Nijenhuis Geometry

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(based on joint works with A. Konyaev and V. Matveev)

Geometry, Dynamics, and Mechanics Seminar

March 30, 2021

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Plan

- What is Nijenhuis Geometry? Our motivation
- Nijenhuis operators: definitions and basic facts
- Splitting theorem
- Nijenhuis operators with complex eigenvalues
- Linearisation and left-symmetric algebras
- Non-degenerate LSAs and stable singular points
- Nijenhuis perturbations of Jordan block and versal deformations
- $\mathfrak{gl}$-regular Nijenhuis operators
- Some global results
- ...
- Open problems

Albert Nijenhuis

Albert Nijenhuis (November 21, 1926 – February 13, 2015),

Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics. https://en.wikipedia.org/wiki/Albert_Nijenhuis

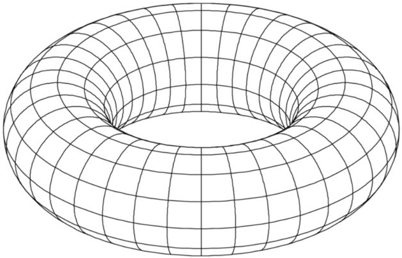
A. Nijenhuis, X_{n-1}-forming sets of eigenvectors. Proc. Kon. Ned. Akad. Amsterdam 54 (1951), 200–212.

Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

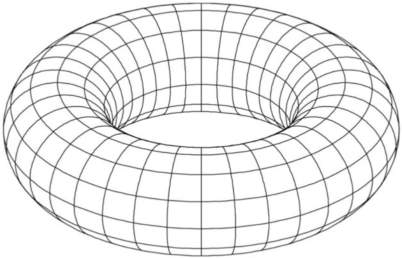
What is GEOMETRY?

Space, manifold M^n



What is GEOMETRY?

Space, manifold M^n $\quad + \quad$ Structure



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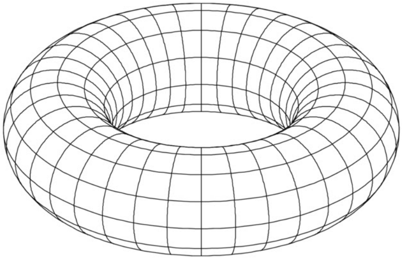

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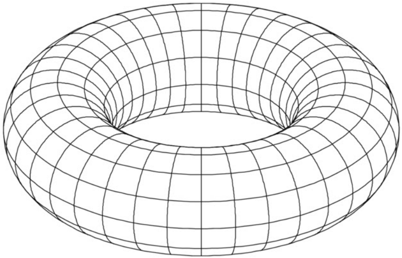

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Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \left(a_{ij}(x)\right)$ whose entries depend on coordinates $x = (x^1, \dots, x^n)$ and satisfy some algebraic and differential conditions.

Nijenhuis geometry. Our motivation

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Definition

By Nijenhuis operators we understand $(1, 1)$-tensors $L = \left(L^i_j(x)\right)$ with vanishing Nijenhuis torsion. A manifold M endowed with such an operator is called a Nijenhuis manifold.

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- In the theory of projectively equivalent Riemannian metrics, such metrics are related by means of a Nijenhuis operator.

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- In integrable systems on Lie algebras, the algebraic Nijenhuis operators are used in the study of Lie-Poisson pencils.
- In algebra, left-symmetric algebras (also known as pre-Lie algebras) are in one-to-one correspondence with Nijenhuis operators whose entries are linear in coordinates.

Research programme: where we are and what we want?

Poisson geometry	Nijenhuis geometry
< 1970	current situation
Local description at regular points	Local description at regular points
Tool in Hamiltonian Mechanics	Tool in integrable systems
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Shall we try?

New research agenda:

from local normal forms

to singularities and global analysis

Many open problems !!!

Nijenhuis operators: definitions and basic facts

Let L be a $(1,1)$-tensor field (operator) on a smooth manifold M. The Nijenhuis torsion $\mathcal{N}_L$ of the operator L is a $(1,2)$-tensor that can be defined in several equivalent ways.

Definition

- As a vector-valued 2-form:

$$\mathcal{N}_L(\xi, \eta) = L^2[\xi, \eta] + [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta].$$

- As a map from "vector fields" to "endomorphisms":

$$\mathcal{N}_L : \ \xi \mapsto L\mathcal{L}_\xi L - \mathcal{L}_{L\xi} L.$$

- As a map from "1-forms" to "2-forms":

$$\mathcal{N}_L : \ \alpha \mapsto \beta, \quad \text{where}$$

$$\beta(\cdot\,,\cdot) = \mathrm{d}\,(L^{*2}\alpha)(\cdot\,,\cdot) + \mathrm{d}\,\alpha(L\cdot\,, L\cdot) - \mathrm{d}\,(L^*\alpha)(L\cdot\,,\cdot) - \mathrm{d}\,(L^*\alpha)(\cdot\,, L\cdot).$$

- In local coordinates:

$$(\mathcal{N}_L)^i_{jk} = L^l_j \frac{\partial L^i_k}{\partial x^l} - L^l_k \frac{\partial L^i_j}{\partial x^l} - L^i_l \frac{\partial L^l_k}{\partial x^j} + L^i_l \frac{\partial L^l_j}{\partial x^k}.$$

Elementary examples

- Constant operator:
$$L(x) = \left(L^i_j\right)$$
with L^i_j being constant for all i, j
- Scalar operator:
$$L(x) = f(x) \cdot \mathrm{Id},$$
where $f(x)$ is an arbitrary smooth function
- Complex structure on a smooth manifold
- $L(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$
- $L(x) = \begin{pmatrix} x_5 & x_4 & x_3 & x_2 & x_1 \\ & x_5 & x_4 & x_3 & x_2 \\ & & x_5 & x_4 & x_3 \\ & & & x_5 & x_4 \\ & & & & x_5 \end{pmatrix}$

Basic facts

Let L be a Nijenhuis operator, then

- For any polynomial $p(\cdot)$ with constant coefficients, the operator $p(L)$ is also Nijenhuis. Moreover, the same holds true for any real analytic matrix function $f(L)$.
- $$\mathrm{d}\,(\mathrm{tr}\,L^k) = k(L^*)^{k-1}\mathrm{d}\,\mathrm{tr}\,L,$$
  where $L^* : T_q^*M \to T_q^*M$ denotes the operator dual to L.
- $$L^*\mathrm{d}\,(\det L) = \det L \cdot \mathrm{d}\,\mathrm{tr}\,L.$$
- Let $\lambda(x)$ be a smooth eigenvalue of L. Then
  $$\big(L - \lambda(x)\cdot \mathrm{Id}\big)^*\mathrm{d}\,\lambda(x) = 0.$$
- If $L(p_0)$ is diagolisable over $\mathbb{R}$ and has no multiple eigenvalues, then L is locally diagonalisable, i.e., there is a local coordinate system on $U(p_0)$ such that $L = \mathrm{diag}\big(\lambda_1(x_1), \dots, \lambda_n(x_n)\big)$.

Generic and singular points

Definition

- A point $p \in M$ is called *algebraically generic*, if the algebraic type of L does not change in some neighbourhood $U(p) \subset M$.
- A point $p \in M$ is called *singular*, if it is not algebraically generic.
- An operator $L(p)$ (and the corresponding point $p \in M$) is called gl-*regular*, if its $GL(n)$-orbit $\mathcal{O}\big(L(p)\big) = \{X\, L(p)\, X^{-1} \mid X \in GL(n)\}$ has maximal dimension, namely, $\dim O\big(L(p)\big) = n^2 - n$.
- A point $p \in M$ is called *differentially non-degenerate*, if the differentials $\mathrm{d}\,\sigma_1(x), \dots, \mathrm{d}\,\sigma_n(x)$ of the coefficients of the characteristic polynomial of $L(x)$ are linearly independent at this point.
- A singular point $p \in M$ is called *(C^k-) stable*, if for any perturbation
$$L(x) \quad \mapsto \quad \widetilde{L}(x) = L(x) + R_k(x)$$
such that $\widetilde{L}(x)$ is Nijenhuis and $R_k(x)$ has zero of order k at the point $p \in M$, there exists a local diffeomorphism $\phi : U(p) \to \widetilde{U}(p)$, $\phi(p) = p$, that transforms $L(x)$ to $\widetilde{L}(x)$.

Splitting theorem

Let $\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t)$ be a factorisation of the characteristic polynomial of L at a point $p \in M$ into two factors with no common roots (over $\mathbb{R}$). We call such factorisations *admissible*. This factorisation can be naturally extended to a neighbourhood of p.
Consider the distributions $\mathcal{D}_i = \operatorname{Ker}\chi_i(L)$ $(i = 1, 2)$ that provide a natural decomposition of the tangent bundle $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

Theorem

The distributions $\mathcal{D}_i = \operatorname{Ker}\chi_i(L)$ are both integrable. Moreover, in any adapted coordinate system $(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$:

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}. \tag{1}$$

In other words, L splits into a direct sum of two Nijenhuis operators: $L(x, y) = L_1(x) \oplus L_2(y)$.

Corollary

Every Nienhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \dots$ each of which at the point $p \in M$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Generalised Nirenberg-Newlander theorem

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Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

Generalised Nirenberg-Newlander theorem

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Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

1. *M is a complex manifold w.r.t. a complex structure J canonically associated with L.*
2. *L is a complex holomorphic tensor field on M w.r.t. J, i.e. can be written in the form*
$$L^{\mathbb{C}} = \sum_{i,j=1}^{n} l^i_j(z)\, \mathrm{d}\, z^j \otimes \partial_{z^i}$$
with all the functions $l^i_j(z)$ being holomorphic in complex coordinates $z_1, \dots, z_n$.
3. *The complex Nijenhuis torsion of L vanishes, i.e.*
$$\left(\mathcal{N}_L^{\mathbb{C}}\right)^i_{jk} = l^m_j \frac{\partial l^i_k}{\partial z^m} - l^m_k \frac{\partial l^i_j}{\partial z^m} - l^i_m \frac{\partial l^m_k}{\partial z^j} + l^i_m \frac{\partial l^m_j}{\partial z^k} = 0.$$

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Key point: $J = f(L)$ where f is an analytic function on $\mathbb{C} \setminus \mathbb{R}$

Linearisation and left-symmetric algebras

Let $p \in M$ be a singular point of *scalar type*, i.e., $L(p) = \lambda \cdot \mathrm{Id}$. Assume (w.l.o.g.) that $\lambda = 0$. Then locally:

$$L(x) = 0 + L_1(x) + L_2(x) + L_3(x) + \dots$$

where the entries of $L_k(x)$ are homogeneous polynomials in $x^1, \dots, x^n$ of degree k.

Proposition (Definition)

The linear part $L_{\mathrm{lin}} = L_1 = \left(l^i_{j,k} x^k \right)$ is itself a Nijenhuis operator that is called the *linearisation* of L at the point $p_0 \in M$.

The corresponding tensor $l^i_{j,k}$ defines a structure of a left-symmetric algebra $(\mathfrak{a}_L, *)$ on T_pM called the *isotropy LSA of L* at the singular point p (the converse is also true).

Reminder: An algebra $(\mathfrak{a}, *)$ is called *left-symmetric* if:

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \eta * (\xi * \zeta) - (\eta * \xi) * \zeta, \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}.$$

Two natural problems: linearisation and non-degeneracy.

Non-degenerate LSAs and stable singular points

Definition

A left-symmetric algebra $\mathfrak{a}$ is called *non-degenerate* if any Nijenhuis operator L, whose isotropy left-symmetric algebra $\mathfrak{a}_L$ at a singular point $p \in M$ is isomorphic to $\mathfrak{a}$, is linearisable at this point.

Theorem (A. Konyaev)

In dimension two, there are 12 types of real LSAs, six of which are non-degenerate in the smooth sense (with parameters appropriately chosen). In the real analytic case the list of non-degenerate LSAs is different (slightly larger).

Theorem (Real analytic or formal)

Let $L(x) = L_{\mathrm{lin}}(x) + L_2(x) + L_3(x) + \dots$ *with*

$$L_{\mathrm{lin}}(x) = \mathrm{diag}(x_1, x_2, \dots, x_n)$$

Then $L(x)$ is linearisable. In other words, the diagonal left-symmetric algebra is non-degenerate.

Jordan block: typical behaviour of Nijenhuis operators and versal deformations

Let $p \in M$ be a singular point for a Nijenhuis operator L (for example $L(p)$ is conjugate to a Jordan block but it is not true any more at neighbouring points).

Jordan block: typical behaviour of Nijenhuis operators and versal deformations

Let $p \in M$ be a singular point for a Nijenhuis operator L (for example $L(p)$ is conjugate to a Jordan block but it is not true any more at neighbouring points).

Theorem

Assume that L is differentially non-degenerate at a point $p \in M$. Then there exists a local coordinate system $x_1, \dots, x_n$ in which L takes the following canonical form (universal for both semisimple and non-semisimple cases!):

$$L = \begin{pmatrix} x_1 & 1 & & & \\ x_2 & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n-1} & 0 & \dots & 0 & 1 \\ x_n & 0 & \dots & 0 & 0 \end{pmatrix} \qquad (2)$$

Corollary

Differentially non-degenerate singular points are C^2-stable.

gl-regular operators: first companion form

Theorem

Let L be a gl*-regular Nijenhuis operator with characteristic polynomial* $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$.
Then in a suitable coordinate system $x = (x^1, \dots, x^n)$*, this operator takes the form*

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{3}$$

where $f_i = f_i(x)$ *are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the following PDE system:*

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \tag{4}$$

gl-regular operators: second companion form

Theorem

Let L be gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$.
Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ f_n & f_{n-1} & \dots & f_2 & f_1 \end{pmatrix}, \tag{5}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the PDE system that can be written in the form

$$\mathrm{d}\,\omega = 0, \quad \mathrm{d}\left(L^*\omega\right) = 0, \tag{6}$$

where $\omega = f_n \mathrm{d}\,x^1 + \dots + f_1 \mathrm{d}\,x^n$.

Local classification in dimension 2

First three types of non-singular points

Local classification in dimension 2

First three types of non-singular points

- Two real eigenvalues:

$$L = \begin{pmatrix} f(x) & 0 \\ 0 & g(y) \end{pmatrix},$$

  where $f(x)$ and $g(y)$ are smooth functions such that $f(x) \neq g(y)$ for all (x, y). In real analytic case, $f(x)$ is either constant of reduces to $f(x) = f_0 \pm x^m$ by coordinate transformation. Similar for $g(y)$.

- Two complex conjugate eigenvalues:

$$L = \begin{pmatrix} f(x,y) & -g(x,y) \\ g(x,y) & f(x,y) \end{pmatrix},$$

  where $h = f + \mathrm{i}\, g$ is holomorphic in $z = x + \mathrm{i}\, y$, $g(x, y) \neq 0$ for all (x, y). Here $h(z)$ is either constant or reduces to $h(z) = h_0 + z^m$, $m \in \mathbb{N}$ by a coordinate transformation.

- Jordan block:

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix},$$

  where $f(y)$ is a smooth function. In real analytic case, $f(y)$ is either constant or reduces to $f(x) = f_0 \pm x^{2m}$.

Local classification in dimension 2 (continued)

Theorem

Let L be a Nijenhuis operator such that $L(\mathrm{p}_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then in suitable local coordinates (x, y), this operator takes one of the following forms:

1. *Series L:* $L_{\mathrm{nil}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $L_{\mathrm{nd}} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$,
2. *Series M:* $M_{2k-1} = \begin{pmatrix} 0 & 1 \\ 0 & y^{2k-1} \end{pmatrix}$, $M_{2k}^{\epsilon} = \begin{pmatrix} 0 & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix}$,
3. *Series N:* $N_{2k-1} = \begin{pmatrix} y^{2k-1} & 1 \\ 0 & y^{2k-1} \end{pmatrix}$, $N_{2k}^{\epsilon} = \begin{pmatrix} \epsilon y^{2k} & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix}$.

Here $k \in \mathbb{N}$, $\epsilon = \pm 1$.

(to be continued ...)

Local classification in dimension 2 (continued)

Important property: L can be uniquely recovered from the coefficients of its characteristic polynomial $v = \operatorname{tr} L$ and $u = -\det L$ as follows:

$$L = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \qquad v = \operatorname{tr} L, \;\; u = -\det L.$$

Local classification in dimension 2 (continued)

Important property: L can be uniquely recovered from the coefficients of its characteristic polynomial $v = \operatorname{tr} L$ and $u = -\det L$ as follows:

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4. Series $O^{d,\epsilon}_{k,\mathrm{c}}$, $k \geq 1$, $d \geq 2k+1$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$ and we set $\epsilon = 1$, if $d = 2m+1$ is odd:

$$v = \alpha x y^{2k-1} + y^k \left(c_{k-1} y^{k-1} + \cdots + c_1 y + c_0 \right), \quad u = \epsilon\, y^d, \quad \alpha = k c_0^2 \left(1 - \tfrac{k}{d}\right) \neq 0.$$

5. Series $P^{k,\epsilon}_{s,\mathrm{c}}$, $k \geq 1$, $s \geq 2k$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^s + y^{s-k+1} \left(c_{k-1} y^{k-1} + \cdots + c_1 y + c_0 \right) + 2\epsilon\, y^k, \quad u = -y^{2k}, \quad \alpha = 2\epsilon\, k c_0 \neq 0.$$

6. Series $S^{2k,\epsilon}_{\mathrm{c}}$ and S^{2k+1}_{c}, $k \geq 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^{2k-1} + y^k \left(c_{k-1} y^{k-1} + \cdots + c_1 y + c_0 \right), \quad u = \epsilon\, y^{2k}, \quad \alpha = \tfrac{k}{2} \left(c_0^2 + 4\epsilon \right) \neq 0,$$

$$v = \alpha x y^{2k} + y^{k+1} \left(c_{k-1} y^{k-1} + \cdots + c_1 y + c_0 \right), \quad u = y^{2k+1}, \quad \alpha = 2k+1.$$

Global example: two-dimensional torus T^2

Example

- Two constant eigenvalues λ_1 and λ_2. Let ξ and η be two vector fields on T^2, that are linearly independent at each point. Then we define L as follows:

$$L(\xi) = \lambda_1 \xi \quad \text{and} \quad L(\eta) = \lambda_2 \eta. \tag{7}$$

- One of eigenvalues is constant (and equals zero), the other is not. In usual angle coordinates (ϕ_1, ϕ_2), we set

$$L = \begin{pmatrix} 0 & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}. \tag{8}$$

- Two non-constant eigenvalues λ_1 and λ_2. Obvious example:

$$L = \begin{pmatrix} f(\phi_1) & 0 \\ 0 & g(\phi_2) \end{pmatrix}, \qquad f(\phi_1) < c < g(\phi_2). \tag{9}$$

  This example can be generalised by taking finite covering over this "standard" torus. On the covering torus such a *global diagonalisation* is not always possible.

Global examples: torus T^2 (continued)

Example

Assume that L is a gl-regular operator on T^2 with double eigenvalue λ. If $\lambda = \mathrm{const}$, then we may set $\lambda = 0$, i.e. think of L as a nilpotent operator.

- Consider two vector fields ξ and η on T^2 that are linearly independent at each point and define L as follows:

$$L(\xi) = 0, \quad L(\eta) = \xi.$$

  Then L is a gl-regular nilpotent Nijenhuis operator on T^2 (notice that every nilpotent operator in dimension 2 is Nijenhuis automatically).

- The case with a non-constant eigenvalue can be modelled on T^2 as follows:

$$\begin{pmatrix} f(\phi_2) & g(\phi_1,\phi_2) \\ 0 & f(\phi_2) \end{pmatrix}, \qquad g(\phi_1,\phi_2) > 0,$$

  where ϕ_1, ϕ_2 are standard angle coordinates on the torus.

Global results in dimension 2

Theorem

Let (M^2, L) *be a closed connected* gl*-regular Nijenhuis manifold. Then we have one of the following possibilities:*

1. M^2 *is orientable and* $L = \alpha \,\mathrm{Id} + \beta J$*, where* J *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$ *are constants,* $\beta \neq 0$*.*
2. M^2 *is homeomorphic to either torus or Klein bottle, and* L *has two distinct real eigenvalues at each point of* M^2*.*
3. M^2 *is homeomorphic to a torus, and* L *is similar to a Jordan block at each point of* M^2*.*
4. M^2 *is homeomorphic to either torus or Klein bottle, and one of eigenvalues of* L *is constant.*

Global results in dimension 2

Theorem

Let (M^2, L) be a closed connected gl-regular Nijenhuis manifold. Then we have one of the following possibilities:

1. M^2 *is orientable and* $L = \alpha \,\mathrm{Id} + \beta J$, *where* J *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$ *are constants,* $\beta \neq 0$.
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3. M^2 *is homeomorphic to a torus, and* L *is similar to a Jordan block at each point of* M^2.
4. M^2 *is homeomorphic to either torus or Klein bottle, and one of eigenvalues of* L *is constant.*

Corollary

1. *Let* M^2 *be orientable and* $\mathrm{M}^2 \not\approx \mathrm{T}^2$. *Then on* M^2 *there are no* gl*-regular Nijenhuis operators except for* $L = \alpha \,\mathrm{Id} + \beta J$.
2. *Let* M^2 *be non-orientable and* $\mathrm{M}^2 \not\approx \mathrm{K}^2$. *Then on* M^2 *there are no* gl*-regular Nijenhuis operators at all.*

Some other global results

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least at one point. Then this number λ is an eigenvalue of L with the same algebraic multiplicity at every point of M. Shortly: a Nijenhuis operator on a closed manifold may not have non-constant complex eigenvalues.

Corollary

A Nijenhuis operator L on a closed manifold cannot have differentially non-degenerate singular points (like e.g. versal deformations of Jordan blocks).

Corollary

The eigenvalues of a Nijenhuis operator on the 4-dimensional sphere S^4 are all real.

Open problems

- Describe the topology of closed $\mathfrak{gl}$-regular Nijenhuis manifolds.
- Construct real analytic examples of Nijenhuis operators on closed two-dimensional surfaces whose eigenvalues are real and generically distinct.
- Let us fix a certain algebraic type of a linear operator, i.e. its Segre characteristic. Does there exist a Nijenhuis operator L in $\mathbb{R}^n$ with the following two properties:
  1. L has compact support;
  2. in a certain domain $U \subset \mathbb{R}^n$, the algebraic type of L does not change and coincides with the given one?
- Classify left-symmetric algebras of low dimension.
- Find/construct examples of non-degenerate LSAs of arbitrary dimension.
- Describe/classify LSAs with algebraically independent coefficients of the characteristic polynomial.
- etc.