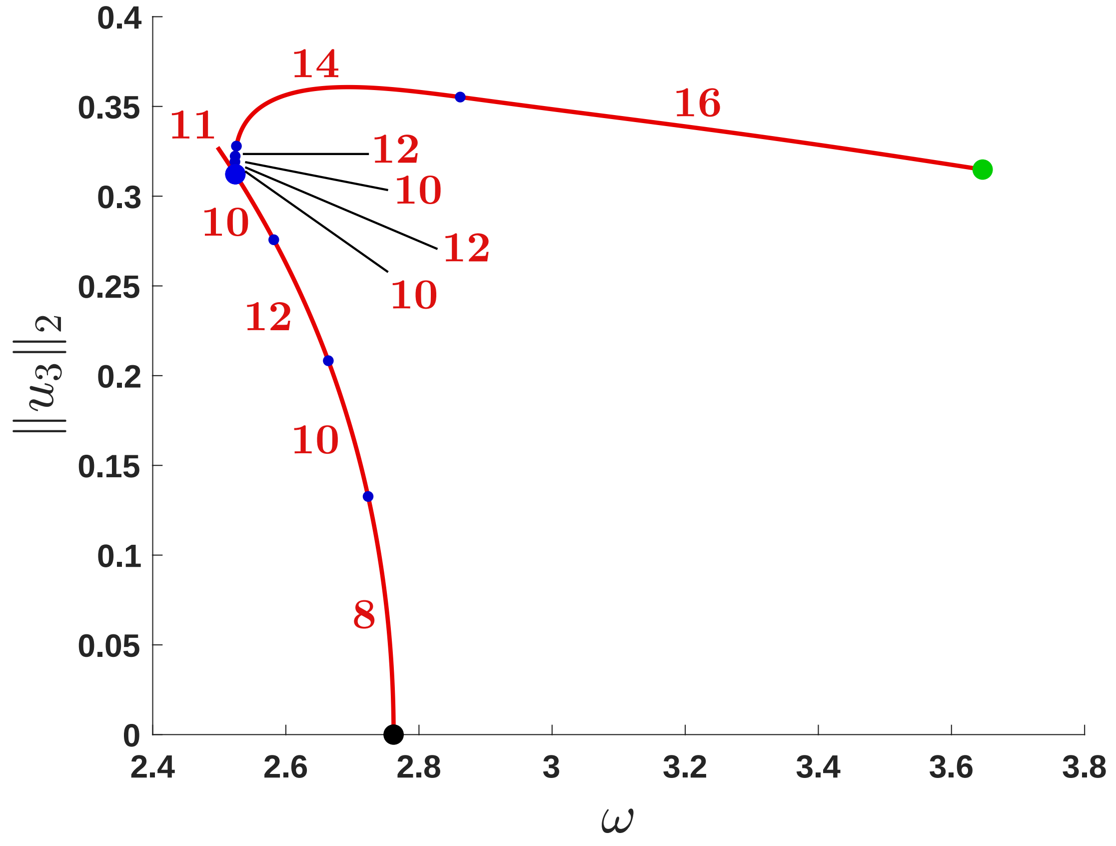
N = 3



N = 5



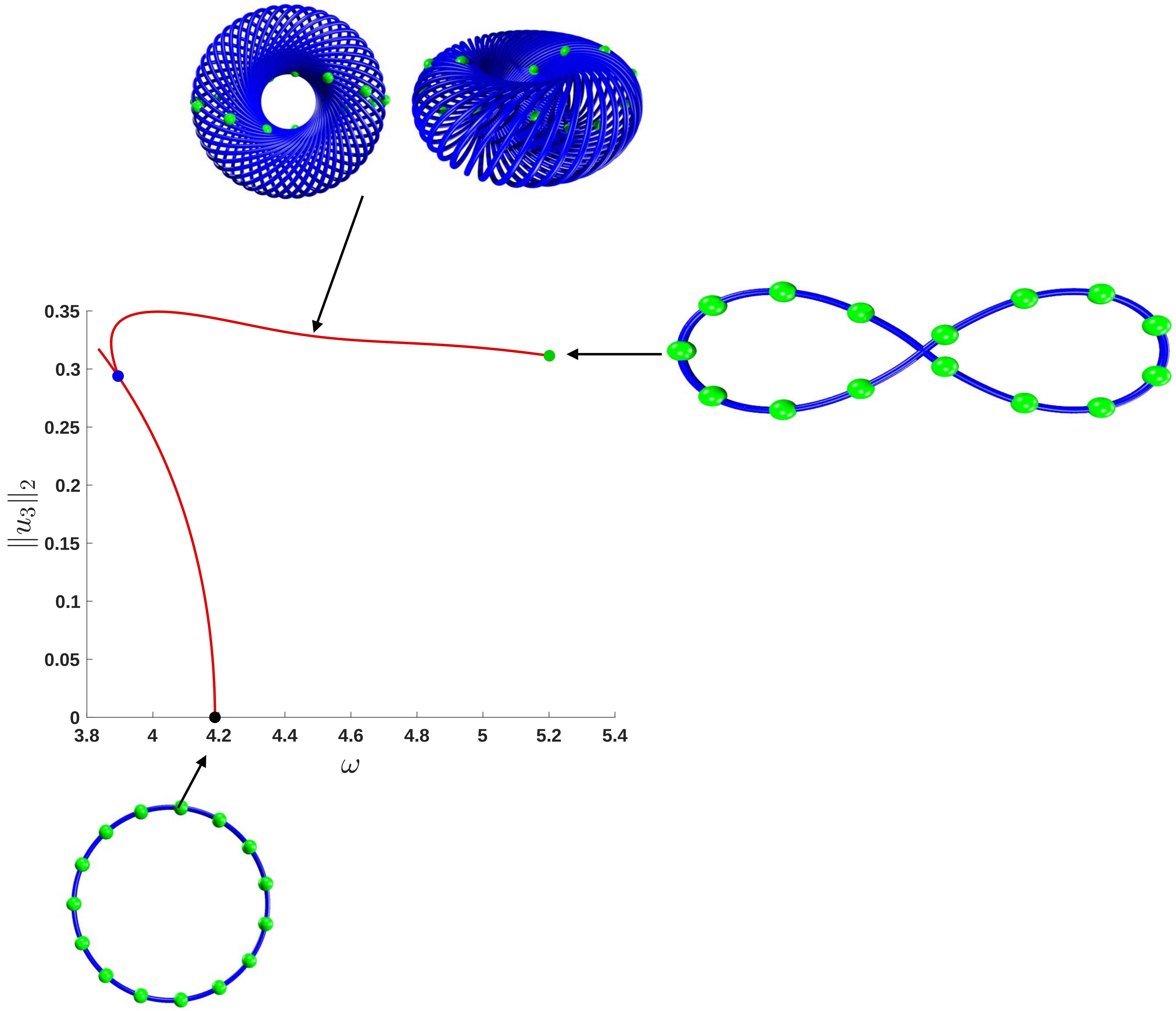
N = 7

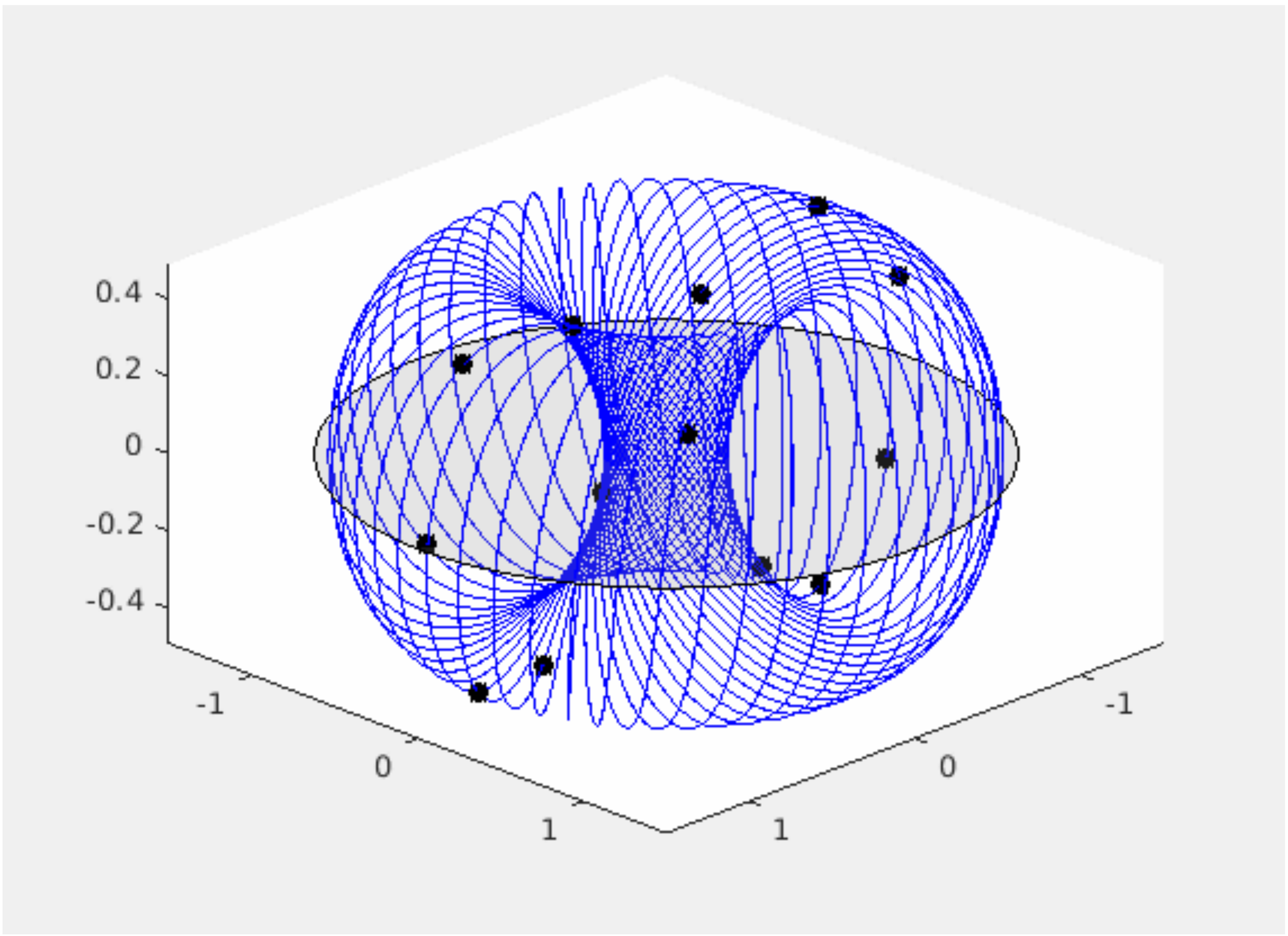


N = 9

- R. C., C. García-Azpeitia, J.P. Lessard, and J.D. Mireles-James, From the Lagrangre polygons to the figure eight I, Celestial Mech. Dynam. Astronom., volume 133, Article number: 10 (2021)

The eight when n > 3 is not stable





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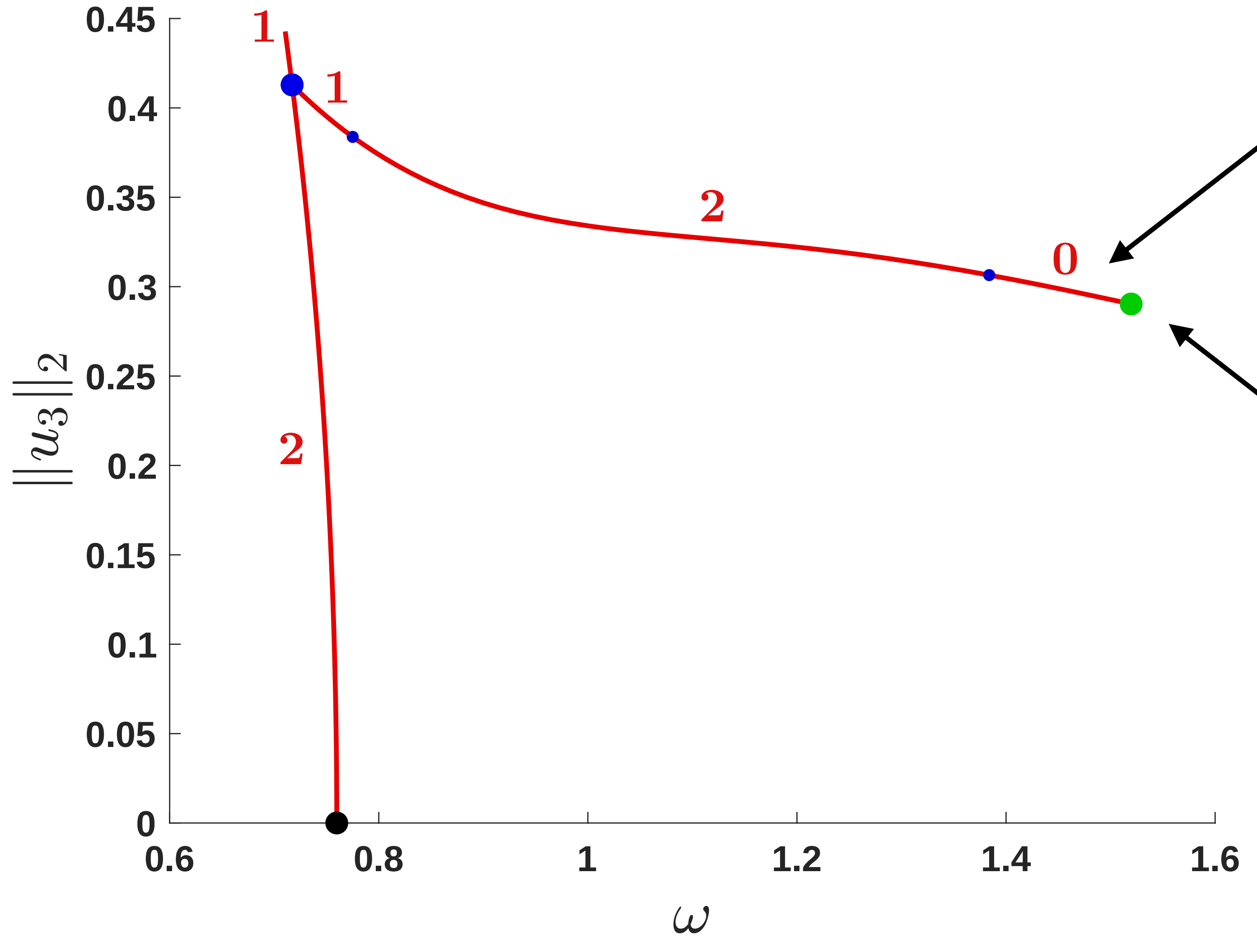
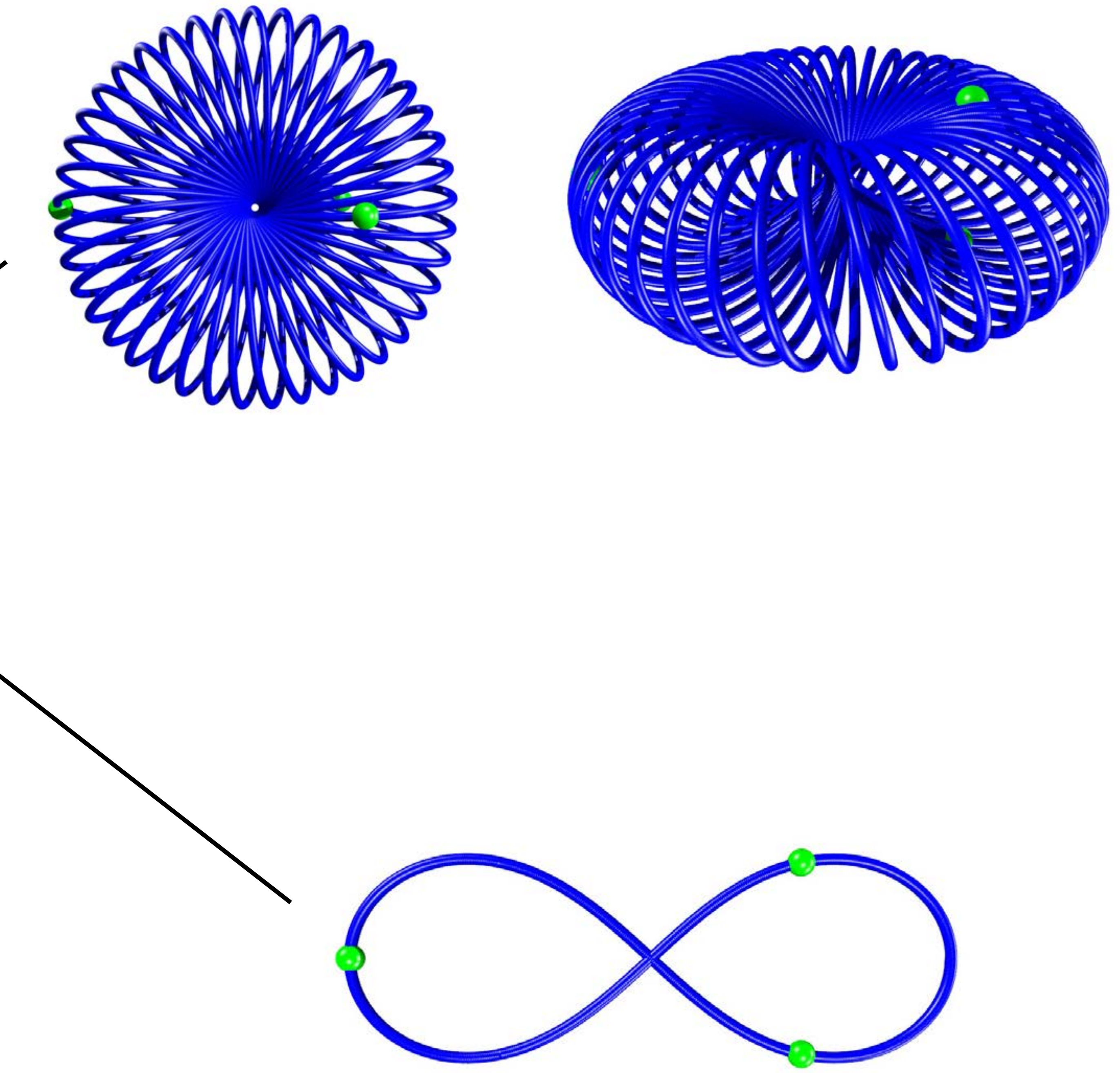
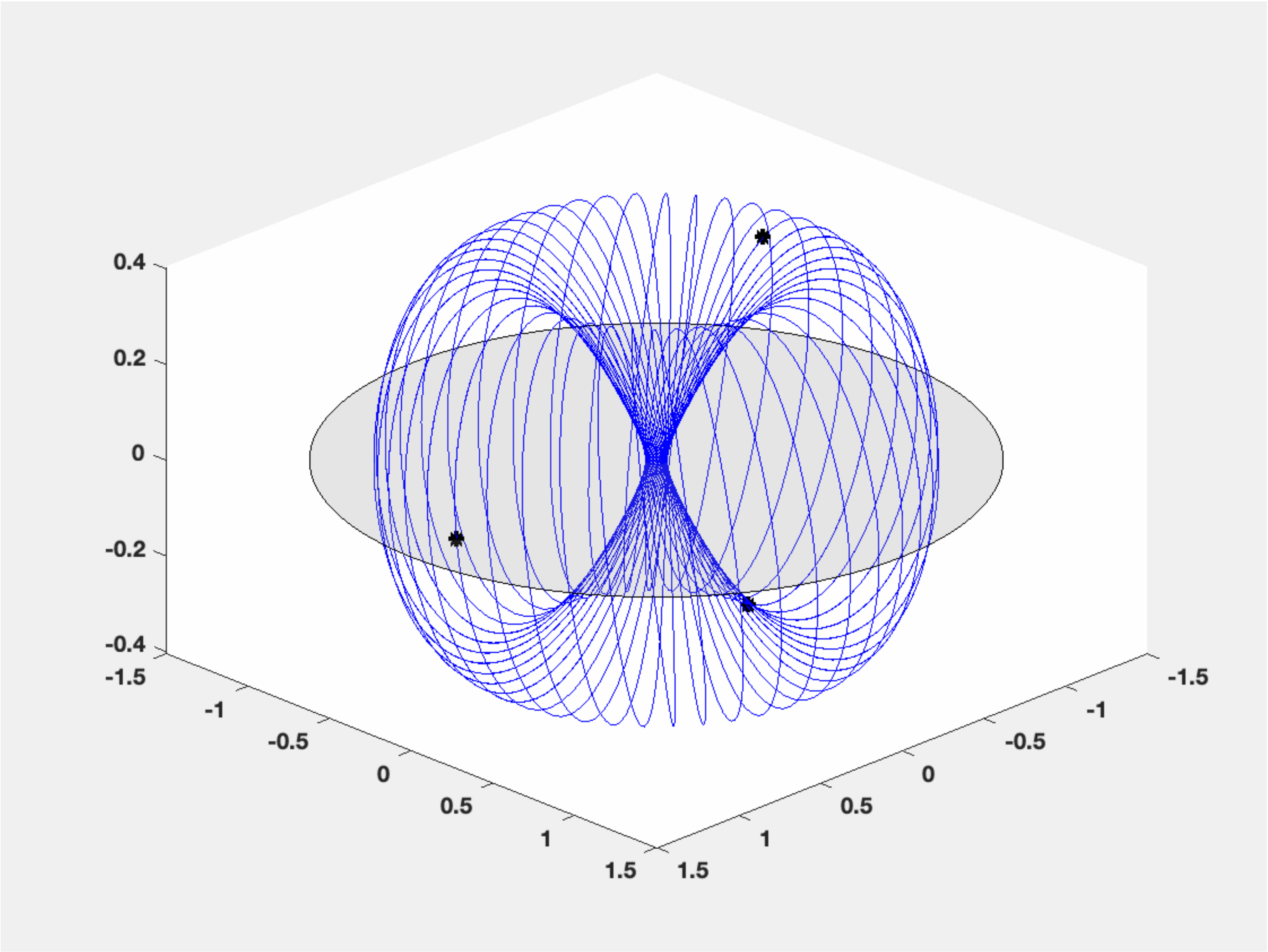
The eight when n = 3 is stable







A spatial choreography that is stable numerically

https://mym.iimas.unam.mx/renato/choreographies/index.html

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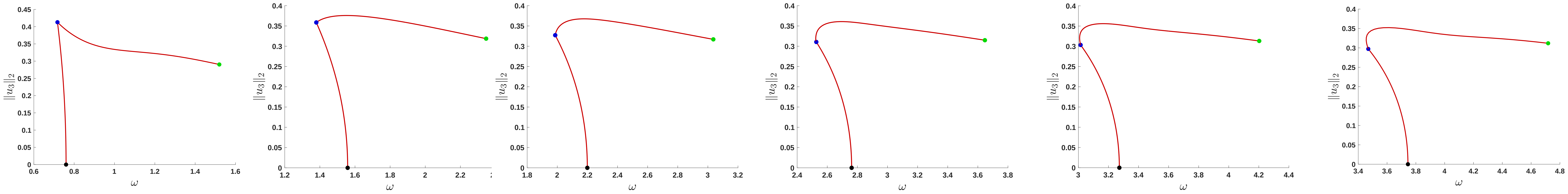
An extension of the conjecture

(Generalized Marchal's Conjecture)

For any odd number of bodies, the n-gon choreography and the n-body figure eight are in the same continuation class.



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Summary

- We give mathematically rigorous proofs of the existence of torus knot choreographies
- Functional analytic and computer assisted approach
- A choreography is the zero of a nonlinear operator on a Banach algebra
- We give a proof of the existence of a trefoil knot choreography and countably many choreographies close to it
- The method is systematic and can be applied to spatial choreographies
- We give numerical evidence for a conjecture of Marchal
- We extend Marchal's conjecture
- We find numerically stable spatial choreographies with $n = 3$

Thank you

https://mym.iimas.unam.mx/renato/choreographies/index.html

- R. C., E. Doedel, and C. García-Azpeitia, **Symmetries and choreographies in families that bifurcate from the polygonal relative equilibrium of the n -body problem**, Celestial Mech. Dynam. Astronom. 130 (2018), 130:48.
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