Lyapunov center theorem for Hamiltonian systems with symmetries

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Plan

D. Strzelecki.
Periodic solutions of symmetric Hamiltonian systems.
Arch. Rational Mech. Anal., 237:921–950, 2020.

Symmetries in differential equations

Assume that Γ acts on $\mathbb{R}^n$ by

$$\Gamma \times \mathbb{R}^n \ni (\gamma, z) \to \rho(\gamma)z \in \mathbb{R}^n,$$

where $\rho : \Gamma \to O(n) \subset GL(n)$ is a group homomorphism. $\gamma z := \rho(\gamma)z$.

The orbit of z_0: $\Gamma(z_0) = \{\gamma z_0 : \gamma \in \Gamma\}$
The stabilizer of z_0: $\Gamma_{z_0} = \{\gamma \in \Gamma : \gamma z_0 = z_0\}$
Invariant map: $f : \mathbb{R}^n \to \mathbb{R}$, $f(\gamma z) = f(z)$.
Equivariant map: $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(\gamma z) = \gamma h(z)$.

$$\dot{z}(t) = h(z(t)) \tag{1}$$

$$\dot{z}(t) = \nabla f(z(t))$$

Remark

If $z(t)$ is a solution of (1) then $\gamma z(t)$ is a solution for any $\gamma \in \Gamma$.

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What's the problem with symmetries?

$$\dot{x}(t) = h(x(t))$$

Noether's theorem - positive impact of symmetries

Every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

The problem

If $h(z_0) = 0$, then $\Gamma(z_0) \subset h^{-1}(0)$. Critical points are not isolated in general!

$$\Gamma(z_0) \approx \Gamma / \Gamma_{z_0}$$

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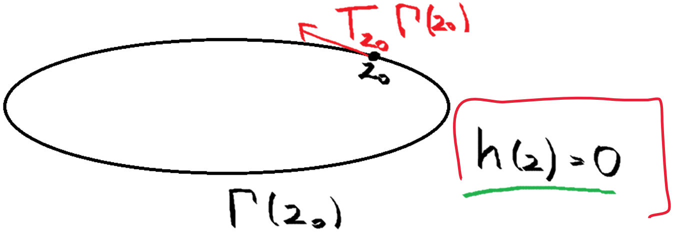

$$T_{z_0}\Gamma(z_0) \subset \ker h'(z_0)$$

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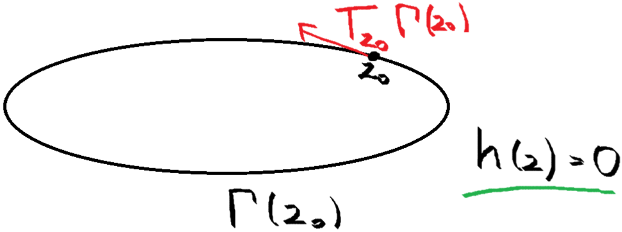

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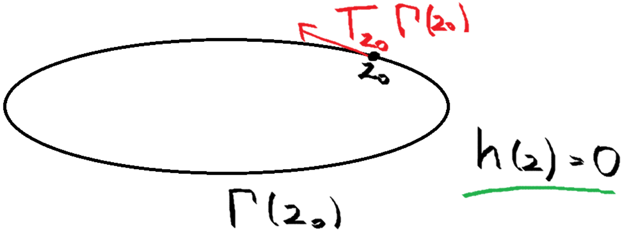

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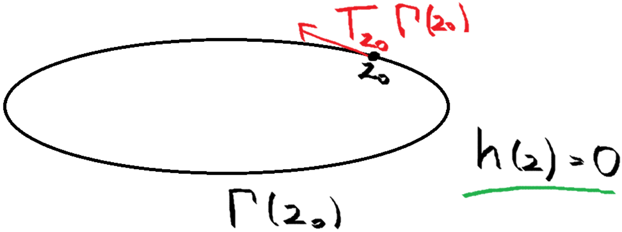

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det$(h'(z_0)) \neq 0$,

Hamiltonian systems with symmetries

We assume that Γ acts unitary on $\mathbb{R}^{2N}$ ($\rho : \Gamma \to U(N) = Sp(2n) \cap O(2n)$).

$$\dot{z}(t) = J\nabla H(z(t)), \tag{2}$$

where $J = \begin{bmatrix} 0 & Id_N \\ -Id_N & 0 \end{bmatrix}$, $z : \mathbb{R} \to \mathbb{R}^{2N}$ and $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ is Γ-invariant. $\nabla H(z_0) = 0$.

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The goal

Prove the existence of periodic solutions of the system (2) in a neighborhood of stationary solution z_0.

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Motivation - Liapunov center theorem

$$\dot{z}(t) = J\nabla H(z(t))$$

$D^2H(0) > 0$

Theorem

Let $H \in C^2(\mathbb{R}^{2N}, \mathbb{R}), \nabla H(0) = 0$. If $\sigma(J\nabla^2 H(0)) = \{\pm i\beta_1, \ldots, \pm i\beta_m\}$ for $\beta_j \geq 0$ then for β_{j_0} satisfying $\beta_j/\beta_{j_0} \notin \mathbb{N}$ for $j \neq j_0$ there is a smooth two-dimensional manifold passing through 0 and intersecting each energy level near 0 in a periodic trajectory with minimal period near $2\pi/\beta_{j_0}$.

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Challenges

- Study $z_0 \in (\nabla H)^{-1}(0)$ such that $\dim \Gamma_{z_0} < \dim \Gamma$. Then the orbit $\Gamma(z_0)$ is at least one dimensional manifold and **critical points** from this orbit **are not isolated**.
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Plan

1. Problem
2. Lypunov center theorem with symmetries
3. The ideas of proof
4. Applications

General assumptions

$$\dot{z}(t) = J\nabla H(z(t)) \tag{3}$$

(A1) $H : \mathbb{R}^{2N} \to \mathbb{R}$ is a Γ-invariant Hamiltonian of the class C^2,

(A2) $z_0 \in \mathbb{R}^{2N}$ is a critical point of H such that the isotropy group Γ_{z_0} is trivial,

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(A4) $\pm i\beta_1, \ldots, \pm i\beta_m$, $0 < \beta_m < \ldots < \beta_1$, $m \geq 1$ are the purely imaginary eigenvalues of $J\nabla^2 H(z_0)$,

(A5) $\deg(\nabla H_{|T^{\perp}_{z_0}\Gamma(z_0)}, B(z_0, \epsilon), 0) \neq 0$ for sufficiently small ϵ,

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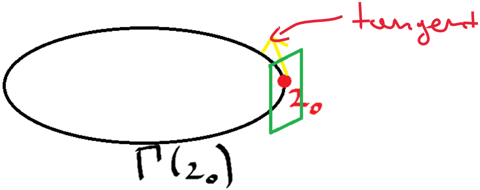

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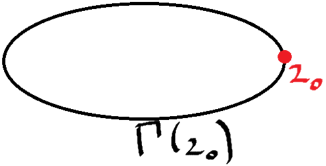
The most general theorem

Theorem 1.

Under the assumptions **(A1)–(A7)** there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 (with amplitude tending to 0) such that minimal periods of solutions in a small neighborhood of z_0 are close to $2\pi/\beta_{j_0}$.



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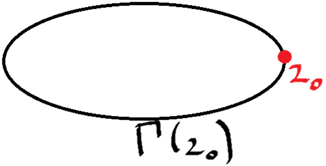

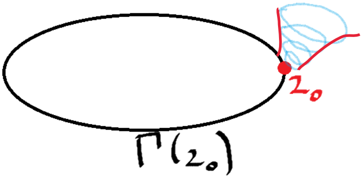

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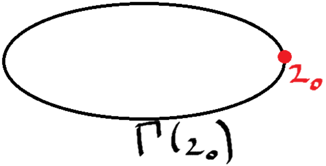

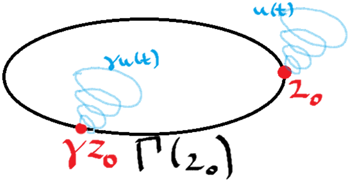

Modifications

(A5) Brouwer degree $\deg_B(\nabla\left(H_{|T_{z_0}^{\perp}\Gamma(z_0)}\right), B^d(z_0,\epsilon), 0) \neq 0$ for sufficiently small ϵ and $d = \dim T_{z_0}^{\perp}\Gamma(z_0)$.

Fact

The assumption **(A5)** is satisfied for non-degenerate orbit i.e. under the assumption $\dim \Gamma(z_0) = \dim \ker \nabla^2 H(z_0)$.

(A7) $m^-\left(\begin{bmatrix} -\lambda\nabla^2 H(z_0) & -J \\ J & -\lambda\nabla^2 H(z_0)\end{bmatrix}\right)$ changes at $\lambda = \frac{1}{\beta_{j_0}}$ when λ varies.

Fact

The assumption **(A7)** is implied by the condition
(A7') $\quad m^+(\nabla^2 H(z_0)) > N \quad$ or $\quad m^-(\nabla^2 H(z_0)) > N$.
Moreover, under **(A7')** the assumption **(A4)** is satisfied.

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Moreover, under **(A7')** the assumption **(A4)** is satisfied.

Modifications

(A5) Brouwer degree $\deg_B(\nabla \left(H_{|T_{z_0}^{\perp}\Gamma(z_0)} \right), B^d(z_0, \epsilon), 0) \neq 0$ for sufficiently small ϵ and $d = \dim T_{z_0}^{\perp}\Gamma(z_0)$.

Fact

The assumption **(A5)** is satisfied for non-degenerate orbit i.e. under the assumption $\dim \Gamma(z_0) \overset{\leq}{=} \dim \ker \nabla^2 H(z_0)$.

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$\mathbb{R}^{2N}$

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Modifications

Theorem 2.

Under the assumptions **(A1), (A2), (A3), (A5)** and **(A7')** there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 with periods (not necessarily minimal) close to $2\pi/\beta_j$, where $i\beta_j$, $\beta_j > 0$ is some eigenvalue of $J\nabla^2 H(z_0)$.

Newtonian systems

$$\ddot{q}(t) = -\nabla U(q(t)) \quad \text{(UN)}$$

$$H(p,q) = \frac{1}{2}||p||^2 + U(q)$$

$\gamma(p,q) = (\gamma p, \gamma q)$ unitary action

$$z_0 = (p_0, q_0)$$

- $\pm i\beta \in \sigma(J\nabla^2 H(z_0)) \Leftrightarrow \beta^2 \in \sigma(\nabla^2 U(q_0))$
- the assumption **(A7)** is always satisfied!
- what about **(A5)**?
  - non-degenerate orbit
  - q_0 is a minimum of the potential U

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Plan

Variational problem

Periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$

$$\Updownarrow$$

z_0

Variational problem

Periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$

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2π-periodic solutions of the parameterized system $\dot{z}(t) = \lambda J\nabla H(z(t))$

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Variational problem

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WLOG we assume a growth condition $|\nabla H(z)| \leq a_1 + a_2|z|^s$ for some $a_1,\ a_2 > 0$ and $s \in [1, \infty)$.

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$\Gamma \times S^1 \ni (\gamma, e^{i\theta})\, z(t) \longrightarrow \gamma z(t+\theta)$

Fakt

The space $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N})$ is an orthogonal representation of the group $G = \Gamma \times S^1$. Φ is G-invariant.

$$z_0 \quad \to \quad \{z_0\} \times (0, \infty) \quad \to \quad G(z_0) \times (0, \infty)$$

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Bifurcation problem

$$\nabla_z \Phi(z, \lambda) = 0 \qquad \text{(ZB)}$$

Trivial set of solutions

$$\mathcal{T} = G(z_0) \times (0, \infty)$$

Let $\lambda_0 \in \{\frac{k}{\beta_j} : k \in \mathbb{N}\}$.

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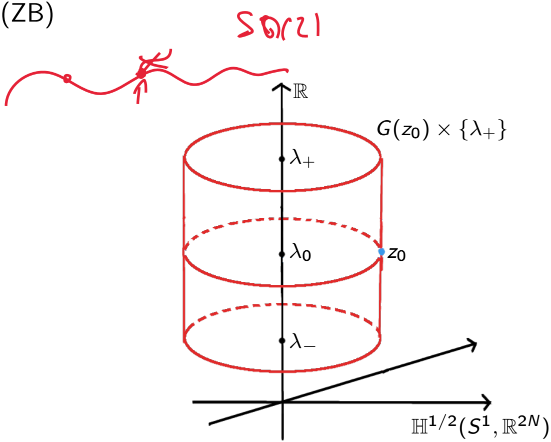

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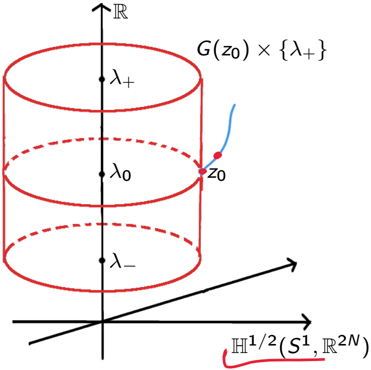

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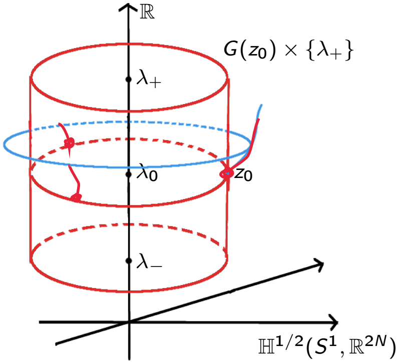

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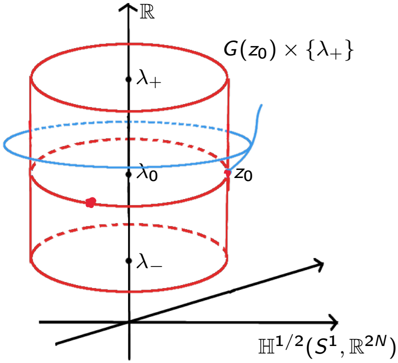
We use bifurcation theorem for degree for gradient equivariant map (Gołębiewska and Rybicki, 2011).



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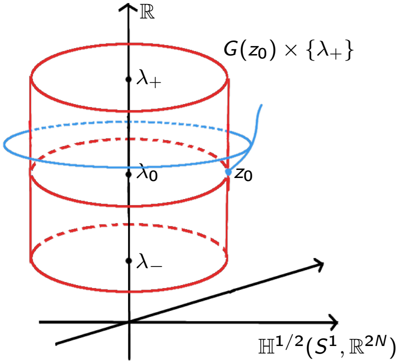
$$\nabla_G - deg(\nabla\Phi, \mathcal{O}) = \Upsilon_G(\mathcal{CJ}_G(\mathcal{O}, -\nabla\Phi)).$$



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GS - G-homotopy types of G-spectra

$U(G)$ - Euler ring of a compact Lie group G

$\mathcal{CI}_G \in GS$ (Izydorek, 2002) for compact pert. of linear

$\Upsilon_G : GS \to U(G)$
(Gołębiewska i Rybicki, 2013)

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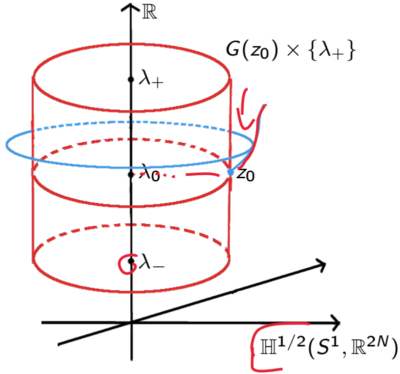

$$\Upsilon_G\left(\mathcal{CI}_G\left(G(z_0), -\nabla\Phi(\cdot, \lambda_+)\right)\right) \neq \Upsilon_G\left(\mathcal{CI}_G\left(G(z_0), -\nabla\Phi(\cdot, \lambda_-)\right)\right).$$

The end of a proof

- We prove the bifurcation of solutions of $\nabla_z \Phi(z, \lambda) = 0$ on the function space $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N})$ from any point of the orbit $G(z_0) \times \{\lambda_0\}$.
- These solutions solve the original problem $\dot{z}(t) = J\nabla H(z(t))$ with periods close to $2\pi\lambda_0$.
- The period $2\pi\lambda_0$ is minimal since there is no bifurcation on the level λ_0/k for any $k \in \mathbb{N}$.

$\frac{2\pi\lambda_0}{3}$

Let $z(t) \times \{\lambda\}$ be a solution close to $z_0 \times \{\lambda_0\}$ in $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N}) \times \mathbb{R}$.
The map $z(t) \to \nabla H(z(t))$ is continuous from $L^2(S^1, \mathbb{R}^{2N})$ to $L^2(S^1, \mathbb{R}^{2N})$.
Let $\varepsilon > 0$ and we choose $0 < \delta < \varepsilon$, such that $||z - z_0||_{L^2(S^1,\mathbb{R}^{2N})} \leq ||z - z_0||_{\mathbb{H}^{1/2}(S^1,\mathbb{R}^{2N})} < \delta$ implies

$$||\nabla H(z)||_{L^2(S^1,\mathbb{R}^{2N})} = ||\nabla H(z) - \nabla H(z_0)||_{L^2(S^1,\mathbb{R}^{2N})} < \varepsilon.$$

Then

$$\begin{aligned} ||z - z_0||^2_{L^\infty(S^1,\mathbb{R}^{2N})} &\leq c||z - z_0||^2_{\mathbb{H}^1(S^1,\mathbb{R}^{2N})} = \\ &= c\left(||z - z_0||^2_{L^2(S^1,\mathbb{R}^{2N})} + ||(z - z_0)'||^2_{L^2(S^1,\mathbb{R}^{2N})}\right) = \\ &= c\left(||z - z_0||^2_{L^2(S^1,\mathbb{R}^{2N})} + ||\lambda J \nabla H(z)||^2_{L^2(S^1,\mathbb{R}^{2N})}\right) \leq \\ &\leq c(1 + (\lambda_0 + \theta)^2)\varepsilon^2. \end{aligned}$$

Let $z(t) \times \{\lambda\}$ be a solution close to $z_0 \times \{\lambda_0\}$ in $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N}) \times \mathbb{R}$.
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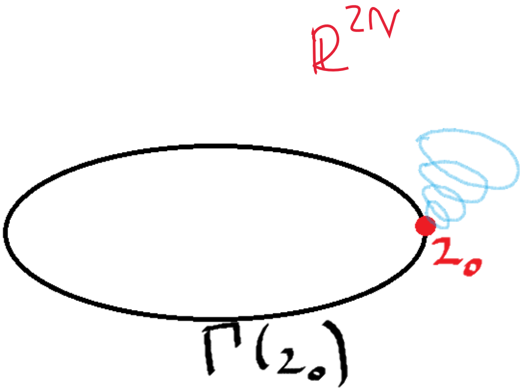

R^{2N}
z₀
Γ(z₀)

Plan

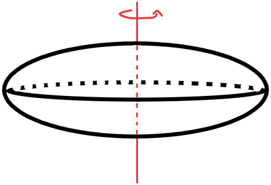
1. Problem
2. Lypunov center theorem with symmetries
3. The ideas of proof
4. Applications

Quasi-periodic motions close to geostationary orbit

We study motions near the geostationary orbit of an oblate spheroid.

$$U_G(r,\theta) = -G\frac{E}{r}\left(1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right),$$

E - mass, R - equatorial radius, P_n - n-th Legendre polynomial, J_n - coefficients ($J_2 = 1.0826359 \cdot 10^{-3}$ for the Earth).



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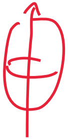


E - mass, R - equatorial radius, P_n - n-th Legendre polynomial, J_n - coefficients ($J_2 = 1.0826359 \cdot 10^{-3}$ for the Earth).
Approximation:

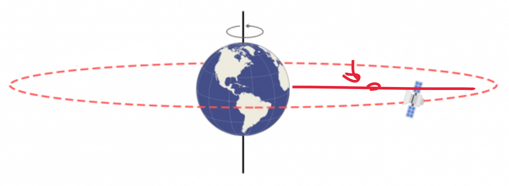
$$U(r,\theta) = -\frac{GE}{r}\left(1 - \frac{J_2 R^2}{r^2} P_2(\cos\theta)\right).$$

By the change of coordinates ($c = \frac{1}{2}R^2 J_2 > 0$, $d = \sqrt{r^2 + z^2}$):

$$V(r,z) = -\frac{1}{d}\left(1 - \frac{c}{d^2}\left(3\frac{z^2}{d^2} - 1\right)\right) = -\frac{1}{d} - \frac{c}{d^3} + \frac{3cz^2}{d^5},$$

Fact

Any oblate spheroid possesses exactly one geostationary orbit in a plane perpendicular to rotation axis.



https://www.everythingrf.com/community/what-is-a-geostationary-orbit

ω - angular velocity of a particle on geostationary orbit

Hamiltonian reformulation

$$\mathcal{H}(q_1,q_2,q_3,p_1,p_2,p_3)=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+\omega(q_1p_2-q_2p_1)-\frac{1}{d}-\frac{c}{d^3}+\frac{3cq_3^2}{d^5},$$

where $c=\frac{1}{2}R^2J_2>0,\ d=\sqrt{q_1^2+q_2^2+q_3^2},\ z_0=(d_0,0,0,0,-\omega d_0,0).$

(A1) $H:\mathbb{R}^{2N}\to\mathbb{R}$ is a Γ-invariant Hamiltonian of the class C^2,

(A2) $z_0\in\mathbb{R}^{2N}$ is a critical point of H such that the isotropy group Γ_{z_0} is trivial,

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(A7') $m^+(\nabla^2H(z_0))>N$ lub $m^-(\nabla^2H(z_0))>N$.

$$m^+(\nabla^2\mathcal{H}(z_0))=4,\ m^-(\nabla^2\mathcal{H}(z_0))=1,\ \dim\ker\nabla^2\mathcal{H}(z_0)=1$$

Hamiltonian reformulation

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where $c = \frac{1}{2}R^2 J_2 > 0$, $d = \sqrt{q_1^2 + q_2^2 + q_3^2}$, $z_0 = (d_0, 0, 0, 0, -\omega d_0, 0)$.

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Hamiltonian reformulation

N=3

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SO(2)

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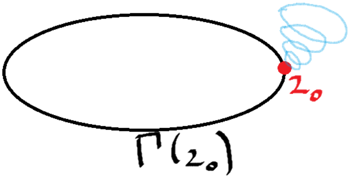
.

There exist a family of periodic solutions with trajectories arbitrarily close to z_0 (in the rotating frame!)



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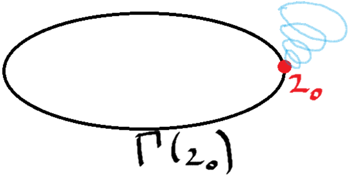

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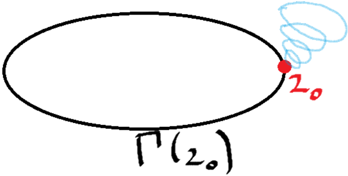
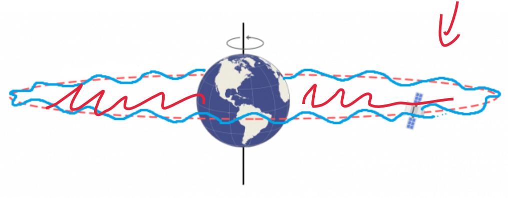


In the original problem there are quasi-periodic, solutions with trajectories arbitrarily close to geostationary orbit.

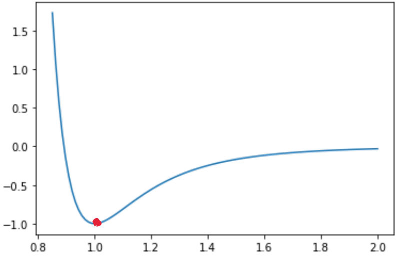


Lennard-Jones N-body problem

$$\ddot{q}(t) = -\nabla U(q(t))$$

$$U(q) = \sum_{1 \le i < j \le N} \left(\frac{1}{\mid q_i - q_j \mid^{12}} - \frac{2}{\mid q_i - q_j \mid^{6}} \right),$$

where $q = (q_1, q_2, \ldots, q_N) \in (\mathbb{R}^2)^N$ and $q_i \in \mathbb{R}^2$ is a position of the i-th particle.



$\frac{1}{r^{12}} - \frac{2}{r^{6}}$

$N = 2$ i $N = 3$

$$U(q_1, q_2) = \frac{1}{\mid q_1 - q_2 \mid^{12}} - \frac{2}{\mid q_1 - q_2 \mid^{6}}$$

M. Corbera, J. Llibre, and E. Pérez-Chavela. Equilibrium points and central configurations for the Lennard-Jones 2- and 3-body problems. Celestial Mech. Dynam. Astronom., 89(3):235–266, 2004.

- families of equilibria
- relative equilibria

Periodic solutions

- $N = 2$: the existence of non-stationary periodic solutions emanating from equilibrium $(0, \frac{1}{2}, 0, -\frac{1}{2})$ with minimal periods close $\pi/6$.
- $N = 3$: the existence of two families emanating from triangle equilibrium, minimal periods: $\pi/(3\sqrt{3})$ and $\pi/(3\sqrt{6})$.

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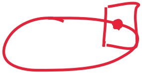

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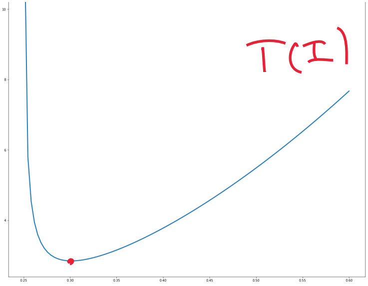
Quasi-periodic solutions, $N = 2$

Twierdzenie, Corbera, Llibre, Pérez-Chavela, 2004

For any $I \in \left(\frac{1}{4}, \infty\right)$ the set

$$CC = \{(q_1, q_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ q_2 = -q_1, |q_1 - q_2| = 2\sqrt{I}\}$$

is a family of central configurations generating periodic solutions with period $T(I)$.



Quasi-periodic solutions, $N = 2$

Rotating frame $\rightarrow$ $SO(2)$-equivariant Hamiltonian system with equilibria.

Theorem

For $l \in \left(\frac{1}{4}, \frac{1}{4}\sqrt[3]{\frac{5}{2}}\right) \setminus \left\{\frac{1}{4}\sqrt[3]{\frac{7}{4}}\right\}$ there exists a family of quasi-periodic solutions of the Lennard-Jones 2-body problem emanating from relative equilibrium.

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D. Strzelecki.
Periodic solutions of symmetric Hamiltonian systems.
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Bifurcations of quasi-periodic solutions from relative equlibria in the Lennard-Jones 2-body problem.
Celestial Mech. Dynam. Astronom, art. 44, 2021.

Thank you for your attention!

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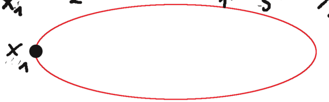
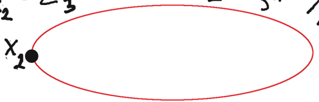
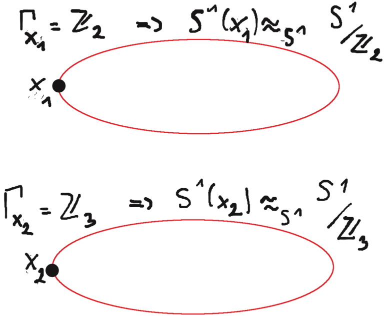
$$\Gamma_{x_1} = \mathbb{Z}_2 \Rightarrow S^1(x_1) \approx_{S^1} S^1/\mathbb{Z}_2$$



$$\Gamma_{x_2} = \mathbb{Z}_3 \Rightarrow S^1(x_2) \approx_{S^1} S^1/\mathbb{Z}_3$$





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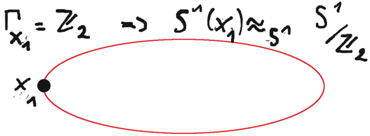

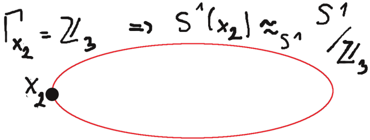

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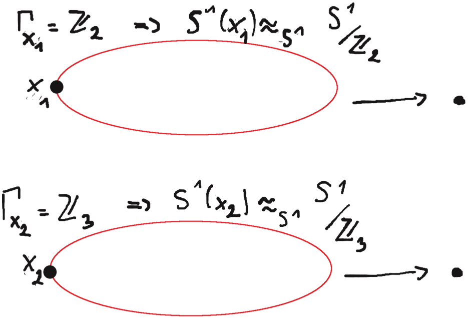

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$$\{*_1\} \approx \{*_2\}$$

Julian I. Palmore

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Uwaga 1

Dla zwartej grupy Liego G oraz gładkiej rozmaitości M, jeżeli działanie grupy G jest **wolne**, to przestrzeń orbit M/G jest gładką rozmaitością.

Uwaga 2

Jeżeli działanie zwartej grupy Liego nie jest wolne, to spójne składowe przestrzeni orbit M/G tworzą stratyfikację Whitney'a.

J. J. Duistermaat, J. A. C. Kolk, *Lie Groups*, Springer-Verlag, Berlin Heidelberg, 2000.