Symplectic invariants for structurally stable singularities of integrable systems

Elena Kudryavtseva
Moscow State University

(based on a joint work with Andrey Oshemkov)

GDM Seminar

01.02.2022

An *integrable system* on a symplectic $2n$-manifold (M^{2n}, ω) is defined by n functions $f_1, \dots, f_n$ satisfying two properties:

- they pairwise Poisson commute: $\{f_i, f_j\} = 0$, $\{f, g\} := \omega(X_f, X_g)$, $\omega(\cdot, X_f) = df$;
- they are functionally independent on M^{2n} almost everywhere.

Integral (momentum) map $F = (f_1, \dots, f_n) : M^{2n} \to \mathbb{R}^n$.

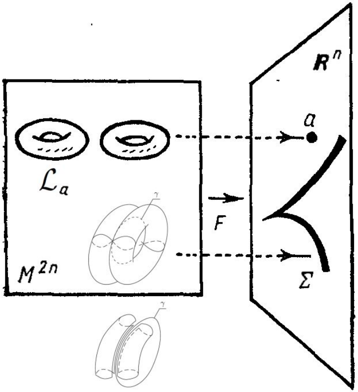
Singular Lagrangian fibration (Liouville fibration) on M, whose regular fibres are invariant tori with quasi-periodic dynamics.

The fibres are connected components $\mathcal{L}_a$ of integral surfaces $F^{-1}(a)$. Assume that all fibres are compact.

Singular set S $= \{x \in M^{2n} \mid \operatorname{rank} dF(x) < n\}$.
Bifurcation diagram $\Sigma = F(S) \subset \mathbb{R}^n$.

Then F generates the Hamiltonian action of $\mathbb{R}^n$ on M^{2n} by F-preserving symplectomorphisms $\phi^{t_1}_{X_{f_1}} \circ \dots \circ \phi^{t_n}_{X_{f_n}} : M \to M$, $(t_1, \dots, t_n) \in \mathbb{R}^n$. $\mathcal{L} = \bigcup_i \mathcal{O}_i$.

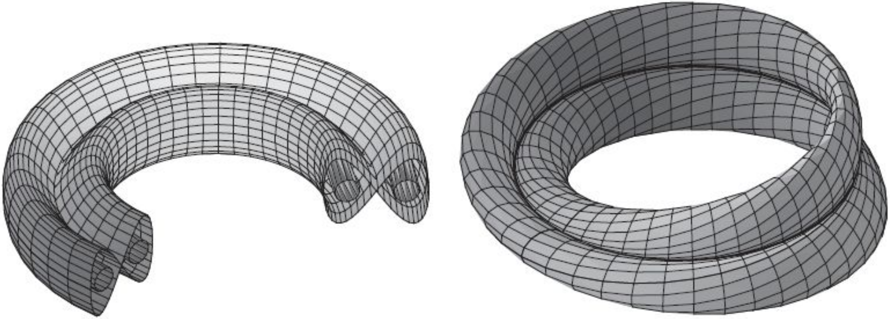
A local (resp., semilocal) singularity is the fibration germ at a singular orbit $\mathcal{O}$ (resp., singular fiber $\mathcal{L}$).



A singularity is called structurally stable if the topology of the fibration is preserved after any (small enough) real-analytic integrable perturbation of the system.

Illustration: Liouville fibration. Singular Lagrangian fibrations

For simplicity: two degrees of freedom.
(M^4, ω, F), $F = (H, K) : M^4 \to \mathbb{R}^2$, $F(x) = (H(x), K(x))$.



The object we want to study is just a singular Lagrangian fibration

$$\hat{F} : M^{2n} \to B^n,$$

which locally can be given by commuting functions.

It is more convenient to replace the image $F(M^{2n})$ of the momentum map by the set of fibres B^n which, in general, is not a smooth manifold.
However in all interesting examples, B has a structure of a stratified n-manifold.

Equivalent integrable systems

Given two integrable systems $F_1 : M_1^{2n} \to B_1$ and $F_2 : M_2^{2n} \to B_2$ (singular Lagrangian fibrations), we want to find/discuss/study conditions for the existence of *fibrewise maps* Φ between them:

$$\begin{array}{ccc} M_1^{2n} & \xrightarrow{\Phi} & M_2^{2n} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_2} \\ B_1 & \xrightarrow{\hat{\phi}} & B_2. \end{array}$$

Does **homeomorphic** $\Longrightarrow$ **symplectomorphic**?

If not, what are additional **symplectic invariants**?

2 options for M_1 and M_2:

- local (neighbourhood of a singular point or a singular orbit);
- semilocal (neighbourhood of a singular fibre).

Two options for Φ (fibrewise map between M_1 and M_2):

- topological;
- symplectic.

Allowed types of local singularities:

- non-degenerate singularities (direct products of ellyptic, hyperbolic, focus-focus, regular).

What is known about topological/symplectic invariants?

- Local
  - J. Vey 1978, H. Eliasson 1990:
    Non-degenerate singularities: no local symplectic invariants,
  - E. Miranda and N.T. Zung, 2004:
    Equivariant version of this result (near a non-degenerate orbit).
- Semi-local
  - A. Fomenko and H. Zieschang, 1990: Topology of hyperbolic corank 1 singularities (2 d.f.),
    N.T. Zung, 1996: Topology of nondegenerate singularities,
    A.S. Lermontova, 2005: Topology of hyperbolic corank 1 singularities,
  - J.-P. Dufour, P. Molino and A. Toulet, 1994: Simple hyperbolic singularities (1 d.f.),
  - S. Vũ Ngọc, 2007: Simple focus-focus singularities (two d.f., pinched torus),
  - H. Dullin and S. Vũ Ngọc: Hyperbolic (saddle-saddle) singularities (2 d.f.),
  - A. Bolsinov and S. Vũ Ngọc (2005, unpublished): Non-degenerate singularities.
- Global
  - A. Fomenko and H.Zieschang, 1990: Topology of fibrations on 3D isoenergy manifolds (2.d.f.),
  - J. Duistermaat, 1987: Regular case (no singular fibres), Mishachev, 1996 (2 d.f.),
  - T. Delzant, 1988: Toric actions,
  - A. Pelayo, S. Vũ Ngọc, 2009: Semitoric manifolds (2 d.f.),
  - N.T. Zung, 2003: Very general case (topological and symplectic classifications).
- Structurally stable non-degenerate singularities
  - Local: K. 2021 (compact orbits), K. and Oshemkov (arbitrary orbits).
  - Semilocal: Oshemkov and Tuzhilin (structural stablity under "component-wise" integrable perturbations for saddle-saddle fibers satisfying connectedness condition).

Definition

A singular point $x \in M^{2n}$ of rank 0 is called non-degenerate if

- the linear operators A_{f_i}, which are the linearizations of X_{f_i} at x, are linearly independent, i.e. generate an n-dimensional subalgebra of $sp(T_x M) \approx sp(\mathbb{R}, 2n)$,
- there exists a linear combination $\sum \lambda_i A_{f_i}$, $\lambda_i \in \mathbb{R}$, having only simple eigenvalues.

Example: in dimension 4, there are 4 conjugacy classes of such subalgebras:

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$$\begin{pmatrix} 0 & 0 & -A & 0 \\ 0 & 0 & 0 & -B \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix}$$

центр-седло

$$\begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & 0 & 0 & -B \\ 0 & 0 & A & 0 \\ 0 & B & 0 & 0 \end{pmatrix}.$$

седло-седло

$$\begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

фокус-фокус

$$\begin{pmatrix} -A & -B & 0 & 0 \\ B & -A & 0 & 0 \\ 0 & 0 & A & -B \\ 0 & 0 & B & A \end{pmatrix}$$

Example

The canonical foliation L_{can} of the Williamson type (k_e, k_h, k_f) and rank r is given by the following quadratic functions on $(\mathbb{R}^{2n}, \omega = \sum d\lambda_i \wedge d\varphi_i + \sum dx_i \wedge dy_i)$:

$$
\begin{array}{lll}
h_i = \lambda_i & (\text{regular type}), & 1 \le i \le r, \\
h_{r+i} = x_i^2 + y_i^2 & (\text{elliptic type}), & 1 \le i \le k_e, \\
h_{r+i} = x_i y_i & (\text{hyperbolic type}), & k_e + 1 \le i \le k_e + k_h, \\
h_{r+i} = x_i y_i + x_{i+1} y_{i+1} & (\text{focus-focus type}), & 1 \le j \le k_f,\ i = k_e + k_h + 2j - 1, \\
h_{r+i+1} = x_i y_{i+1} - x_{i+1} y_i. & & \text{Here } k_e + k_h + 2k_f + r = n.
\end{array}
$$

Local non-degenerate singularities = symplectic direct products

Theorem 1 (J. Vey 1978, local symplectic classification)

In real-analytic case, the Liouville foliation in a neighborhood U of a non-degenerate singular point m_0 of rank r is locally symplectomorphic to a canonical foliation L_{can}, which is the direct product of basic foliations: regular, elliptic, hyperbolic, and focus-focus ones.

- regular: $h_i = \lambda_i$,
- elliptic: $h_{r+i} = x_i^2 + y_i^2$,
- hyperbolic: $h_{r+i} = x_i y_i$,
- focus-focus: $h_{r+i} = x_j y_j + x_{j+1} y_{j+1}$, $h_{r+i+1} = x_j y_{j+1} - x_{j+1} y_j$, $\quad r + k_e + k_h + 2k_f = n$.


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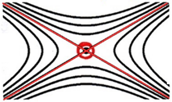
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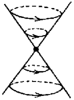
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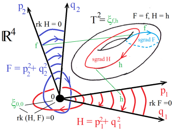


$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathbb{R}^{2n} \\ \downarrow F & & \downarrow F_{can} \\ \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^n. \end{array}$$

In other words, there exist a real-analytic symplectomorphism $\Phi = (\lambda_1, \varphi_1, \dots, \lambda_r, \varphi_r, x_1, y_1, \dots, x_{n-r}, y_{n-r}) : (U, \omega) \hookrightarrow (\mathbb{R}^{2n}, \omega_{can})$ and a real-analytic diffeomorphism germ $\phi = (\phi_1, \dots, \phi_n) : (\mathbb{R}^n, F(m_0)) \to (\mathbb{R}^n, 0)$ s.t. $\Phi(m_0) = 0$ and $\phi \circ F \circ \Phi^{-1} = F_{can} = (h_1, \dots, h_n)$, where $\omega_{can} = \sum_{s=1}^{r} \mathrm{d}\lambda_s \wedge \mathrm{d}\varphi_s + \sum_{j=1}^{n-r} \mathrm{d}x_j \wedge \mathrm{d}y_j$.

The map $\phi \circ F = F_{can} \circ \Phi$ is called the Vey momentum map (it generates $(S^1)^n$-action near $\mathcal{O}$).

Semilocal non-degenerate singularities = almost direct products

Definition ([Zung 1996, Def. 6.3])

We say that a nondegenerate semilocal singularity at $\mathcal{L}$ satisfies the non-splitting condition if the singular value set of the momentum map $F|_{U(\mathcal{L})}$ coincides with the singular value set of the momentum map $F|_{U(\mathcal{O})}$ for any compact orbit $\mathcal{O} \subset \mathcal{L}$.

Theorem (Nguyen Tien Zung, decomposition Theorem for semi-local non-degenerate singularities, 1996)

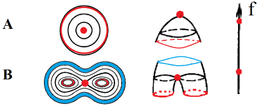
Each non-degenerate singularity satisfying the non-splitting condition is topologically equivalent to a singularity of almost-direct product type $(V_1 \times \cdots \times V_k)/G$ whose factors have one of the following four types:

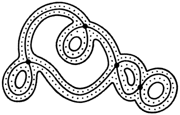
1) an elliptic singularity V_{ell} with one degree of freedom (i.e., 2-atom A);

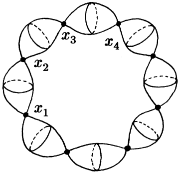
2) a hyperbolic singularity V_{hyp} with one degree of freedom (2-atom);

3) a focus-focus singularity V_{foc} with two degrees of freedom;

4) a regular Liouville foliation $V_{reg} = D^1 \times S^1$ without singularities with 1 degree of fr.







Structural stability test for semilocal singularities

Definition

We say that a compact nondegenerate rank-r singular fiber $\mathcal{L}$ satisfies connectedness condition if there exists a compact orbit $\mathcal{O} \subset \mathcal{L}$ of rank r such that each critical set

$$\mathbb{K}_i := \{m \in U(\mathcal{L}) \mid \mathrm{d}(\phi_i \circ F)(m) = 0\}, \qquad r+1 \leq i \leq n,$$

is connected and contains all compact orbits in $\mathcal{L}$. Here $\phi \circ F$ is a Vey momentum map at $\mathcal{O}$.

Theorem 1

Suppose, for a real-analytic integrable system, $\mathcal{L}$ is a compact nondegenerate singular fiber satisfying the connectedness condition (e.g. contains a unique compact orbit). Then the semilocal singularity at $\mathcal{L}$ is structurally stable under real-analytic (but not necessarily under C^∞) integrable perturbations.

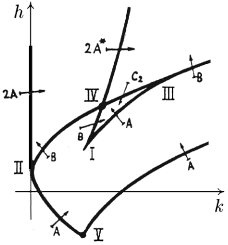
Example

Structurally stable semilocal singularities: almost-direct products of elliptic, regular,

- simple singularities
  $\mathcal{B} \times \mathcal{B}$, $(\mathcal{B} \times \mathcal{C}_2)/\mathbb{Z}_2$, $(\mathcal{B} \times \mathcal{D}_1)/\mathbb{Z}_2$, $(\mathcal{C}_2 \times \mathcal{C}_2)/(\mathbb{Z}_2 + \mathbb{Z}_2)$, $\mathcal{B} \times \mathcal{F}_1$, $(\mathcal{B} \times \mathcal{F}_2)/\mathbb{Z}_2$,
- singularities of complexity 2:
  $(\mathcal{D}_1 \times \mathcal{D}_1)/\mathbb{Z}_2$, $(\mathcal{P}_4 \times \mathcal{P}_4)/D_4$, $(\mathcal{C}_2 \times \mathcal{C}_2)/\mathbb{Z}_2$, $(\mathcal{C}_1 \times \mathcal{I}_1)/\mathbb{Z}_4$, $(\mathcal{C}_2 \times \mathcal{P}_4)/(\mathbb{Z}_2 + \mathbb{Z}_2)$, $(\mathcal{C}_1 \times \mathcal{J}_1)/\mathbb{Z}_4$, $(\mathcal{K}_3 \times \mathcal{K}_3)/(\mathbb{Z}_4 + \mathbb{Z}_2)$, $(\mathcal{C}_1 \times \mathcal{K}_3)/\mathbb{Z}_4$, $(\mathcal{C}_1 \times \mathcal{P}_4)/\mathbb{Z}_4$, $(\mathcal{D}_1 \times \mathcal{C}_2)/\mathbb{Z}_2$, $(\mathcal{D}_1 \times \mathcal{F}_2)/\mathbb{Z}_2$, $(\mathcal{C}_1 \times \mathcal{F}_4)/\mathbb{Z}_4$, $(\mathcal{C}_2 \times \mathcal{F}_2)/\mathbb{Z}_2$, ... (blue = structurally stable under C^∞ integrable perturb's)

Example

Structurally stable singularities in Mechanics:



$\mathcal{B}\times\mathcal{B}$ and $(\mathcal{B}\times\mathcal{C}_2)/\mathbb{Z}_2$ appear in the Kovalevskaya top,
$(\mathcal{B}\times\mathcal{D}_1)/\mathbb{Z}_2$ appears in the Goryachev-Chaplygin-Sretenskii case,
non-simple $(\mathcal{C}_2\times\mathcal{C}_2)/\mathbb{Z}_2$ occurs in the Clebsch case.
Simple *focus-focus* singularity $\mathcal{F}_1$ appears in the Lagrange case,
$\mathcal{F}_2$ occurs in the Clebsch case.

Thus, the saddle-saddle singularity $IV \approx (\mathcal{B}\times\mathcal{C}_2)/\mathbb{Z}_2$ (and the arrangement of singularities IV and $\mathcal{C}_2$) of the Kovalevskaya top is structurally stable under real-analytic integrable perturbations, but not under smooth integrable perturbations.

Principle Lemma

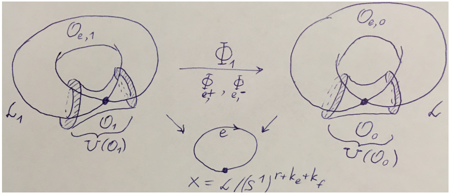
Under the above assumptions, every orbit $\mathcal{O}'\subseteq\mathcal{L}$ has the following properties.
(a) *The Vey momentum map $\phi\circ F$ at the compact orbit $\mathcal{O}$ can serve as a Vey momentum map at $\mathcal{O}'$, with the same regular, elliptic, hyperbolic and focus-focus components apart from some of the hyperbolic and/or focus-focus components at $\mathcal{O}$ which are regular components at $\mathcal{O}'$.*
(b) *The perturbed Vey momentum map $\tilde\phi\circ\tilde F$ near $\mathcal{O}$ can serve as a perturbed Vey momentum map near $\mathcal{O}'$.*

Semilocal symplectic invariant

Suppose $\mathcal{L}$ is a compact non-degenerate fibre of a real-analytic 'reference system', and $\mathcal{L}_1$ is a compact non-degenerate fibre of a real-analytic integrable system. Suppose there exists a fiberwise homeomorphism $\Phi_1 : U(\mathcal{L}_1) \to U(\mathcal{L})$ which is fibrewise isotopic to a symplectomorphism near a compact orbit $\mathcal{O}_1 \subset \mathcal{L}_1$, $\Phi_1(\mathcal{L}_1) = \mathcal{L}$.
We define a 1-cocycle [Dufour, Molino, Toulet 1994], [Zung 2003], [Vũ Ngọc 2007]

$$\mu_{\mathcal{L}_1} \in H^1(X, Z^1) \cong Hom(H_1(X), Z^1), \qquad X = \mathcal{L}/(S^1)^{r+k_e+k_f}, \qquad (*)$$

with coefficients in the group Z^1 of converging (near the origin) power series in n variables with real coefficients, vanishing at the origin ($\mu_{\mathcal{L}_1}$ is an obstruction for a symplectic extendability of $\Phi_1|_{U(\mathcal{O}_1)}$ to $U(\mathcal{L}_1)$, see below). We call $\mu_{\mathcal{L}_1}$ the Lagrange-Vey class of the singularity at $\mathcal{L}_1$ (w.r.t. the reference singularity at $\mathcal{L} = \mathcal{L}_0$).



For each directed edge $e \approx (-1, 1)$ of the reduced fiber $X = \mathcal{L}/(S^1)^{r+k_e+k_f}$, consider the corresponding orbit $\mathcal{O}_{e,i} \subset \mathcal{L}_i$, $i = 0, 1$, its neighbourhood $U(\mathcal{O}_{e,i}) \approx (-1, 1) \times B_e$, and two fibrewise symplectomorphisms $\Phi_{e,\pm} : U(\mathcal{O}_{e,1}) \to U(\mathcal{O}_{e,0})$ coinciding with Φ_1 on $(\pm(1-\varepsilon), \pm 1) \times B_e \subseteq U(\mathcal{O}_{e,1}) \cap U(\partial_\pm \mathcal{O}_{e,1})$.

Assign $e \mapsto \Phi_{e,+} \circ \Phi_{e,-}^{-1} = \phi^1_{S_e \circ \phi \circ F}$, F-preserving symplectomorphism $\Rightarrow$ the time-1 map of the Hamiltonian flow generated by a first integral (always having the form $S_e \circ \phi \circ F$, for some function S_e in n variables, $S_e(0) = 0$). Here $\phi \circ F$ is a Vey momentum map at $\mathcal{O}_0 = \Phi_1(\mathcal{O}_1) \subset \mathcal{L}$. Define 1-cochain $\mu_{\mathcal{L}_1} := [e \mapsto S_e]$. **Lemma:** $\mu_{\mathcal{L}_1}$ is a cocycle!

Theorem 2 (Semilocal symplectic classification)

Suppose (M_k, ω_k, F_k), $k = 1, 2$, *are two real-analytic integrable Hamiltonian systems, and* $\mathcal{L}_k \subset M_k$ *a compact non-degenerate singular fiber satisfying the connectedness condition. Suppose* $\Phi_k : U(\mathcal{L}_k) \to U(\mathcal{L})$ *is a fiberwise homeomorphism being fibrewise isotopic to a real-analytic symplectomorphism on* $U(\mathcal{O}_k)$, $\mathcal{O}_k \subset \mathcal{L}_k$, $\Phi_k(\mathcal{L}_k) = \mathcal{L}$. *Then*

- *The fiberwise homeomorphism* $\Phi = \Phi_2^{-1} \circ \Phi_1 : U(\mathcal{L}_1) \to U(\mathcal{L}_2)$ *can be deformed to a real-analytic symplectomorphism in the space of fiberwise homeomorphisms if and only if* $\mu_{\mathcal{L}_1} = \mu_{\mathcal{L}_2}$.
- *For each 1-cocycle* $\mu \in H^1(X, Z^1)$, $\exists$ *a semilocal singularity* $\mathcal{L}_1$ *s.t.* $\mu_{\mathcal{L}_1} = \mu$.

Theorem 3 (On action variables)

Under the hypothesis of Theorem 2, suppose there exists a real-analytic diffeomorphism germ $\phi : (\mathbb{R}^n, F_1(\mathcal{L}_1)) \to (\mathbb{R}^n, F_2(\mathcal{L}_2))$ *such that* $\phi \circ F_1 = F_2 \circ \Phi$. *Suppose there exists a finite-sheeted covering of a neighbourhood of* $\mathcal{L}_1$ *that is fibrewise homeomorphic to the direct product of several regular, elliptic, hyperbolic and focus-focus semilocal singularities of dimensions* 2, 2, 2 *and* 4, *resp., where all hyperbolic components (2-atoms) have genus 0. Then the fiberwise homeomorphism* $\Phi : U(\mathcal{L}_1) \to U(\mathcal{L}_2)$ *can be deformed to a real-analytic symplectomorphism in the space of fiberwise homeomorphisms if and only if the map* $\hat{\phi}$ *preserves action variables.*