Flexibility and rigidity aspects of the dynamics of the steady Euler flows

Daniel Peralta-Salas

Instituto de Ciencias Matemáticas (ICMAT), Madrid

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Ideal fluids on Riemannian manifolds

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$$\Rightarrow \; L_u d\alpha = 0 \,, \qquad \operatorname{div} u = 0 \,, \qquad \text{(Helmholtz's transport of vorticity)}$$

Eulerisable flows

Definition

A volume-preserving vector field u on M is Eulerisable if there exists a Riemannian metric g on M such that u is a steady Euler flow on (M, g). The vector field $\omega :=$ curl u is the vorticity, and is defined as $i_\omega \mu = d\alpha$.

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The geometric wealth of the 3D Eulerisable flows has been unveiled recently:

- Non-vanishing Beltramisable fields with constant factor $\Longleftrightarrow$ Reeb flows of a contact structure (Sullivan and Etnyre & Ghrist).
- Non-vanishing Beltramisable fields with nonconstant factor $\Longleftrightarrow$ volume-preserving geodesible flows (Rechtman).
- Eulerisable flows with nonconstant Bernoulli function are not geodesible in general (Cieliebak & Volkov).

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The vorticity

When $\dim M = 2n+1$, the vorticity is the vector field defined as $i_\omega \mu = (d\alpha)^n$, where μ is the Riemannian volume-form. If $\dim M = 2n$, the vorticity is the scalar function $\frac{(d\alpha)^n}{\mu}$. In both cases $L_u \omega = 0$, i.e., it is a volume-preserving vector field that commutes with u in odd dimensions, or a first integral in even dimensions (this leads to a connection with integrable systems developed by Ginzburg & Khesin). In any dimension a volume-preserving geodesible field is Eulerisable.

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Warning: We can define a Beltrami field in dimension $2n+1$ as $\omega = fu$ for some function f, and $\operatorname{div} u = 0$. However, when $n > 1$, the field u does not need to be Eulerisable (R. Cardona).

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Remark

One expects a certain flexibility in Eulerisable fields because the metric is not fixed. If the metric is fixed (e.g. the round metric on $\mathbb{S}^n$ or the flat metric on $\mathbb{R}^n$), all these questions become much harder: rigidity.

A Sullivan type characterization of Euler

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Reminder

On compact manifolds, the space of p-currents $\mathcal{Z}^p$ is the continuous dual of the space of smooth p-forms on M. The cone (with compact convex base) of foliation currents $\mathcal{Z}_X$ of X is the set of 1-currents that can be approximated arbitrarily well by tangent 1-chains. The cone of foliation cycles $\mathcal{C}_X$ of X is the set of closed foliations currents (in 1-1 correspondence with the invariant measures of X).

A Sullivan type characterization of Euler (II)

An observation

If u is an Eulerisable flow on a 3-manifold M, then its vorticity ω commutes with u, and by Stokes theorem we have that $\int_L u dl = \int_S \omega \cdot \nu d\sigma$, where the curve L is the boundary of the surface S, $\partial S = L$. Accordingly, if the circulation of u along L is positive, the flux of ω through S cannot be zero.

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This motivates to introduce the following definition:

$$\mathcal{F}_X := \{\partial c : \ c \text{ is a 2-chain with } \int_c i_X \mu = 0\}$$

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Theorem (P-S, Rechtman & Torres de Lizaur, 2019)

Let u be a non-vanishing volume-preserving vector field on a compact 3-manifold M with trivial first cohomology group. Then u is Eulerisable if and only if there exists a (non-identically zero) vector field ω that commutes with u, i.e. $[u, \omega] = 0$, such that $\overline{\mathcal{F}_\omega} \cap \mathcal{C}_u = \{0\}$.

Sketch of the proof

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Step 1: The assumption implies that the compact convex base K of $\mathcal{Z}_u$ satisfies $\overline{\mathcal{F}_\omega} \cap K = \varnothing$. Since $\overline{\mathcal{F}_\omega}$ is a closed vector subspace of $\mathcal{Z}^1$, it follows from Hahn-Banach that there exists a 1-form α such that $\alpha(u) > 0$ and $c(d\alpha) = 0$ for any 2-current c with $\partial c \in \overline{\mathcal{F}_\omega}$ (in particular, $i_\omega d\alpha = 0$).

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Step 2: Assume that $d\alpha = T i_\omega \mu$ for some constant $T \neq 0$. A few computations show that this condition and the fact that $[u, \omega] = 0$, imply that $i_u d\alpha$ is closed (and hence exact because $H^1(M; \mathbb{R}) = 0$).

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Step 3: To prove the previous condition, first notice that α cannot be closed because it is nondegenerate (and $H^1(M; \mathbb{R}) = 0$). Consider now the 1-dimensional subspace $\mathcal{Y}$ of 2-forms proportional to $i_\omega \mu$. If $d\alpha \notin \mathcal{Y}$, the Hahn-Banach theorem implies that there is $c \in \mathcal{Z}^2$ such that $c(d\alpha) > 0$ and $c(i_\omega \mu) = 0$. It can then be argued that $\partial c \in \overline{\mathcal{F}_\omega}$, so $\partial c(\alpha) = c(d\alpha) = 0$, thus contradicting that $c(d\alpha) > 0$.

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Definition of plug

A plug is (usually) a cylinder $C := \mathbb{D} \times [0, 1]$ together with a vector field X with the following dynamics:

- X is equal to the vertical field ∂_t on ∂C, $t \in [0, 1]$.
- X has at least one trapped orbit: there is a point p in $\mathbb{D} \times \{0\}$ whose orbit $\Gamma_p(t)$ remains always in the interior of C for all $t > 0$.
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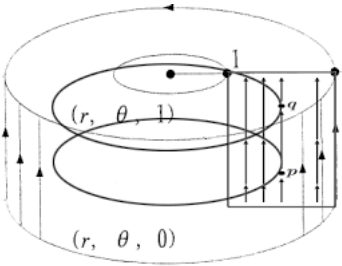
Open: are there C^∞ volume-preserving plugs without periodic orbits?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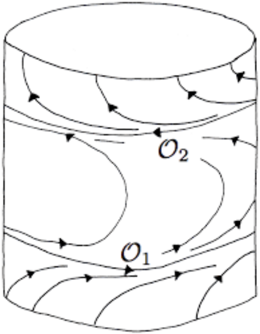
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Theorem (P-S, Rechtman & Torres de Lizaur, 2019)

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Idea of the proof: First, using Sullivan's theorem, we show that the geodesible flows cannot exhibit plugs. Then, if u is an Eulerisable non-geodesible flow exhibiting a plug, we prove that there exists a 2-current A such that ∂A is a foliation cycle of u and

$$\int_A i_\omega \mu = 0$$

where $\omega = \operatorname{curl} u$ is the vorticity (which commutes with u). Therefore, there is a non-trivial element in $\overline{\mathcal{F}_\omega} \cap \mathcal{C}_u$, which contradicts the characterization theorem.

Obstructions: Euler flows cannot exhibit plugs (III)

Corollary

On any 3-manifold endowed with a volume-form μ, there exists an L^2-dense set of (C^∞) volume-preserving fields that are not homeomorphic to Eulerisable flows.

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Proof: Inserting (C^∞) volume-preserving Wilson plugs, we obtain flows that are not Eulerisable. The plugs can be as small as desired, so the change in the L^2 norm is as small as one wishes.

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Remark

The fact that the Eulerisable flows cannot exhibit plugs holds on any dimension $n \geqslant 3$ (of course, in dimension $n = 2$ a plug cannot be inserted into a non vanishing vector field).

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Problem: How is the space of steady solutions on $(\mathbb{S}^3, g_0)$? In other words, can we characterize the volume-preserving fields on $\mathbb{S}^3$ that are Eulerisable with the round metric, up to a volume-preserving diffeomorphism?

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Etnyre & Ghrist proved that there exists an Eulerisable flow on $\mathbb{S}^3$ (of Beltrami type) realizing all knot and link types at the same time. The corresponding metric is not the round one. The conjecture holds true in Euclidean space (Enciso & P-S, 2012).

Beltrami fields on the round sphere

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Theorem (Enciso, P-S & Torres de Lizaur, 2017)

Let L be a finite link in $\mathbb{S}^3$. Then for any large enough integer $|\lambda|$, there exists a λ-eigenvalue Beltrami field u on $\mathbb{S}^3$ with a collection of closed integral curves that is isotopic to L.

Beltrami fields on the round sphere (II)

This flexibility actually occurs at small scales: the set of integral curves isotopic to L is contained in a ball of size $|\lambda|^{-1}$.

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Theorem (P-S & Slobodeanu, 2020)

Let u be a non-vanishing Beltrami field on $(\mathbb{S}^3, g_0)$ of eigenvalue λ. Then $|\lambda|$ is even. Conversely, for each even $\lambda = 2m$, $m \geqslant 1$, there exists a non-vanishing λ-eigenfield, and the associated contact structure is tight if $m = 1$ and overtwisted (OT) otherwise. This realizes only two homotopy classes of OT contact structures, depending on whether m is even or odd. In the case of odd m the class is the trivial (tight) one.

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Remark: These Beltrami fields are explicit (they can be expressed in terms of Jacobi polynomials in Hopf coordinates).

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Remark: Etnyre, Komendarczyk and Massot (2012) showed that the round metric cannot be compatible with an OT contact structure (compatible $\Longrightarrow$ $|u|$ =const). The theorem above shows that it can be weakly compatible.

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Example (Khesin, Kuksin & P-S, 2014)

$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ can be endowed with Hopf coordinates $(z_1, z_2) = (\cos s \exp i\phi_1, \sin s \exp i\phi_2)$, $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$. If $R := \partial_{\phi_1} + \partial_{\phi_2}$ and $R' := \partial_{\phi_1} - \partial_{\phi_2}$ are the Reeb fields of the standard (resp. anti-standard) contact form on $\mathbb{S}^3$, the following field is a steady Euler flow:

$$u = F(\cos^2 s)R + G(\cos^2 s)R'$$

for any smooth functions F and G. The Bernoulli function $B \equiv B(\cos^2 s)$ is not generally a constant.

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Choosing $F = p$ and $G = q$, p, q coprime integers, this family realizes all the Seifert foliations of $\mathbb{S}^3$ (foliations by circles). On the other hand, if p, q are rationally independent real numbers, all the orbits of u are quasi-periodic, except for the Hopf link (a couple of periodic orbits).

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Open problem

An analytic (C^ω) steady Euler flow on the round sphere with nonconstant Bernoulli function admits a Killing symmetry (Riemannian version of Grad's conjecture, 1967).

Thanks a lot for your attention!