Broken book decompositions and Reeb dynamics in dimension 3

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Introduction

A closed 3-manifold (M, ξ) is a contact manifold if ξ is a plane field that is non-integrable.
Then there is a 1-form α such that $\xi = \ker(\alpha)$ and $\alpha \wedge d\alpha \neq 0$.
Observe that if f is a non-zero function, then $f\alpha$ defines the same plane field.
The Reeb vector field of α is defined by the equations

$$\alpha(X) = 1 \qquad \iota_X d\alpha = 0.$$

It depends on the contact form.

Introduction

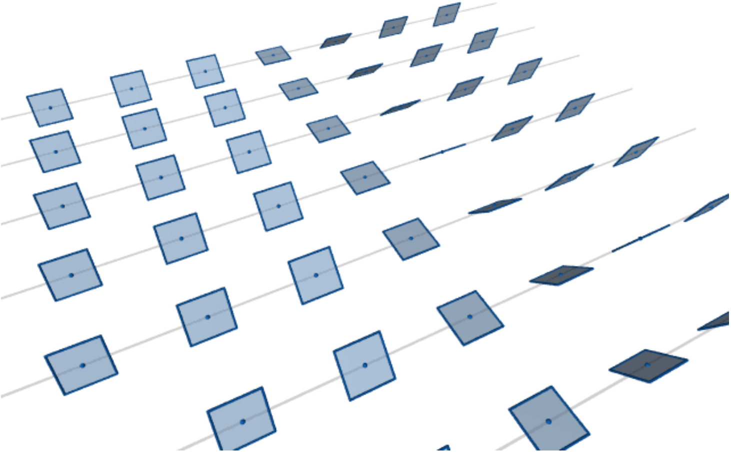


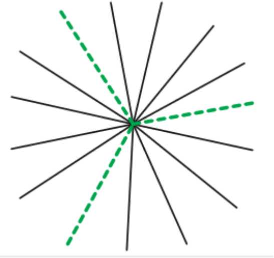
Image by Patrick Massot

Introduction

M denotes a closed oriented 3-manifold.

Definition (Open book decomposition)

An open book decomposition of M is a pair (K, π) with K an oriented link and $\pi : M \setminus K \to \mathbb{S}^1$ is a fibration and $\pi^{-1}(t)$ is the interior of a compact surface whose boundary is K.



Introduction

E. Giroux

Every contact structure ξ is carried by an open book decomposition, *i.e* for a contact form α defining ξ the Reeb vector field is tangent to K and transverse to the pages $\pi^{-1}(t)$.

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A page is Birkhoff section for this Reeb vector field: a transverse surface with boundary, whose boundary is made of periodic orbits.

- Birkhoff proved that geodesic flows of negatively curved surfaces admit Birkhoff sections.
- There are several existence results for some classes of Reeb vector fields by Hofer, Wysocki, Zehnder, Hryniewicz, Salomão,...
- Ghys defined left-handed (or right-handed) vector fields: in these classes every collection of periodic orbits bounds a Birkhoff section.

First result

Theorem 1

Every nondegenerate Reeb vector field X is carried by a broken book decomposition of M.

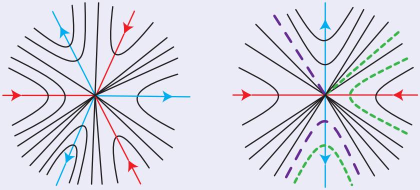
Nondegenerate – that the Poincaré maps of the periodic orbits of X and their powers, have all their eigenvalues different from 1.

Broken books

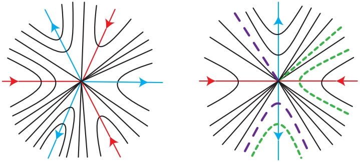
Definition

A broken book decomposition of M is a pair $(K, \mathcal{F})$ such that K is a oriented link and

- $\mathcal{F}$ is a cooriented foliation of $M \setminus K$ whose leaves are properly embedded in $M \setminus K$;
- $K = K_r \cup K_b$, near K_r transversely the foliation is radial, near K_b transversely the foliation is



Broken books



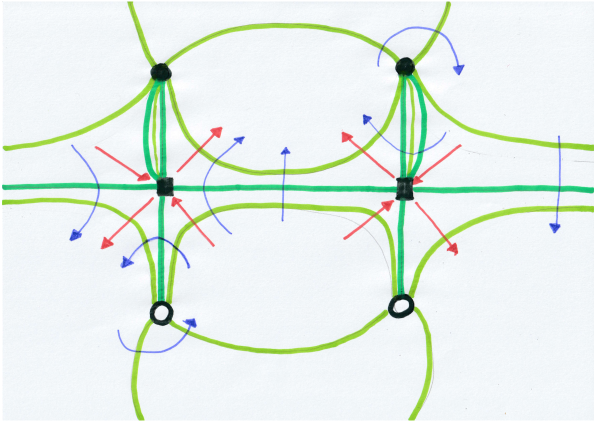
Remarks

A periodic orbit of X that belongs to K can be elliptic or hyperbolic with respect to the dynamics of the flow, and with respect to the foliation $\mathcal{F}$. A periodic orbit that belongs to K_b has to be dynamically hyperbolic.

- The nondegeneracy implies that near K_b there are exactly 4 sectors foliated by hyperbolas (right hand figure).
- The monodromy along K_b is the identity or half a turn.
- If $K_b = \emptyset$, the broken book is an open book.

Broken books

Broken books



Definition

A leaf of $\mathcal{F}$ is rigid if it doesn't belongs to the interior of $\mathbb{R}$-family of diffeomorphic leaves.

The complement of the union of the rigid pages fibers over $\mathbb{R}$.

An application to the existence of periodic orbits

Some previous results:

- Every Reeb vector field has at least two periodic orbits (Taubes 2007 for one periodic orbit, Cristofaro-Gardiner and Hutchings 2016).
- If a nondegenerate Reeb vector field has exactly two periodic orbits, then M is $\mathbb{S}^3$ or a lens space (Hutchings and Taubes 2009).
- Finite energy foliations by Hofer, Wysocky and Zehnder (2003) are broken book decompositions. They proved that every strongly nondegenerate Reeb vector field on $\mathbb{S}^3$ has either 2 or infinitely many periodic orbits.
- Cristofaro-Gardiner, Hutchings and Pomerleano (2018) proved the existence of 2 or infinitely many periodic orbits if the first Chern class of the contact structure is a torsion element.

An application to periodic orbits

Theorem 2

Every nondegenerate Reeb vector field has either 2 or infinitely many periodic orbits.

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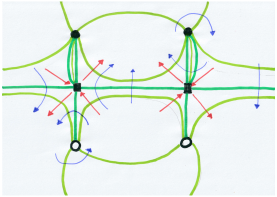

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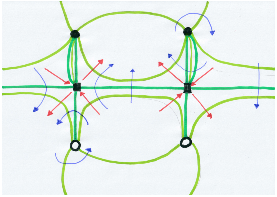

If $K_b = \emptyset$ the broken book decomposition is an open book decomposition.
Let S be a page. If S is a disk or an annulus, then X has either 2 or infinitely many periodic orbits. If not, we prove that the first return map to S has infinitely many periodic points (the result for homeomorphisms was established by Le Calvez and Sambarino).

Sketch of the proof of Theorem 2

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The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

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Lemma

The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

Proof. Let $W^s(k)$ be half of the stable manifold of k and assume it does not intersects the unstable manifold of any orbit in K_b.
It is an injectively immersed cylinder, that has to intersect one rigid page R infinitely many times. This intersection is then a infinite collection C of embedded circles in R.

Sketch of the proof of Theorem 2

Lemma

The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

Proof. There are finitely many of these circles that bound a disc in R. Take two, c_1 and c_2, bounding two discs D_1 and D_2 in R. Let $A \subset W^s(k)$ be the annulus bounded by c_1 and c_2, then

$$0 = \int_{D_1 \cup A \cup D_2} d\alpha = \int_{D_1} d\alpha - \int_{D_2} d\alpha$$

implying that the discs have the same $d\alpha$-area. Since $\overline{R}$ is compact, we conclude.

Sketch of the proof of Theorem 2

Lemma

The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

Proof. Hence there are infinitely many circles not bounding a disc, and we can find 2 of them bounding an annulus A' that does not contains other circles in C. Let now c_3 and c_4 be the bounding circles and $A'' \subset W^s(k)$ be the annulus bounded by c_3 and c_4. Then

$$0 = \int_{A' \cup A''} d\alpha = \int_{A'} d\alpha,$$

a contradiction.

Sketch of the proof of Theorem 2

The lemma allows to prove the existence of a heteroclinic cycle.

If all the intersections in it (or at least one) is transversal, we obtain a homoclinic intersection providing infinitely many periodic orbits of X.

Sketch of the proof of Theorem 2

The proof without the strongly nondegenerate hypothesis is quite technical:

1. We first prove the result if all the stable/unstable manifolds coincide pairwise. We conclude that in this case there are infinitely many periodic orbits.

2. We then prove that a stable/unstable manifold that does not coincide with another unstable/stable manifold, contains a crossing intersection.

Another application of BBD

Theorem 3

If M is a non-graphed closed 3-manifold carrying a non-degenerate Reeb vector field, then X has positive topological entropy.

Back to Theorem 1

Theorem 1

Every nondegenerate Reeb vector field X is carried by a broken book decomposition of M.

1. ECH (in particular the U-map) together with desingularising the projected pseudoholomorphic curves, provide a section through every point $z \in M$.

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2. A compacity argument allows to choose a finite number of these sections that intersect all orbits. Let K be the union of their boundaries.
3. The complement of this system of sections fibers over $\mathbb{R}$.

Sketch of the proof of Theorem 1

The main input from ECH-holomorphic curve theory is the following. There is a collection of periodic orbits $\mathcal{P}$ such that,

Lemma

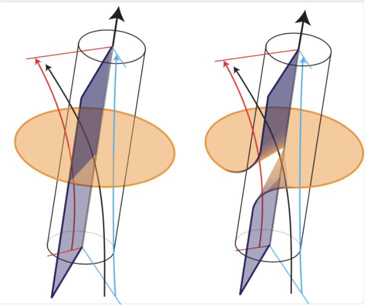
For every z in $M \setminus \mathcal{P}$, there exists an immersed singular compact surface u with boundary such that:

- *$z \in u$;*
- *∂u is made of periodic orbits;*
- *has a finite number of singularities;*
- *away from the singular points, its interior is transverse to X.*

If z belongs to $\mathcal{P}$, it is either in the interior of such a curve or in a boundary component.

Sketch of the proof of Theorem 1

Each surface u can be transformed into a (possibly disconnected) section:



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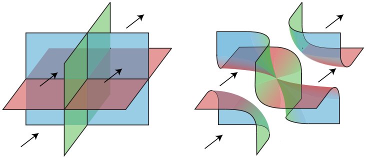

From each projected curve, we obtain connected sections with boundary.