Geometry and dynamics of circulant systems

Anthony Bloch Matt Kvalheim

- **The Sprott System**
- **Repressilator**
- **Existence of families of periodic orbits**
- **Poincaré maps and continuation**

The repressilator:

$$\begin{aligned}
\dot{x} &= \frac{\mu}{1+y^n} - x \\
\dot{y} &= \frac{\mu}{1+z^n} - y \\
\dot{z} &= \frac{\mu}{1+x^n} - z.
\end{aligned} \tag{0.1}$$

Exhibits stable periodic orbits.

Motivating example: Sprott System

$$\begin{aligned} \dot{x} &= y^2 - z - \mu x \\ \dot{y} &= z^2 - x - \mu y \\ \dot{z} &= x^2 - y - \mu z \end{aligned} \tag{0.2}$$

on $\mathbb{R}^3$ which depend on the parameter μ.

This system is inspired by various systems that have been analyzed in the synthetic biology literature in particular the repressilator and its generalizations.

The repressilator is an example of a monotone cyclic feedback system. Such systems have been analyzed by many people – see for example Mallet-Paret and Smith 1990.

The Sprott system is not monotone however even though it is cyclic.

We have examined a class of systems which include both these systems.

Proving the existence of periodic orbits particularly in dimension greater than 2 is difficult. Our approach based on continuation and Poincaré maps.

Goes back to work of Sotomayor 1973 who showed existence of a residual set of systems with orbits having one of three normal forms. Such a system is called generic.

Other work: work with lab of Rajapakse, Brockett, Surana, Chen and others on control of biological systems – control of stem cell dynamics, control of general networks and tensor control systems.

Transcription factors a la Yamanaka used in control of large systems.

Reduction using topologically associated domains etc.

Structure of tensors capturing gene expression and geometry – Hi-C.

We study families of periodic orbits of a C^1 autonomous ordinary differential equation (ODE) with one parameter

$$\dot{x} = f(x, \mu) =: f_\mu(x), \qquad (x, \mu) \in Q \times \mathbb{R} \tag{0.3}$$

on a smooth manifold Q. Our primary contributions are (i) a theorem on the *global continuation* of periodic orbits for this system (as the parameter μ is varied), and (ii) theorems on existence of periodic orbits for the system based on our global continuation theory. A key hypothesis for our theorems is the existence of a closed 1-form η on $Q \times \mathbb{R}$ satisfying certain properties.

Also much previous work in this area including work of Fuller, Alexander and Yorke, Alexander, Alligood and Yorke, Mallet-Paret – various papers, Fiedler in the case of symmetries

One difficulty in proving existence using continuation results is that a priori upper bounds on the periods (or "virtual periods") of periodic orbits are required.

We show that the existence of a closed 1-form η on $Q \times \mathbb{R}$ satisfying certain properties enables the a priori period upper bound to be replaced with a conditions such as $\eta((f, 0)) > 0$ which are in principle computable.

We briefly note that η satisfying this last condition can be viewed as a Lyapunov 1-form.

Equilibria, periodic orbits (POs), Poincaré maps, and Floquet multipliers (Figure from "Foundations of Mechanics" by Abraham and Marsden.)

- **If $f(x_0) = 0$, then the solution with initial condition x_0 has zero velocity at x_0 and hence stays at x_0 forever; say x_0 is an *equilibrium.***
  - **The eigenvalues of the Jacobian $\mathsf{D}_{x_0} f$ give stability information; e.g., if they all have negative real part then all initial conditions near x_0 approach x_0 exponentially fast.**
- **If $\gamma(t)$ is a solution satisfying $\gamma(T) = \gamma(0)$ for some $T \in \mathbb{R}$, say γ is a *periodic orbit (PO).* If $T = \inf\{t \geq 0 \colon \gamma(t) = \gamma(0)\}$ say that T is the *(least) period* of γ and, if $T > 0$, say that γ is *nonstationary.***
- **Let Γ be the image of γ and consider a hypersurface S transversely intersecting Γ as in the figure. There is a neighborhood $U \subset S$ of $\gamma_0 \coloneqq \Gamma \cap S$ and a well-defined *Poincaré map***

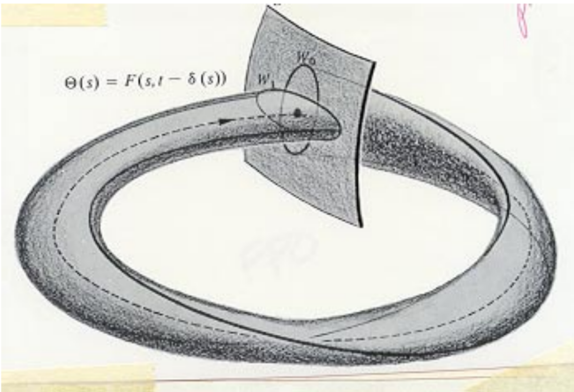


FIGURE 4.3.8. Poincaré maps

$P\colon U \to S$ defined by following trajectories.

- **The *(Floquet) multipliers* of Γ are defined to be the eigenvalues of $\mathsf{D}_{\gamma_0} P\colon \mathsf{T}_{\gamma_0} S \to \mathsf{T}_{\gamma_0} S$, and are independent of the choices of γ_0 and S.**
  - **Multipliers also give stability information; e.g., if they are all inside the open unit disk then all initial conditions near Γ approach Γ exponentially fast.**

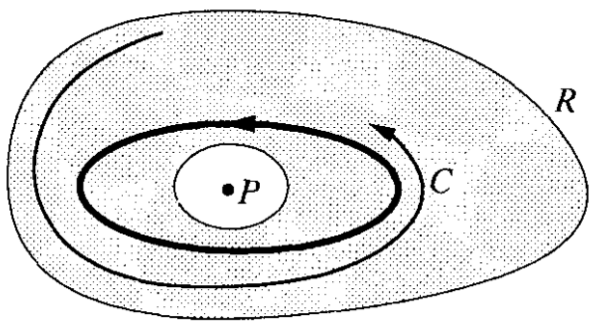
Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).



Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, *R contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

Classical PO existence results, pt. I: Poincaré-Bendixson theorem

From "Nonlinear Dynamics and Chaos", 1st ed. by Strogatz.

How to prove POs exist, pt. II: Hopf bifurcation theorem (source: [Kuz])

- **Consider a** C^3 **1-parameter family** $f\colon Q \times \mathbb{R} \to \mathsf{T}Q$ **of vector fields on** Q**, write** $f_\mu(\,\cdot\,) \coloneqq f(\mu,\,\cdot\,)$ **and let** $x_0 \in f_{\mu_0}^{-1}(0)$ **be such that** $A \coloneqq \mathsf{D}_{x_0} f_{\mu_0}$ **has precisely one pair** $\lambda_{\mu_0}^{\pm} \coloneqq \pm i\omega_0$ **of purely imaginary eigenvalues.**
- **By the implicit function theorem, there is a unique** C^3 **family** $\mu \mapsto x(\mu) \in f_\mu^{-1}(0)$ **such that** $x(\mu_0) = x_0$ **and unique** C^3 **families** $\mu \mapsto \alpha(\mu), \omega(\mu)$ **satisfying** $\alpha(\mu_0) = 0$**,** $\omega(\mu_0) = \omega_0$ **and such that** $\lambda^{\pm}(\mu) \coloneqq \alpha(\mu) \pm i\omega(\mu)$ **are eigenvalues of** $\mathsf{D}_{x(\mu)} f_\mu$**.**
- **Assume that** $\alpha'(\mu) \neq 0$ **and define, in local coordinates, the bilinear and trilinear maps** $B \coloneqq \mathsf{D}^2_{x_0} f_{\mu_0}$ **and** $C \coloneqq \mathsf{D}^3_{x_0} f_{\mu_0}$**. Fix** $p, q \in \mathbb{C}^n$ **satisfying** $Aq = i\omega_0 q$**,** $A^T p = -i\omega_0 p$**,** $\bar{p} \cdot q = 1$**, and assume that the following quantity is nonzero:**

$$l_1(0) = \frac{1}{2\omega_0}\mathrm{Re}\left[\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle \right.$$
$$\left. + \ \langle p, B(\bar{q}, (2i\omega_0 I_n - A)^{-1}B(q,q))\rangle\right].$$

- **Say a generic Hopf bifurcation occurs. Fix $T > 0$ and a neighborhood $U \ni x_0$.**
- **Subcritical case: if $\mathrm{sign}(\ell_1(0)) = \mathrm{sign}(\alpha'(\mu_0))$, then $\exists \epsilon > 0$ such that, $\forall \mu \in (\mu_0 - \epsilon, \mu_0)$, $\exists!$ nonstationary PO with image in U and period less than T.**
- **Supercritical case: if $\mathrm{sign}(\ell_1(0)) \neq \mathrm{sign}(\alpha'(\mu_0))$, then $\exists \epsilon > 0$ such that, $\forall \mu \in (\mu_0, \mu_0 + \epsilon)$, $\exists!$ nonstationary PO with image in U and period less than T.**
- **Moreover, the PO is hyperbolic and the number of Floquet multipliers which are inside/outside the unit disk are determined by $\mathrm{spec}(\mathsf{D}_{x_0} f_{\mu_0})$ and $\mathrm{sign}(\alpha'(0))$.**
- **Powerful qual. theorem, easy numerically verify hypotheses.**

- **Problem for understanding any fixed f_{μ_1}: how small does $\epsilon > 0$ need to be? Theorem cannot tell us, e.g., whether $f_{\mu_0 \pm 10^{-23}}$ has a PO...**

Classical PO existence results, pt. III: additional selected results

- One technique: find clever conditions under which n-dimensional dynamics can be faithfully projected to those of a two-dimensional system where the Poincaré-Bendixson theorem applies [Gra77, Smi80].
  - Notable example: Poincaré-Bendixson theorem for the class of monotone cyclic feedback systems [MPS90] relevant for various applications in biology.
- For Hamiltonian systems: rich literature on PO existence; notable examples include [Rab78, Wei79] and the solution of the celebrated Arnold conjecture [Zeh86, Zeh19].
- For general n-dimensional ODEs: "torus principle" [Li81] based on Brouwer's fixed point theorem is widely used to prove existence of POs; application is made easier by recent work of Brockett and Byrnes [Byr07, Byr10].

Sprott system:

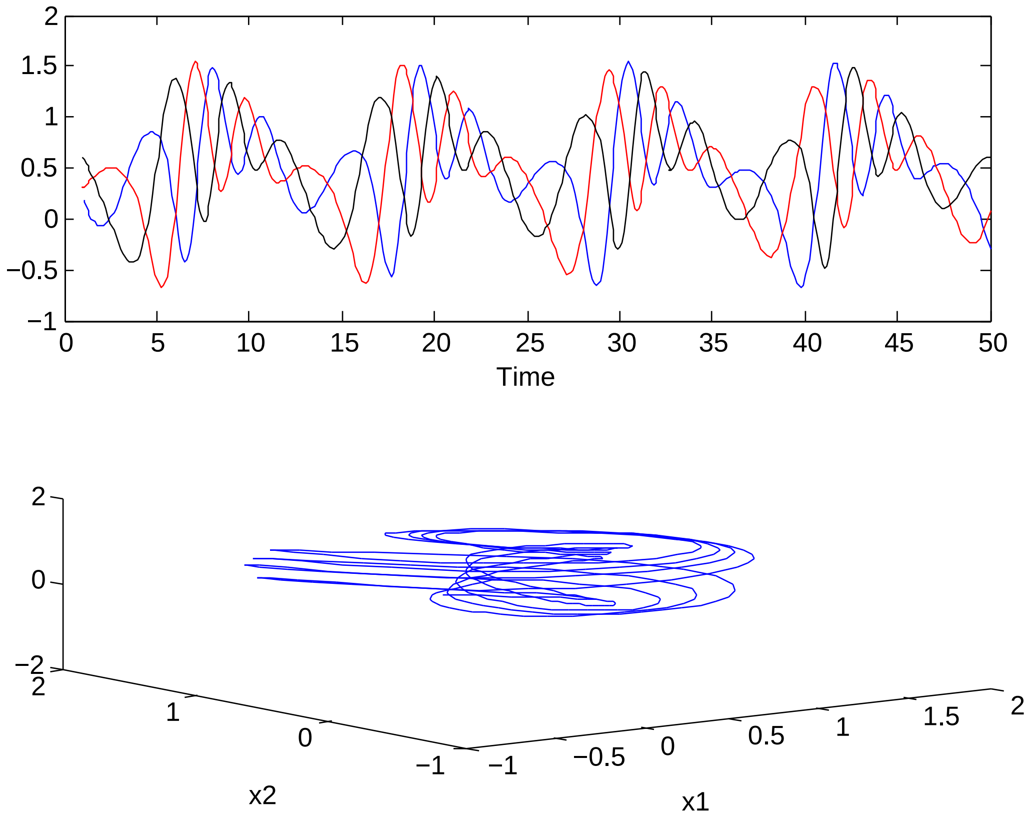


Figure 0.1: Sprott system with no damping

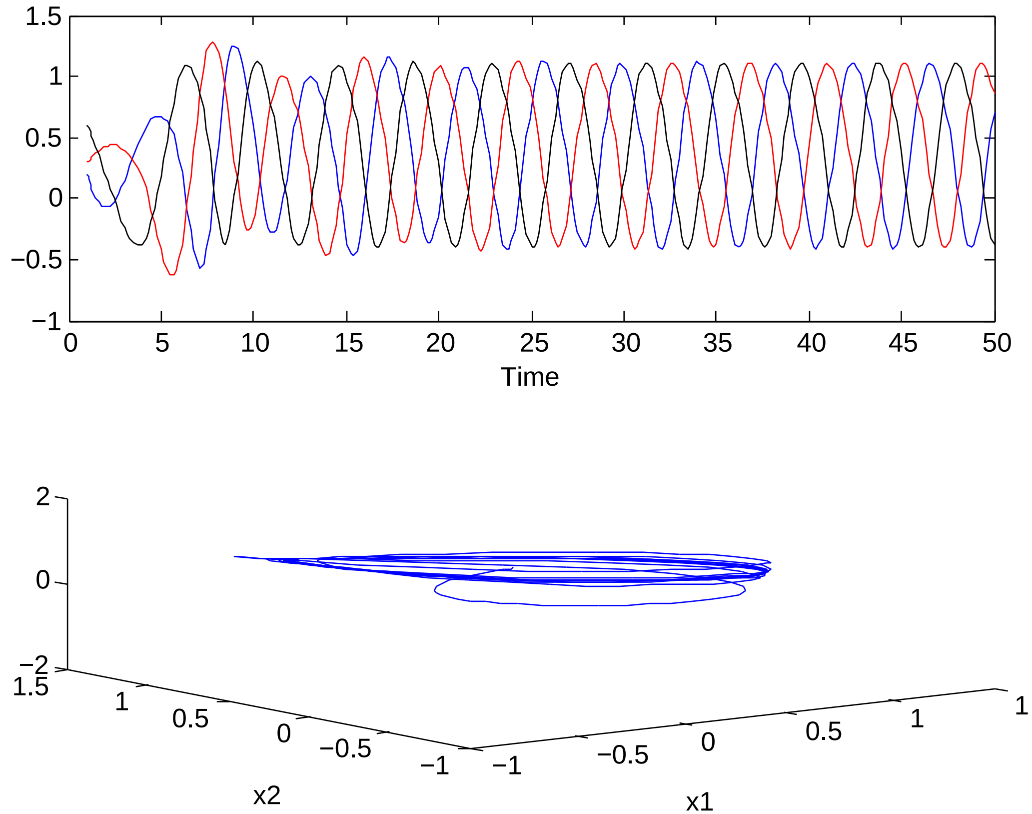


Figure 0.2: Sprott system with no damping

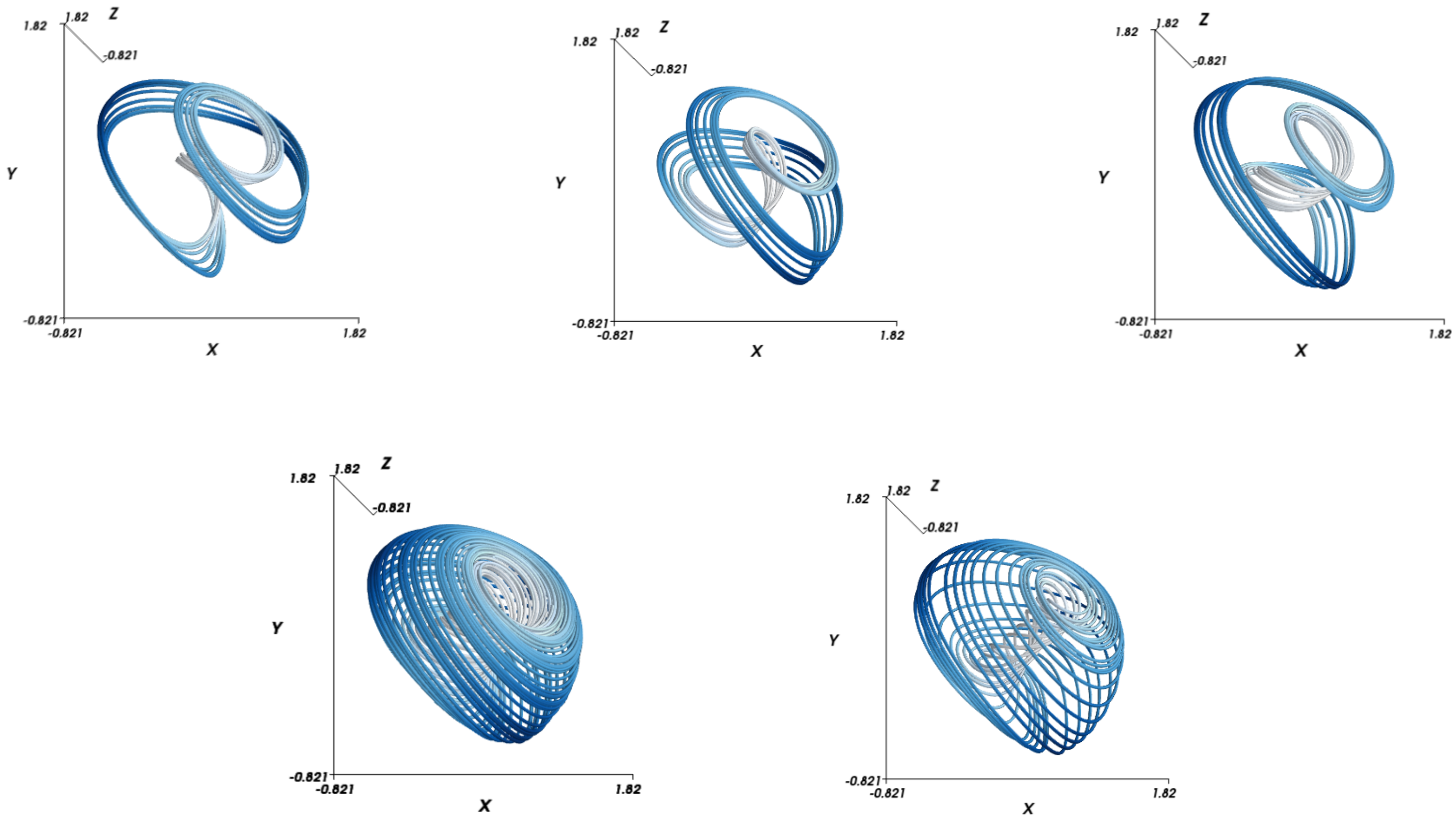


Figure 0.3: Shown here are trajectory segments (each of length 150 time units) of the Sprott system for $\mu = 0$. Each of the top three figures consists of a single trajectory segment, with initial condition (x_0, y_0, z_0) given from left to right by: $(1.2, 0.7, 0.6)$, $(0.7, 0.6, 1.2)$, $(0.6, 1.2, 0.7)$. These three trajectory segments are superimposed in the bottom left figure. The bottom right figure consists of a single trajectory segment with initial condition $(x_0, y_0, z_0) = (0.3, 0.2, -0.3)$. Light portions of trajectory segments indicate where the sum $x + y + z$ is decreasing as a function of time, and dark segments indicate where $x + y + z$ is increasing. As an application of our theory we prove that this dynamical system has a nonstationary periodic orbit (see Theorem 12).

Some trajectories of the dynamics for different values of μ are shown in Figures 0.4, 0.5, and 0.6. The sphere shown is defined below; see Equation (0.4) and Figure 0.7 below.

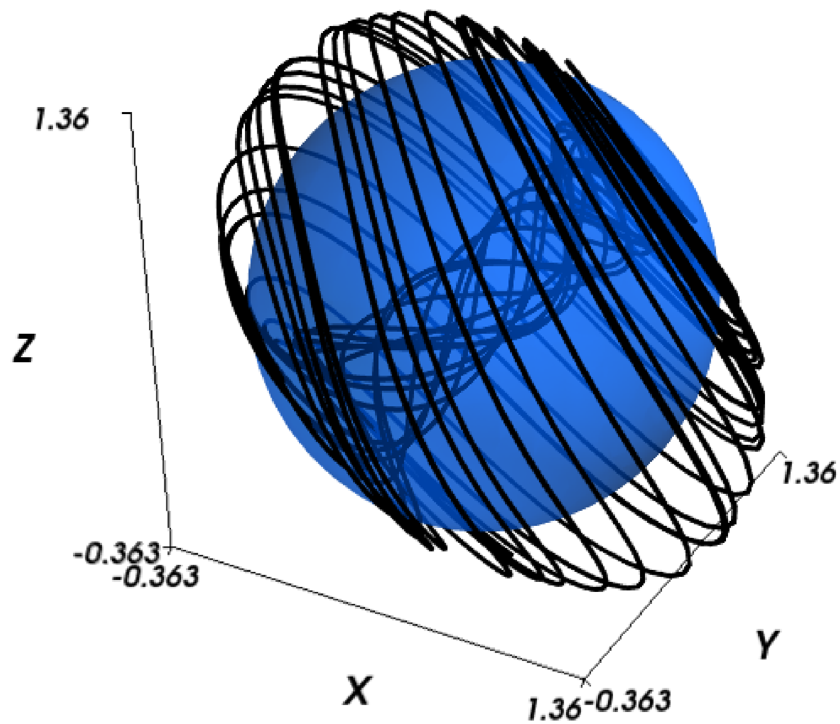


Figure 0.4: Shown here is a trajectory segment of f_0 (i.e., $\mu = 0$) with initial condition $(x, y, z) = (0.3, 0.2, -0.3)$. Also shown is the sphere $\dot{V}^{-1}(0)$ (c.f. Equation (0.4) and Figure 0.7).

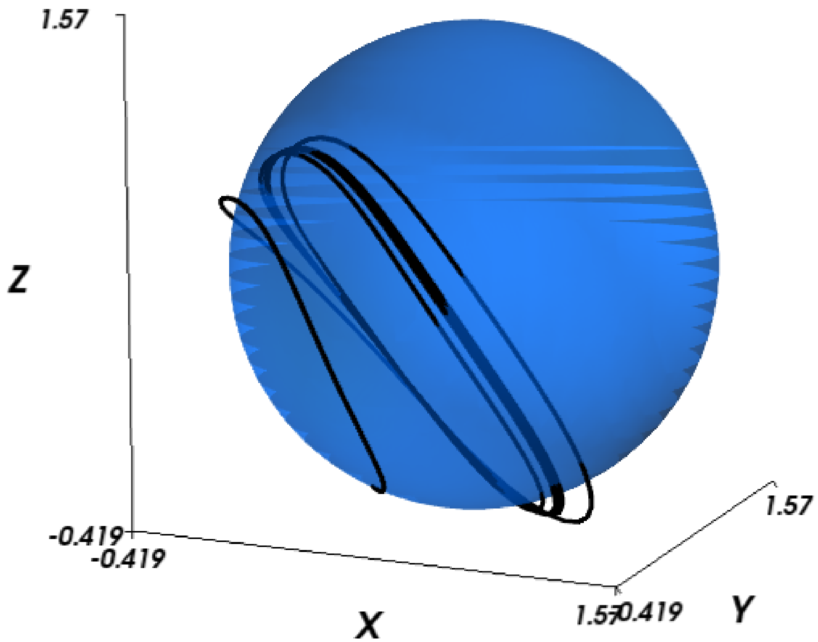


Figure 0.5: Shown here is a trajectory segment of $f_{0.15}$ (i.e., $\mu = 0.15$) with initial condition $(x, y, z) = (0.3, 0.2, -0.3)$. Also shown is the sphere $\dot{V}^{-1}(0)$ (c.f. Equation (0.4) and Figure 0.7).

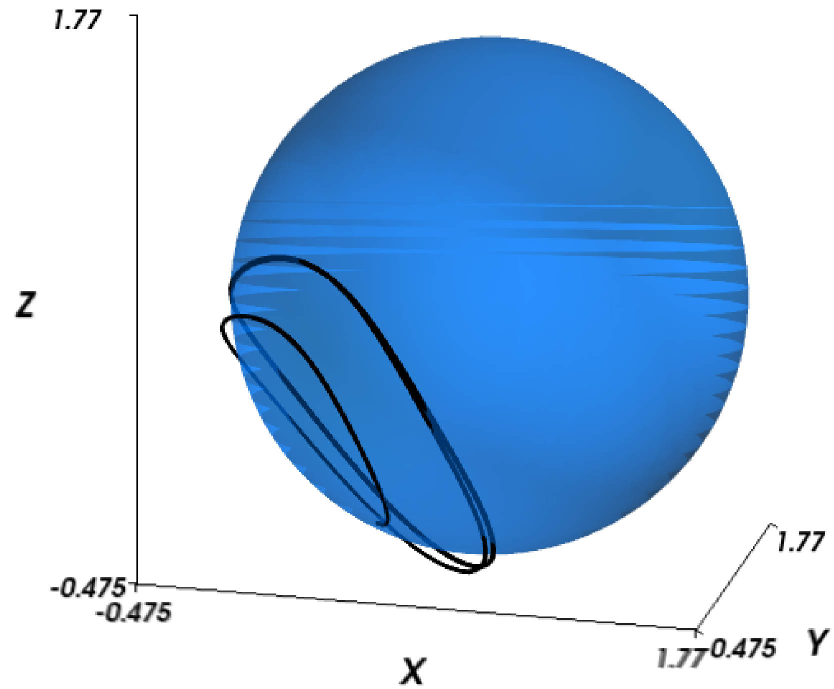


Figure 0.6: Shown here is a trajectory segment of $f_{0.3}$ (i.e., $\mu = 0.3$) with initial condition $(x, y, z) = (0.3, 0.2, -0.3)$. Also shown is the sphere $\dot{V}^{-1}(0)$ (c.f. Equation (0.4) and Figure 0.7).

A compact set containing all compact invariant sets: **Define the function $V\colon \mathbb{R}^3 \to \mathbb{R}$ via $V(x,y,z) \coloneqq x+y+z$. A computation shows that the Lie derivative $\dot{V}$ of V is**

$$\dot{V}(x,y,z) = \|(x,y,z)\|^2 - (\mu+1)(x+y+z) = \mathbf{1}\cdot f_\mu(x,y,z). \qquad (0.4)$$

For any $c \geq -\frac{3}{4}(\mu+1)^2$, the sublevel set $B_{\mu,c} \coloneqq \dot{V}^{-1}(-\infty, c]$ is the closed ball of radius $\frac{\sqrt{3(\mu+1)^2+4c}}{2}$ centered at $(\frac{1+\mu}{2})\mathbf{1}$. In particular, the zero sublevel set of $\dot{V}$ is centered at the midpoint of the two equilibria on the diagonal, with the two equilibria being antipodal points on the bounding sphere. Furthermore, the planes $V^{-1}(0)$ and $V^{-1}(3(1+\mu))$ are tangent to the sphere at these antipodal points. See Figure 0.7.

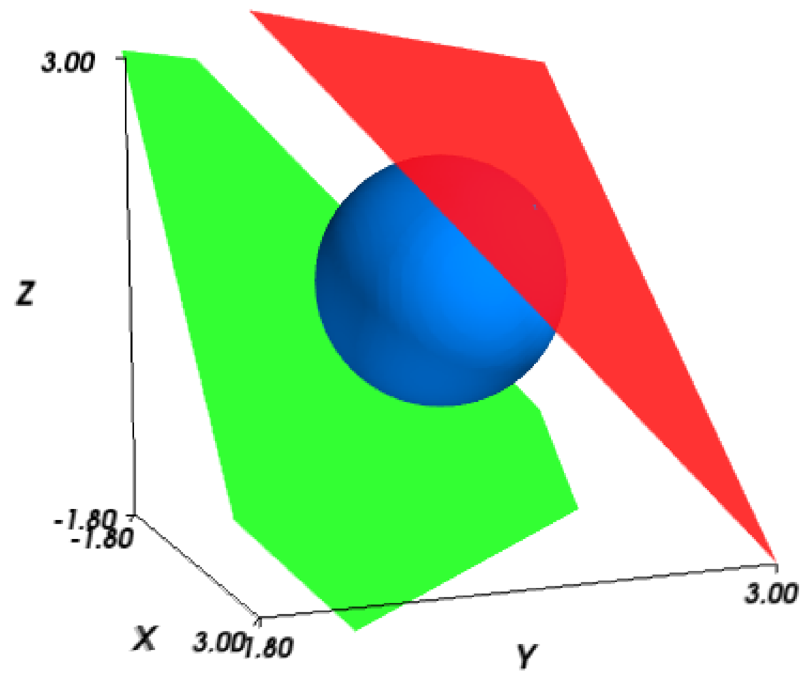


Figure 0.7: Shown here is the spherical level set $\dot{V}^{-1}(0)$ (teal), and the two planes $V^{-1}(0)$ (green) and $V^{-1}(3(1+\mu))$ (red) for $\mu = 0.4$.

The geometry described above implies that the subsets $V^{-1}(-\infty, 0)$ **and** $V^{-1}(3(1+\mu), \infty)$ **are respectively negatively and positively invariant for** $\mu \geq -1$**. Furthermore, trajectories in these regions respectively tend to** ∞ **in negative and positive time. It follows that any compact invariant set must be contained in** $V^{-1}[0, 3(1+\mu)]$ **when** $\mu \geq -1$**. We will further refine these considerations to produce a certain** ***compact*** **set containing all compact invariant sets.**

Define translated coordinates $\mathbf{x}_\mu \coloneqq (x_\mu, y_\mu, z_\mu) \coloneqq (x, y, z) - \frac{1}{2}(1+\mu)\mathbf{1}$ **and define** $r_\mu \coloneqq \|\mathbf{x}_\mu\|$**.**

Theorem 1. *For $\mu \geq -1$, every compact invariant set is contained in the compact set K_μ defined by*

$$K_\mu := \left\{ (x, y, z) \colon V \geq r_\mu - \frac{3^{\frac{3}{4}} + 3^{\frac{1}{4}}}{2}(1+\mu) \arctan\left(\frac{2r_\mu}{(3)^{1/4}(1+\mu)} \right) - \frac{\sqrt{3}}{2}(1+\mu) \right.$$
$$\left. + \frac{3^{\frac{3}{4}} + 3^{\frac{1}{4}}}{2}(1+\mu) \arctan\left(3^{1/4}\right) \right\} \cap V^{-1}[0, 3(1+\mu)]. \tag{0.5}$$

For a visual depiction of K_μ, see Figure 0.8. Note that the sphere $\dot{V}^{-1}(0)$ shown in Figure 0.7 is strictly contained in the interior of K_μ.

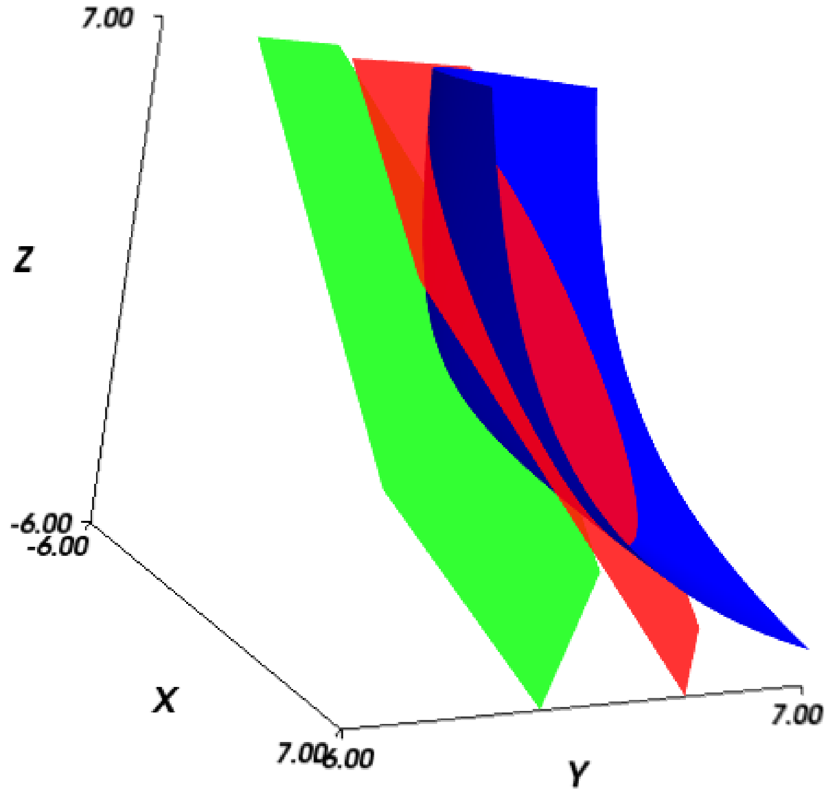


Figure 0.8: The compact set K_μ of Theorem 8 is the region bounded by the blue surface, red plane $V^{-1}(3(1+\mu))$, and green plane $V^{-1}(0)$. Note that the sphere $\dot{V}^{-1}(0)$ shown in Figure 0.7 is strictly contained in the interior of K_μ. This figure was generated using $\mu = 0.4$.

Cylindrical coordinates and rotation of the flow:
Define an orthogonal matrix M via

$$M = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \tag{0.6}$$

and define coordinates $(u, v, w)^T := M^{-1}(x, y, z)^T = M^T(x, y, z)^T$. The w-axis corresponds to Δ in the original coordinates, and the u and v axes determine an orthonormal coordinate system for $\Delta^\perp$.

We further define cylindrical coordinates (ρ, θ, w) **via**

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned} \tag{0.7}$$

Obtain the equations of motion in these new coordinates in closed form:

$$\begin{aligned}
\dot{\rho} &= \rho\left(-\sqrt{2}\rho\sin^3(\theta) + \frac{\sqrt{6}}{3}\rho\cos^3(\theta) - \frac{\sqrt{6}}{2}\rho\cos\left(\theta + \frac{\pi}{3}\right) - \frac{1}{\sqrt{3}}w - \mu + \frac{1}{2}\right) \\
\dot{\theta} &= \rho\left(\frac{\sqrt{6}}{3}\sin^3(\theta) - \frac{\sqrt{6}}{2}\sin\left(\theta + \frac{\pi}{3}\right) + \sqrt{2}\cos^3(\theta)\right) - w - \frac{\sqrt{3}}{2} \\
\dot{w} &= \frac{1}{\sqrt{3}}(\rho^2 + w^2) - (\mu + 1)w.
\end{aligned} \tag{0.8}$$

Theorem 2. *There exists* $\epsilon > 0$ *such that for all* $-1 < \mu \leq 1$, $\dot{\theta} < -\epsilon$ *whenever* $(x, y, z) \in K_\mu \setminus \Delta$.

Corollary 1. *For* $-1 < \mu \leq 1$, *all equilibria of* f_μ *lie in the diagonal* Δ.

Corollary 2. *All periodic orbits are contained in* K_μ, *and the (oriented) winding number of any nonstationary periodic orbit around* Δ *is less than or equal to* -1.

Have Δ is invariant and the dynamics restricted to Δ are given by

$$\dot{x} = x^2 - x - \mu x = x(x - 1 - \mu). \tag{0.9}$$

Theorem 3. *For all $\mu \in \mathbb{R}$, the vector field f_μ has at least two equilibria: $\mathbf{0}$ and $(1+\mu)\mathbf{1}$. For $-1 \leq \mu \leq 1$, these are the only equilibria.*

Theorem 4. *The equilibrium $\mathbf{0}$ undergoes a subcritical Hopf bifurcation at $\mu = -0.5$ and the equilibrium $(1+\mu)\mathbf{1}$ undergoes a supercritical Hopf bifurcation at $\mu - 0.25$. The first bifurcation produces an exponentially stable limit cycle near $\mathbf{0}$ for $0 < 0.5 - \mu \ll 1$, and the second bifurcation produces an exponentially* un*stable limit cycle near $(1+\mu)\mathbf{1}$ for $0 < \mu - (-0.25) \ll 1$.*

More generally consider families of periodic orbits of a C^1 autonomous ordinary differential equation (ODE) with one parameter

$$\dot{x} = f(x, \mu) =: f_\mu(x), \qquad (x, \mu) \in Q \times \mathbb{R} \tag{0.10}$$

on a smooth manifold Q.

Prove general results on global continuation of periodic orbits as the parameter is varied using earlier work of Alligood and Yorke and Alexander and Yorke and others.

In the following definition and below, for each $\mu \in \mathbb{R}$ we let $\iota_\mu \colon Q \hookrightarrow Q \times \mathbb{R}$ be the inclusion $\iota_\mu(x) = (x, \mu)$.

Definition 1 ((η, ℓ)-global continuability)**.** Let $\ell > 0$, η be a closed 1-form on $Q \times \mathbb{R}$, and $A \subset Q \times \mathbb{R}$ be a connected component of nonstationary periodic orbits of (0.10). Define $A_\ell \subset A$ to be the subset of points on periodic orbits α_μ with $\left|\int_{\alpha_\mu} \iota_\mu^* \eta\right| = \ell$ and $A_{\leq \ell}$ the subset with $\left|\int_{\alpha_\mu} \iota_\mu^* \eta\right| \leq \ell$.

Let γ be a periodic orbit with image $\Gamma \subset A_\ell$. Let $\widetilde{A}_{\leq \ell} \subset A_{\leq \ell}$, $\widetilde{A}_\ell \subset A_\ell$ be the connected components of $A_{\leq \ell}, A_\ell$ containing γ. We say that γ is (η, ℓ)-*globally continuable* if at least one of the following holds.

- $\widetilde{A}_{\leq \ell} \setminus \Gamma$ is connected,

or $\widetilde{A}_{\leq \ell} \setminus \Gamma$ and $\widetilde{A}_{\ell} \setminus \Gamma$ each consist of two connected components $\widetilde{A}^1_{\leq \ell}, \widetilde{A}^2_{\leq \ell}$ and $\widetilde{A}^1_{\ell}, \widetilde{A}^2_{\ell}$ with each $\widetilde{A}^i_{\ell}$ satisfying one of the following:

1. $\widetilde{A}^i_{\ell}$ is not contained in any compact subset of $Q \times \mathbb{R}$,
2. the closure $\mathrm{cl}(\widetilde{A}^i_{\ell}) \subset Q \times \mathbb{R}$ of $\widetilde{A}^i_{\ell}$ in $Q \times \mathbb{R}$ contains a generalized center (i.e., a stationary point (x, μ) such that $\partial_x f(x_0, \mu_0)$ has some purely imaginary eigenvalues),
3. the periods of $\widetilde{A}^i_{\ell}$ are unbounded, or
4. $\widetilde{A}^i_{\ell} \neq \widetilde{A}^i_{\leq \ell}$.

Theorem 5 ((η, ℓ)-global continuability for non-generic families)**.** *Let $f \in C^1(Q \times \mathbb{R}, \mathsf{T}Q)$ be a family of vector fields and let η be a closed 1-form on $Q \times \mathbb{R}$. Let $A \subset Q \times \mathbb{R}$ be a component of nonstationary periodic orbits of f. Let γ be a periodic orbit of f_{μ_0} with image $\Gamma \subset A$. Assume that γ does not have $+1$ as Floquet multiplier, define $\ell := \left|\int_\gamma \iota_{\mu_0}^* \eta\right|$, and assume $\ell > 0$. Then γ is (η, ℓ)-globally continuable.*

Theorem 6 (Global existence of periodic orbits)**.** *Assume the hypotheses of Theorem 5 and notation of Definition 1. Let* $(x_0, \mu_0) \in Q \times \mathbb{R}$ *be such that* γ *is a periodic orbit through* (x_0, μ_0)*. Assume that* $\widetilde{A}_{\leq \ell} \setminus \Gamma$ *is disconnected, let* $\widetilde{A}^1_{\leq \ell}$ *be one of its connected components, and assume that* $\widetilde{A}^1_{\leq \ell} = \widetilde{A}^1_{\ell}$*. Further assume that there exists* $\mathcal{C} \subset Q \times \mathbb{R}$ *and* $\mu^* < \mu_0$ *(resp.* $\mu^* > \mu_0$*) satisfying the following properties:*

1. $\widetilde{A}^1_{\ell} \cap (Q \times \{\mu^*\}) = \varnothing$,
2. $\widetilde{A}^1_{\ell} \subset \mathcal{C}$,
3. $\iota^*_{\mu}\eta(f_{\mu}(x)) > 0$ *for all* $(x, \mu) \in \mathcal{C} \cap (Q \times [\mu^*, \infty))$ *(resp.* $(x, \mu) \in \mathcal{C} \cap (Q \times (-\infty, \mu^*])$*), and*
4. *for every* $\mu \geq \mu^*$ *(resp.* $\mu \leq \mu^*$*),* $\mathcal{C} \cap (Q \times [\mu^*, \mu])$ *(resp.* $\mathcal{C} \cap (Q \times [\mu, \mu^*])$*) is compact,*

Then for all $\mu > \mu_0$ *(resp.* $\mu < \mu_0$*),* $\widetilde{A}^1_{\ell} \cap (Q \times \{\mu\}) \neq \varnothing$*. In particular,* f_{μ} *has a periodic orbit for all* $\mu \geq \mu_0$*.*

Theorem 7 (Global existence of periodic orbits following a Hopf bifurcation)**.** *Assume that Q is orientable, and let $N \subset Q \times \mathbb{R}$ be a properly embedded, smooth, orientable, codimension-1 submanifold with boundary $M = \partial N$. Let $f \in C^1(Q \times \mathbb{R}, \mathsf{T}Q)$ be a family of vector fields, and let $\eta \in [\eta] \in H^1((Q \times \mathbb{R}) \setminus M; \mathbb{Z}) \subset H^1_{\mathrm{dR}}((Q \times \mathbb{R}) \setminus M)$ be a closed 1-form representing the (closed) Poincaré dual $[\eta]$ of N in $H^1_{\mathrm{dR}}((Q \times \mathbb{R}) \setminus M)$. Further assume that there exists $\mathcal{C} \subset Q \times \mathbb{R}$, $(x_c, \mu_c) \in M \cap \mathrm{int}(\mathcal{C})$ and $\mu^* < \mu_c$ (resp. $\mu^* > \mu_c$) satisfying the following properties:*

1. *f_{μ^*} has no periodic orbits contained in $\mathcal{C} \cap (Q \times \{\mu^*\})$,*
2. *no periodic orbits of f intersect $\partial\mathcal{C} \cap (Q \times [\mu^*, \infty))$ (resp. $\partial\mathcal{C} \cap (Q \times (-\infty, \mu^*])$),*
3. *For every $\mu > \mu^*$ (resp. $\mu < \mu^*$), there exists $\epsilon > 0$ such that $\iota_\mu^* \eta(f_\mu(x)) \geq \epsilon$ for all $(x, \mu) \in (\mathcal{C} \setminus M) \cap (Q \times [\mu^*, \mu])$ (resp. $(x, \mu) \in (\mathcal{C} \setminus M) \cap (Q \times [\mu, \mu^*])$) ,*
4. *for every $\mu \geq \mu^*$ (resp. $\mu \leq \mu^*$), $\mathcal{C} \cap (Q \times [\mu^*, \mu])$ (resp. $\mathcal{C} \cap (Q \times [\mu, \mu^*])$) is compact,*

5. *f is C^3 on a neighborhood of (x_c, μ_c), $(x_c, \mu_c) \in M \cap \operatorname{int}(\mathcal{C})$ is a point of generic Hopf bifurcation for f, and $\mathcal{C} \cap (Q \times [\mu^*, \infty))$ (resp. $\mathcal{C} \cap (Q \times (-\infty, \mu^*]))$ contains no other generalized centers,*

6. *no nonstationary periodic orbits of f intersect $\mathcal{C} \cap M \cap (Q \times [\mu^*, \infty))$ (resp. $\mathcal{C} \cap M \cap (Q \times (-\infty, \mu^*]))$, and*

7. *letting $E^c \subset \mathsf{T}_{x_c} Q$ be the two-dimensional center subspace for $\mathsf{D}_{(x_c,\mu_c)} f$,*

$$\mathsf{T}_{(x_c,\mu_c)}(Q \times \mathbb{R}) = (\mathsf{D}_{(x_c,\mu_c)} \iota_\mu E^c) \oplus \mathsf{T}_{(x_c,\mu_c)} M.$$

Then for all $\mu \geq \mu_c$ (resp. $\mu \leq \mu_c$), f_μ has a periodic orbit contained in $\mathcal{C} \cap (Q \times \{\mu\})$.

This result applies to both the repressilator and Sprott systems.

A continuation idea:

- **Both the repressilator and Sprott systems provably undergo Hopf bifurcations, so POs exist at *some* parameter values, but the "how small is ϵ?" problem remains.**
- **Can we solve the "how small is ϵ?" problem?**
- **Motivating "continuation" degree-theoretic result: let $D \subset \mathbb{R}^n$ be a compact domain with smooth boundary, $g_\mu \colon D \to \mathbb{R}^n$ a smooth 1-parameter family of maps (a smooth homotopy), and $\mu_0 \in \mathbb{R}$ such that**[1]

  1. **g_{μ_0} has finitely many zeros,**
  2. **$\mathsf{D}_x g_{\mu_0}$ is invertible at every zero x,**
  3. **$g_{\mu_0}|_{\partial D}$ has no zeros, and**
  4. **The Poincaré-Hopf index $\Sigma_{x \in g_{\mu_0}^{-1}(0)} \operatorname{sign}(\det(\mathsf{D}_x g_{\mu_0})) \neq 0$.**

  Then if $\mu_1 > \mu_0$ is such that $g_\mu|_{\partial D}$ has no zeros for all $\mu \in [\mu_0, \mu_1)$, there exists a zero of g_μ in D for all $\mu \in [\mu_0, \mu_1]$.

[1] This can be proved using methods from J.W. Milnor's classic "Topology from the Differentiable Viewpoint."

- This is a “continuation principle” valid of an “index”; other examples include the continuation principles for the Lefschetz and Conley indices.
- If there was an analogous index for POs satisfying a continuation principle, then it seems plausible that this could resolve the “how small is ϵ?” problem for the Hopf bifurcations in the Sprott system.
- In fact there exist multiple indices for POs satisfying continuation principles; their use was recently suggested by Rajapakse and Smale (2017) to study POs in biology.
- However: we will see they do not *practicably* solve the “how small is ϵ?” problem...

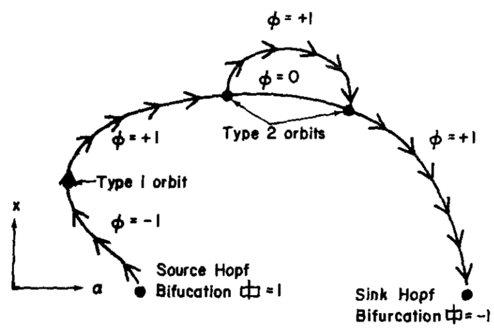
PO continuation theory of Alexander, Alligood, Mallet-Paret, Yorke, and others:

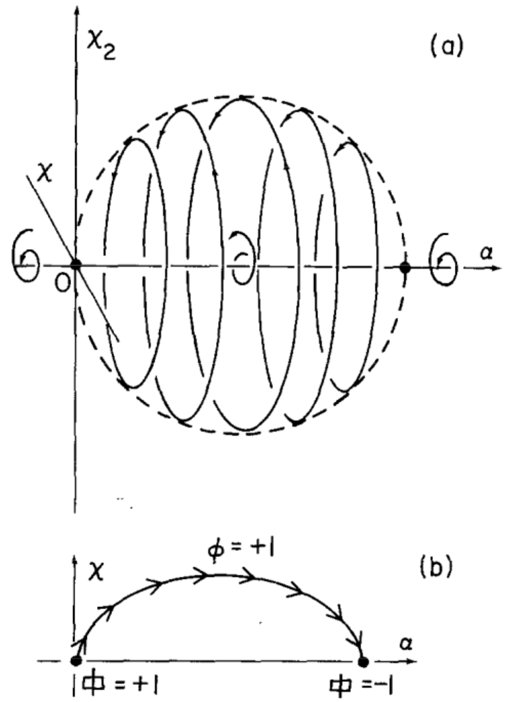
Overview of PO continuation results:

$$\dot{x} = f_\mu(x), \qquad (x, \mu) \in Q \times \mathbb{R}, \qquad f \in C^1(Q \times \mathbb{R}, \mathsf{T}Q) \tag{0.11}$$

1. **Some important early efforts (e.g. the "Fuller index") by [Ful67, AY78, CMP78].**
   - **Studied connected components of POs in (x, μ, τ)-space, where τ is the period of a PO.**
2. **Subsequently several authors showed that more refined information could be obtained by studying components of POs in (x, μ)-space using other techniques [AMPY81, MPY82, CMPY83, AMPY83, AY84].**
   - **Fiedler [Fie88] refined and extended many of these results to ODE families equivariant under certain groups of symmetries.**

3. **Big problem for using all results above to prove PO existence: a priori upper bounds on the periods (or *virtual periods*, to be discussed) of POs are required.**
   - **It seems that there are few general techniques to obtain such bounds.**
4. **However, in [KB19] we show that the existence of a closed 1-form η on $Q \times \mathbb{R}$ satisfying certain properties enables a priori period upper bounds to be replaced with conditions such as $\eta((f, 0)) > 0$ which are in principle computable.**





Snakes: Oriented Families of Periodic Orbits, their Sources, Sinks, and Continuation [MPY82]:

> **"The difficulty of representing families of orbits in high dimensions is considerable. Both (a) and (b) represent [the same family of ODEs]..." - [MPY82]**

Generic and non-generic families:

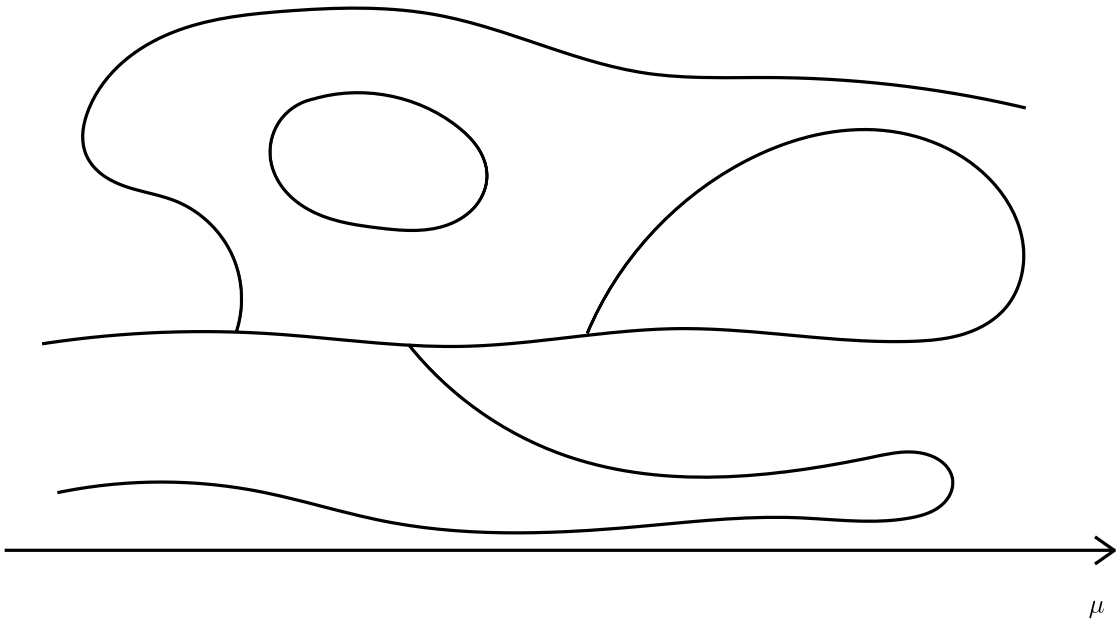


Figure 0.9: An example orbit diagram for a generic family $f \in \mathcal{K}$. Each point corresponds to a single periodic orbit of f. The specific diagram above includes periodic orbits of all three types 0, 1, and 2 (see Figure 0.10).

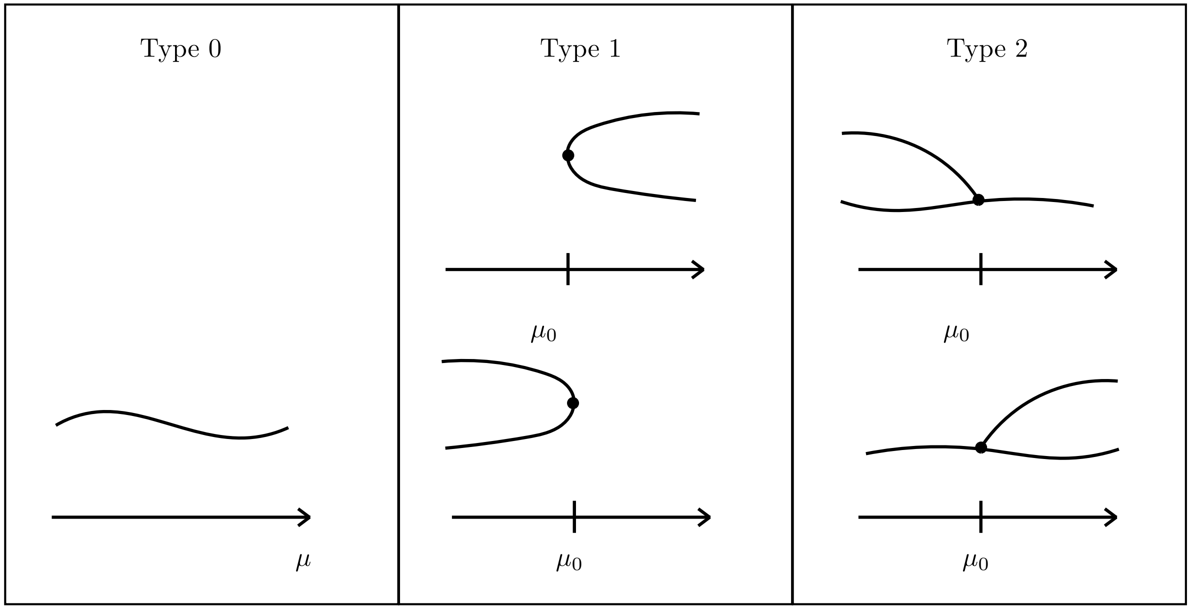


Figure 0.10: Segments of orbit diagrams containing the three types of periodic orbits occurring in generic one-parameter families. In the second and third columns, the dots correspond to type 1 and type 2 orbits occurring at $\mu = \mu_0$.

“Orbit diagrams”:

We can introduce an equivalence relation $\sim$ on the subset $\mathcal{O} \subset Q \times \mathbb{R}$ of periodic orbits of f so that $(x, \mu) \sim (y, \nu)$ if and only if $\mu = \nu$ and x, y lie on the same periodic orbit. Since the natural projection $\pi_2 \colon \mathcal{O} \to \mathbb{R}$ descends to a map $\widetilde{\pi}_2 \colon (\mathcal{O}/\sim) \to \mathbb{R}$, we can “plot” $(\mathcal{O}/\sim)$ as a multi-valued function of μ with each point representing a periodic orbit of f.

A type 0 orbit is one which has no Floquet multipliers that are roots of unity. In particular, since $+1$ is not a Floquet multiplier, applying the implicit function theorem to a Poincaré map shows that a type 0 orbit is locally continuable as a function of μ along a unique branch of orbits on which periods vary continuously.

A type 1 orbit γ has a single (algebraically simple) Floquet multiplier equal to $+1$, no other multipliers which are roots of unity, and we require that the eigenvalue $\lambda_1(\mu)$ satisfying $\lambda_1(\mu_0) = 1$ crosses the unit circle with nonzero velocity: $\lambda_1'(\mu_0) \neq 0$.

A type 2 orbit γ has a single (algebraically simple) Floquet multiplier equal to -1, no other multipliers which are roots of unity, and we require that the eigenvalue $\lambda_1(\mu)$ satisfying $\lambda_1(\mu_0) = -1$ crosses the unit circle with nonzero velocity: $\lambda_1'(\mu_0) \neq 0$.

Definition 2 (Consistently oriented curves in the Möbius band)**.** Let X be the Möbius band (with boundary). Let Γ_1 be the middle circle of the Möbius band, Γ_2 be the boundary circle, and let $\pi\colon X \to \Gamma_1$ be the straight-line retraction of X onto the middle circle. Then (depending on orientations) the degree of $\pi|_{\Gamma_2}\colon \Gamma_2 \to \Gamma_1$ is ± 2. We say that Γ_1 and Γ_2 are *consistently oriented* if the degree of $\pi|_{\Gamma_2}$ is $+2$.

Proposition 1. *Let $Y \subset Q \times \mathbb{R}$ be an arbitrary subset of nonstationary periodic orbits for a generic family $f \in \mathcal{K} \subset C^5(Q \times \mathbb{R}, \mathsf{T}Q)$. Define an equivalence relation $\sim$ on Y so that $(x, \mu) \sim (y, \nu)$ if and only if $\mu = \nu$ and x, y lie on the same periodic orbit. Let $\pi \colon Y \to Y/\sim$ be the quotient map, and let $[(x, \mu)] \coloneqq \pi(x, \mu)$ denote the equivalence class of $(x, \mu) \in Y$. If γ is a periodic orbit for f_μ with image Γ satisfying $\Gamma \times \{\mu\} \subset Y$, then by an abuse of notation we let $[\gamma] \coloneqq [(\gamma(0), \mu)]$.*

We have the following.

1. *The quotient map $\pi \colon Y \to Y/\sim$ is open. If the periods of orbits in Y are uniformly bounded from above, then π is also closed and $Y/\sim$ is Hausdorff.*

2. *Assume that Y is an open subset of a connected component of non-stationary periodic orbits. If $[\gamma] \in Y/\sim$ is a type 0 or type 1 orbit, then there exists $\epsilon > 0$ and a C^5 homotopy $H \colon S^1 \times (-\epsilon, \epsilon) \to Q \times \mathbb{R}$ with the following properties.*

   - *For each $s \in (-\epsilon, \epsilon)$, $H_s \coloneqq H(\,\cdot\,, s)$ is a diffeomorphism onto the* *image of a periodic orbit in Y.*

- *For any $z \in S^1$, the map $(-\epsilon, \epsilon) \to Y/\sim$ given by $s \mapsto \pi \circ H_s(z)$ is a homeomorphism onto a subset $U \subset Y/\sim$ containing $[\gamma]$, and $\pi \circ H_0(z) = [\gamma]$.*
- *For every $N > 0$, there exists a neighborhood $V_N \subset Y/\sim$ of $[\gamma]$ such that $V_N \setminus U$ contains only orbits with periods greater than N.*

3. *Assume that Y is an open subset of a connected component of nonstationary periodic orbits. If $[\gamma] \in Y/\sim$ is a type 2 orbit, then there are three disjoint arcs $S_1, S_2, S_3 \subset Y/\sim$ homeomorphic to open intervals such that the following holds.*

   - *There exists $\epsilon > 0$ and a C^5 homotopy $H\colon S^1 \times (-\epsilon, \epsilon) \to Q \times \mathbb{R}$ satisfying the same properties as the homotopy in 2, except that the map $s \mapsto \pi \circ H_s(z)$ is a homeomorphism onto $U \coloneqq S_1 \cup [\gamma] \cup S_2$.*
   - *If $[\alpha] \in S_3$, then there exists a C^4 embedded Möbius band $X \subset Q \times \mathbb{R}$ such that, when viewed as subsets of $Q \times \mathbb{R}$, the images of γ and α are respectively the middle and boundary circles of X, and*

these images are consistently oriented when given the orientations induced by γ and α.

- *For every $N > 0$, there exists a neighborhood $V_N \subset Y/\sim$ of $[\gamma]$ such that $V_N \setminus ([\gamma] \cup S_1 \cup S_2 \cup S_3)$ contains only orbits with periods greater than N.*

4. *Assume that Y is a connected component of periodic orbits. If (x_n, μ_n) is a sequence of points on the images of periodic orbits γ_n with $(x_n, \mu_n) \notin Y$ but $(x_n, \mu_n) \to (x, \mu) \in Y$ as $n \to \infty$, then the periods τ_n of the γ_n satisfy $\tau_n \to \infty$.*

Global continuation for generic families

We establish in Lemma 4 (a slightly strengthened version of) Theorem 5 in the special case that $f \in \mathcal{K}$ is a generic family. This will enable us to prove Theorem 5 by approximating an arbitrary family f by generic families $g \in \mathcal{K}$.

Lemma 1 (Homotopy invariance)**.** *Let M be a smooth manifold and η be a C^1 closed 1-form on M. If $\alpha, \beta \colon S^1 \to M$ are C^1 maps which are homotopic, then*

$$\int_\alpha \eta = \int_\beta \eta.$$

The following preliminary result is also straightforward.

Lemma 2. *Let X be a C^1 Möbius band (with boundary). Let Γ_1 be the middle circle of the Möbius band and Γ_2 the boundary, and assume Γ_1, Γ_2 are consistently oriented (Definition 2). Then if η is any C^1 closed* 1-*form on X,*

$$\int_{\Gamma_2} \eta = 2 \int_{\Gamma_1} \eta. \tag{0.12}$$

One of the key ideas needed for Lemma 4 is contained in the following Lemma 3 which shows that, for a generic family, the periodic orbit components $\widetilde{A}^i_\ell$ of Definition 1 are topological 1-manifolds if the periods of $\widetilde{A}^i_\ell$ are uniformly bounded.

Recall that $\iota_\mu \colon Q \hookrightarrow Q \times \mathbb{R}$ denotes the inclusion $\iota_\mu(x) = (x, \mu)$ for each $\mu \in \mathbb{R}$.

Lemma 3. *Let* $f \in \mathcal{K} \subset C^5(Q \times \mathbb{R}, \mathsf{T}Q)$ *be a generic family of vector fields and let* η *be a* C^1 *closed 1-form on an open subset* $\operatorname{dom}(\eta) \subset Q \times \mathbb{R}$*. Let* $A \subset \operatorname{dom}(\eta)$ *be a connected component of nonstationary periodic orbits of* $f|_{\operatorname{dom}(\eta)}$*, and let the equivalence relation* $\sim$ *on* A *be as in Proposition 1. For any* $\ell > 0$*, let* $A_\ell \subset A$ *be the subset of points on periodic orbits* α_μ *with* $\left|\int_{\alpha_\mu} \iota_\mu^* \eta\right| = \ell$ *and* $A_{\leq \ell}$ *the subset with* $\left|\int_{\alpha_\mu} \iota_\mu^* \eta\right| \leq \ell$*.*

Fix an open subset $U \subset A/\sim$ *and* $\ell > 0$*, and assume that* $U \cap (A_\ell/\sim) \neq \varnothing$*. Let* V_ℓ *be a connected component of* $U \cap (A_\ell/\sim)$ *and let* $V_{\leq \ell}$ *be the unique component of* $U \cap (A_{\leq \ell}/\sim)$ *containing* V_ℓ*. Assume that the periods of orbits belonging to* $V_{\leq \ell}$ *have a uniform upper bound. Then*

1. V_ℓ *is a topological* 1*-manifold (without boundary).*

2. *If* $V_\ell = V_{\leq \ell}$*, then* V_ℓ *is also closed as a subset of* U*.*

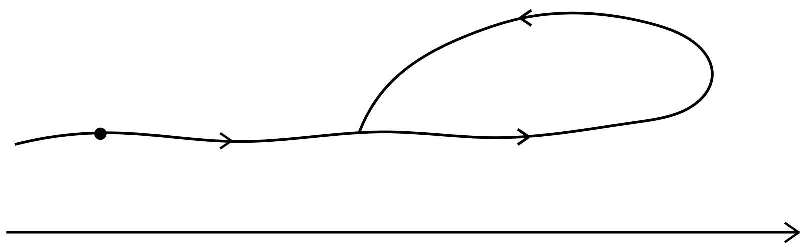


Figure 0.11: The main idea behind Lemma 3 is that its hypotheses imply that portions of orbit diagrams such as the one shown above cannot occur if the corresponding periodic orbits are contained in $\text{dom}(\eta)$ and have uniformly bounded periods. In more detail: if the periodic orbit γ represented by the above dot at μ_0 satisfies $\ell := \left|\int_\gamma \iota_{\mu_0}^* \eta\right| > 0$, then the orbit diagram above cannot occur for a generic one-parameter family. To see this, let β be a periodic orbit represented by a point in the top of the loop at μ_1. There is a homotopy of periodic orbits corresponding to the path indicated by the arrows above, so homotopy invariance (Lemma 1) implies that $\left|\int_\beta \iota_{\mu_1}^* \eta\right| = \ell$. On the other hand, applying Lemma 2 to the branch of bifurcating orbits near the type 2 orbit implies that $\left|\int_\beta \iota_{\mu_1}^* \eta\right| = 2\ell \neq \ell$, a contradiction.

We now state the main result here. Lemma 4 yields a result for general families slightly stronger than Theorem 5, because it does not require the hypothesis that $+1$ is not a Floquet multiplier of the periodic orbit γ.

Lemma 4 ((η, ℓ)-global continuability for generic families)**.** *Let $f \in \mathcal{K} \subset C^5(Q \times \mathbb{R}, \mathsf{T}Q)$ be a generic family of vector fields, and let η be a C^1 closed 1-form on an open subset $\text{dom}(\eta) \subset Q \times \mathbb{R}$. Let $A \subset \text{dom}(\eta)$ be a connected component of nonstationary periodic orbits of $f|_{\text{dom}(\eta)}$. Let γ be*

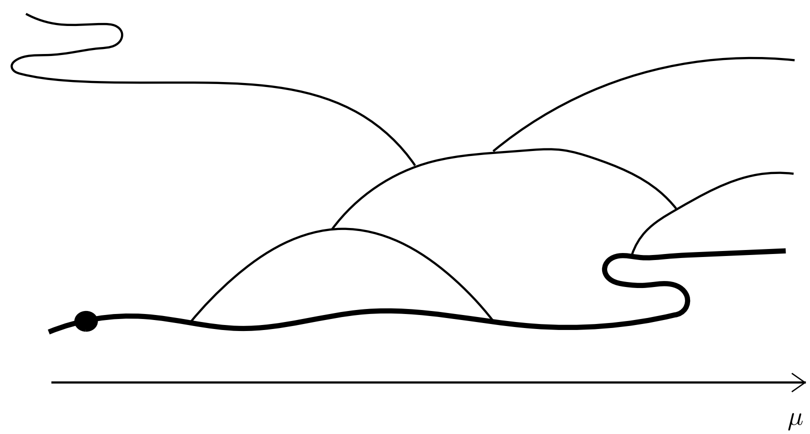


Figure 0.12: A portion of an orbit diagram for a component $A \subset \mathrm{dom}(\eta)$ of a generic family having uniformly bounded periods. In the notation of Lemma 3, the thick curve represents a portion of a set V_ℓ satisfying all hypotheses (including $V_\ell = V_{\leq \ell}$) of Lemma 3 (here $U = A/\sim$, where $\sim$ is the equivalence relation defined in Proposition 1.). Since $\ell > 0$, all periodic orbits α_μ represented by points on the thick curve also satisfy $\left|\int_{\alpha_\mu} \iota_\mu^* \eta\right| = \ell$. V_ℓ is a topological 1-manifold since the situation depicted in Figure 0.11 cannot occur, and V_ℓ is a closed subset of $A/\sim$ since parts 2 and 3 of Proposition 1 imply that the function $[\alpha_\mu] \mapsto \left|\int_{\alpha_\mu} \iota_\mu^* \eta\right|$ is lower semi-continuous on $(A/\sim)$.

a periodic orbit for some f_{μ_0} with image Γ satisfying $\Gamma \times \{\mu_0\} \subset A$, define $\ell := \left|\int_\gamma \iota_{\mu_0}^* \eta\right|$, and assume $\ell > 0$. Then γ is (η, ℓ)-globally continuable.

Non-generic families: Now prove our main theorems on global continuation of periodic orbits for arbitrary C^1 families of vector fields. Before doing this, we require one additional lemma. Lemma 5 enables us to prove Theorem 5 without the consideration of "virtual periods" as required in work of Alligood and Yorke.

Lemma 5. *Let $f_n \in C^1(M, \mathsf{T}M)$ be a sequence of C^1 vector fields on a smooth manifold M which converge in the weak C^1 topology to a C^1 vector field f on M, and let η be a C^1 closed 1-form on M. For each n let γ_n be a periodic orbit of f_n with image Γ_n and (minimal) period τ_n, and let γ be a periodic orbit of f with image Γ and (minimal) period τ. Assume that the periods τ_n have a uniform upper bound, and assume that for each n there exists $x_n \in \Gamma_n$ such that $x_n \to x_0 \in \Gamma$. Then*

1. $\liminf_{n\to\infty} \left|\int_{\gamma_n} \eta\right| \geq \left|\int_\gamma \eta\right|$ *and* $\liminf_{n\to\infty} \tau_n \geq \tau$*;*

2. $\lim_{n\to\infty} \left|\int_{\gamma_n} \eta\right| = \left|\int_{\gamma} \eta\right|$ *if and only if* $\lim_{n\to\infty} \tau_n = \tau$.

Then can prove the theorems stated earlier.

Return to Sprott system: **As far as we know, this is the first time that the existence of nonstationary periodic orbits has been proven rigorously for this system. The equations are given on** $\mathbb{R}^3$ **by**

$$\begin{aligned} \dot{x} &= y^2 - z - \mu x \\ \dot{y} &= z^2 - x - \mu y \\ \dot{z} &= x^2 - y - \mu z, \end{aligned} \tag{0.13}$$

and depend on the parameter $\mu \in \mathbb{R}$**. We note that, unlike the repressilator, the Sprott system is not a monotone cyclic feedback system.**

We will prove that (0.13) has a periodic orbit for all $\mu \in (-0.25, 0.5)$. Just like for the repressilator, the proof will amount to showing that (0.13) satisfies the hypotheses of Theorem 7.

We will construct the ingredients required to do this.

First, we find a certain compact set K_μ which contains all bounded trajectories of (0.13); we will define the set $\mathcal{C}$ of Theorem 7 in terms of K_μ. Unlike the sets K_μ defined for the repressilator, K_μ is not a trapping region and is not even invariant; this illustrates the flexibility allowed by the hypotheses of Theorem 7. Then we derive estimates involving $d\theta(f_\mu)$used to establish hypothesis 3 of Theorem 7. Then we determine the equilibria and associated eigenvalues of $\mathsf{D}f_\mu$. We show that (0.13) exhibits Hopf bifurcations, needed in particular to verify hypothesis 5 of Theorem 7. Finally we combine these ingredients to prove the periodic orbit existence theorem.

A compact set containing all bounded trajectories: Define the function $V\colon \mathbb{R}^3 \to \mathbb{R}$ via $V(\mathbf{x}) \coloneqq x + y + z$. A computation shows that the Lie derivative $\dot{V}$ of V is

$$\dot{V}(x, y, z) = \|\mathbf{x}\|^2 - (\mu + 1)(x + y + z) = \langle \mathbf{1}, f_\mu(\mathbf{x}) \rangle. \tag{0.14}$$

For any $c \geq -\frac{3}{4}(\mu+1)^2$, the sublevel set $B_{\mu,c} \coloneqq \dot{V}^{-1}(-\infty, c]$ is the closed ball of radius $\frac{\sqrt{3(\mu+1)^2+4c}}{2}$ centered at $(\frac{1+\mu}{2})\mathbf{1}$. In particular, the zero sublevel set of $\dot{V}$ is centered at the midpoint of two equilibria on the diagonal (the origin and $(1+\mu)\mathbf{1}$), with the two equilibria being antipodal points on the bounding sphere. Furthermore, the planes $V^{-1}(0)$ and $V^{-1}(3(1+\mu))$ are tangent to the sphere at these antipodal points. See Figure 0.7.

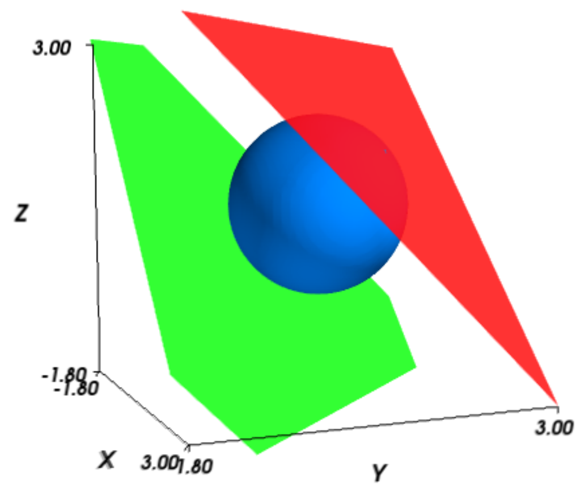


Figure 0.13: Shown here is the spherical level set $\dot{V}^{-1}(0)$ (teal), and the two planes $V^{-1}(0)$ (green) and $V^{-1}(3(1+\mu))$ (red) for $\mu = 0.4$.

This geometry implies that the subsets $V^{-1}(-\infty, 0)$ and $V^{-1}(3(1+\mu), \infty)$ are respectively negatively and positively invariant for $\mu \geq -1$. Furthermore, trajectories in these regions tend to ∞ in negative and positive time, respectively. It follows that any bounded trajectory must be contained in $V^{-1}[0, 3(1+\mu)]$ when

$\mu \geq -1$. **We will further refine these considerations to produce a certain *compact* set containing all bounded trajectories.**

Define translated coordinates $\mathbf{x}_\mu := (x_\mu, y_\mu, z_\mu) := \mathbf{x} - \frac{(1+\mu)}{2}\mathbf{1}$ **and define** $r_\mu := \|\mathbf{x}_\mu\|$.

Theorem 8. *For $\mu > -1$, every bounded trajectory is contained in the compact set K_μ defined by*

$$K_\mu := \left\{ \mathbf{x} \in \mathbb{R}^3 \colon V(\mathbf{x}) \geq r_\mu - \frac{3^{\frac{3}{4}} + 3^{\frac{1}{4}}}{2}(1+\mu)\arctan\left(\frac{2r_\mu}{(3^{1/4})(1+\mu)}\right) - \frac{\sqrt{3}}{2}(1+\mu) \right.$$
$$\left. + \frac{3^{\frac{3}{4}} + 3^{\frac{1}{4}}}{2}(1+\mu)\arctan\left(3^{1/4}\right) \right\} \cap V^{-1}[0, 3(1+\mu)], \tag{0.15}$$

and $\dot{V}^{-1}(-\infty, 0] \subset K_\mu$. For $\mu = -1$, the only bounded trajectory is the equilibrium at the origin; we define $K_{-1} := \{\mathbf{0}\}$.

For a visual depiction of K_μ, **see Figure 0.8. Note that the ball** $\dot{V}^{-1}(-\infty, 0]$ **bounded by the sphere** $\dot{V}^{-1}(0)$ **shown in Figure 0.7 is contained in** K_μ.

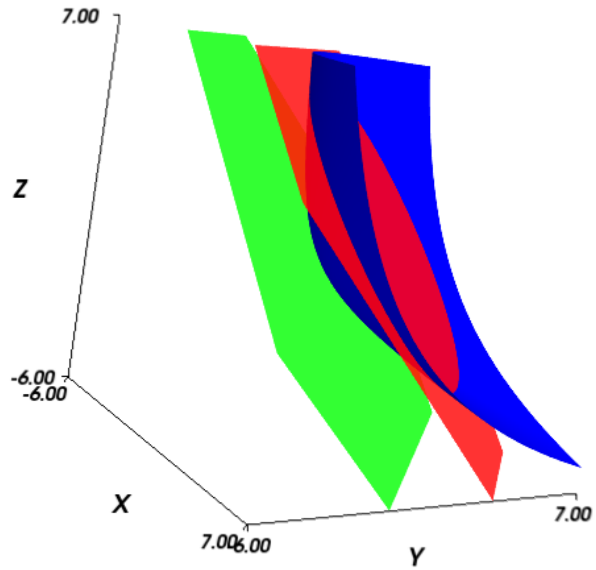


Figure 0.14: The compact set K_μ of Theorem 8 is the region bounded by the blue surface, red plane $V^{-1}(3(1+\mu))$, and green plane $V^{-1}(0)$. Note that the ball $\dot{V}^{-1}(-\infty, 0]$ with boundary $\dot{V}^{-1}(0)$ shown in Figure 0.7 is contained in K_μ. This figure was generated using $\mu = 0.4$.

Cylindrical coordinates and rotation of the flow: **Define an orthogonal matrix** M **via**

$$M = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \tag{0.16}$$

and define coordinates $[u, v, w]^T := M^{-1}[x, y, z]^T = M^T[x, y, z]^T$**. The** w**-axis corresponds to** Δ **in the original coordinates, and the** u **and** v **axes determine an orthonormal coordinate system for** $\Delta^{\perp}$**.**

We further define cylindrical coordinates (ρ, θ, w) **via**

$$\begin{aligned} u &= \rho \cos\theta \\ v &= \rho \sin\theta. \end{aligned} \tag{0.17}$$

Using the symbolic package SymPy, we obtain the equations of motion in these new coordinates in closed form:

$$
\begin{aligned}
\dot{\rho} &= \rho\left(-\sqrt{2}\rho\sin^3(\theta) + \frac{\sqrt{6}}{3}\rho\cos^3(\theta) - \frac{\sqrt{6}}{2}\rho\cos\left(\theta + \frac{\pi}{3}\right) - \frac{1}{\sqrt{3}}w - \mu + \frac{1}{2}\right)\\
\dot{\theta} &= \rho\left(\frac{\sqrt{6}}{3}\sin^3(\theta) - \frac{\sqrt{6}}{2}\sin\left(\theta + \frac{\pi}{3}\right) + \sqrt{2}\cos^3(\theta)\right) - w - \frac{\sqrt{3}}{2}\\
\dot{w} &= \frac{1}{\sqrt{3}}(\rho^2 + w^2) - (\mu + 1)w.
\end{aligned}
\tag{0.18}
$$

Now

$$\dot{\theta} = \left(\frac{udv - vdu}{u^2+v^2}\right)(f_\mu),$$

and (u, v, w) **are orthogonal coordinates adapted to the splitting** $\mathbb{R}^3 = \Delta^\perp \oplus \Delta$ **with** (u, v) **coordinates for** $\Delta^\perp$ **and with** w **a coordinate for** Δ. $\dot{\theta} = d\theta(f_\mu)$, **where** $d\theta$ **is defined above. Because the hypotheses of Theorem 7 are stated in terms of a closed 1-form, we will write** $d\theta(f_\mu)$ **instead of** $\dot{\theta}$ **in the following results.**

Lemma 6. *The following estimate holds on* $\mathbb{R}^3 \backslash \Delta$ *or, equivalently, whenever* $\rho > 0$*:*

$$-0.41\rho - w - \frac{\sqrt{3}}{2} \leq d\theta(f_\mu) \leq 0.41\rho - w - \frac{\sqrt{3}}{2}. \tag{0.19}$$

The following result concerns the rotation rate $\dot{\theta} = d\theta(f_\mu)$ on the compact set K_μ (defined in Theorem 8) which contains all bounded trajectories.

Theorem 9. *There exists $\epsilon > 0$ such that, for all $-1 \leq \mu \leq 0.55$, $d\theta(f_\mu) < -\epsilon$ on $K_\mu \setminus \Delta$.*

Corollary 3. *For $\mu \in [-1, 0.55]$, all equilibria of f_μ belong to the diagonal Δ.*

Corollary 4. *For $\mu \in (-1, 0.55]$, all periodic orbits of f_μ are contained in $K_\mu \setminus \Delta$, and the winding number $\frac{1}{2\pi} \int_\gamma d\theta$ of any nonstationary periodic orbit γ around Δ satisfies $\frac{1}{2\pi} \int_\gamma d\theta \leq -1$. For the case $\mu = -1$, f_{-1} has no nonstationary periodic orbits.*

Δ is invariant and the dynamics restricted to Δ are given by

$$\dot{x} = x^2 - x - \mu x = x(x - 1 - \mu). \tag{0.20}$$

Theorem 10. *For all $\mu \in \mathbb{R}$, the vector field f_μ has the equilibria $\mathbf{0}$ and $(1+\mu)\mathbf{1}$. For $-1 \leq \mu \leq 0.55$, these are the only equilibria.*

Theorem 11. *The equilibrium* $\mathbf{0}$ *undergoes a subcritical generic Hopf bifurcation at* $\mu = 0.5$*, and the equilibrium* $(1+\mu)\mathbf{1}$ *undergoes a supercritical generic Hopf bifurcation at* $\mu = -0.25$*. The first bifurcation produces an exponentially stable limit cycle near* $\mathbf{0}$ *for* $0 < 0.5 - \mu \ll 1$*, and the second bifurcation produces an exponentially unstable limit cycle near* $(1+\mu)\mathbf{1}$ *for* $0 < \mu - (-0.25) \ll 1$*.*

Existence of periodic orbits:

Can put together the preceding results to obtain a periodic orbit existence result for the Sprott vector field (0.13). To do this, we show that the restriction $f|_{(-\infty,0.5)}$ **satisfies the hypotheses of Theorem 7 after a (nonlinear) parameter rescaling.**

Theorem 12. *Let* f_μ *be the Sprott vector field (0.13) and let* K_μ *be defined as in Theorem 8. For all* $\mu \in (-0.25, 0.5)$*,* f_μ *has a periodic orbit contained in* K_μ*.*