When escape trajectories become singular periodic orbits

GDM Seminar

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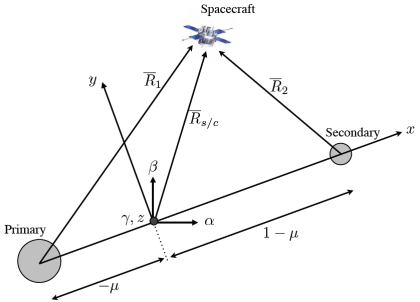
Restricted planar circular 3-body problem

Simplified version of the general 3-body problem:

- One of the bodies has negligible mass.

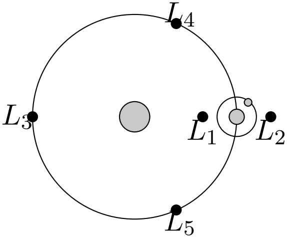
  The other two bodies move in circles following Kepler's laws for the 2-body problem.

  The motion of the small body is in the same plane.



Restricted planar circular 3-body problem

- Time-dependent potential: $U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
  Time-dependent Hamiltonian:
  $H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in \mathbf{R}^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$
  Rotating coordinates $\rightsquigarrow$ Time independent Hamiltonian
  $H(q,p) = \frac{p^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$



H has 5 critical points: L_i Lagrange points $(H(L_1) \leq \cdots \leq H(L_5))$
Periodic orbits of X_H? Perturbative methods (dynamical systems) or.... contact geometry!

Level-sets of Hamiltonians

- $\Sigma \subset (W, \omega)$ be a hypersurface.
  X Liouville vector field $(\mathcal{L}_X \omega = \omega) \pitchfork \Sigma \rightsquigarrow (\Sigma, \alpha = \iota_X \omega)$ is contact.
  If $\Sigma = H^{-1}(c)$, then $R_\alpha \cong X_H|_{H=c}$.

Weinstein conjecture

Let (M, α) closed contact manifold. Then R_α admits periodic orbits.

Contact Geometry of the RC3BP

- For $c < H(L_1)$, $\Sigma_c = H^{-1}(c)$ has 3 connected components: Σ_c^E (the satellite stays close to the earth), Σ_c^M (to the moon), or it is far away.

Albers–Frauenfelder–Koert–Paternain

For $c < H(L_1)$, $X = (q - q_E)\frac{\partial}{\partial q}$ is transverse to Σ_c^E.

$\rightsquigarrow (\Sigma_c^E, \iota_X \omega)$ is contact.

But Weinstein conjecture does not apply because of non-compactness (collision!)

☹

Moser regularization of the restricted 3-body problem

- Via Moser's regularization Σ_c^E can be compactified to $\overline{\Sigma}_c^E \cong \mathbf{R}P(3)$.
  The Liouville vector field $X = (q - q_E)\frac{\partial}{\partial q}$ extends to the regularization.
  Hence $\overline{\Sigma}_c^E$ is contact.

Theorem (Albers–Frauenfelder–Koert–Paternain)

For any value $c < H(L_1)$, the regularized RPC3BP admits at least one closed orbit with energy c.

☺

But...

- Where are those periodic orbits?

  Maybe on the collision set?

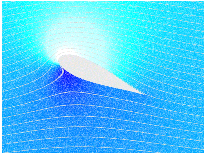
  Keep track of the singularities in the geometric structure?

  ...b^m-symplectic and b^m-contact geometry!

Euler Flows and contact structures

- Euler equations model the dynamics of an **inviscid and incompressible fluid flow**.

$$\begin{cases} \frac{\partial X}{\partial t} + (X \cdot \nabla)X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$



Beltrami fields: $\operatorname{curl} u = fu \quad \operatorname{div} u = 0.$ $\leftrightsquigarrow$ stationary solutions with constant Bernoulli function $B = P + \frac{1}{2} g(u,u)$.

In terms of $\alpha = \iota_X g$, μ volume form **stationary Euler equations** read

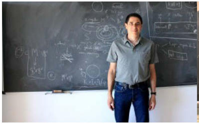
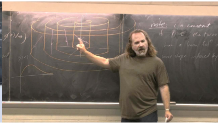
$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$

The contact-Beltrami correspondence

Main trick: If X is non-vanishing rotational Beltrami then $\alpha = \iota_X g$ is contact.

- The Beltrami equation $\leftrightsquigarrow$ $d\alpha = f\iota_X\mu$. Since $f > 0$ and X does not vanish $\rightsquigarrow$ $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.

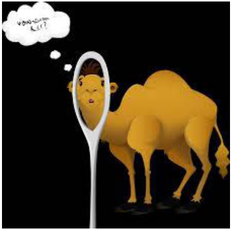
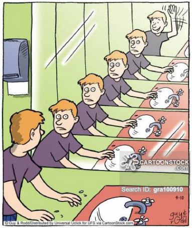
  X satisfies $\iota_X(d\alpha) = \iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$ $\leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.

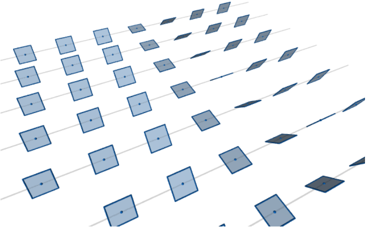





Etnyre-Ghrist: Beltrami fields $\leftrightsquigarrow$ contact structures.

With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.

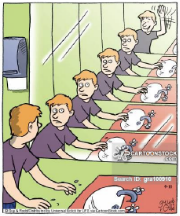
The Symplectic/Contact mirror

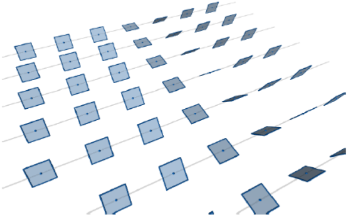





Symplectic	**Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$
	Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H}d\alpha = -dH + R(H)\alpha. \end{cases}$

The Symplectic/Contact mirror "reloaded"





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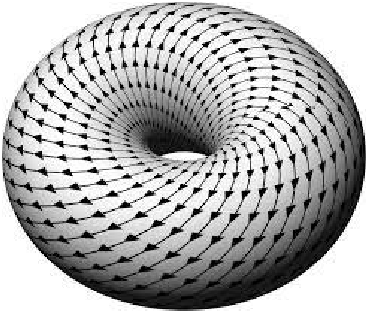
Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



- The b-**cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

A b-**symplectic form** is a closed, nondegenerate, b-form of degree 2.

This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

We can introduce b-**contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.

The b-tangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.

(Singular) symplectic manifolds

b^m-Symplectic

Symplectic

Folded symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

  If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.

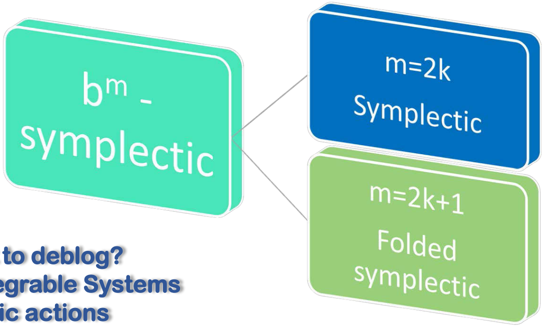
In particular:

- Any b^{2k}-symplectic manifold admits a symplectic structure.

  Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.

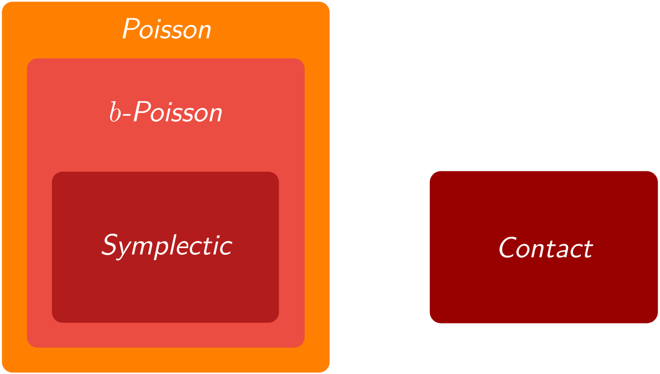
  The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

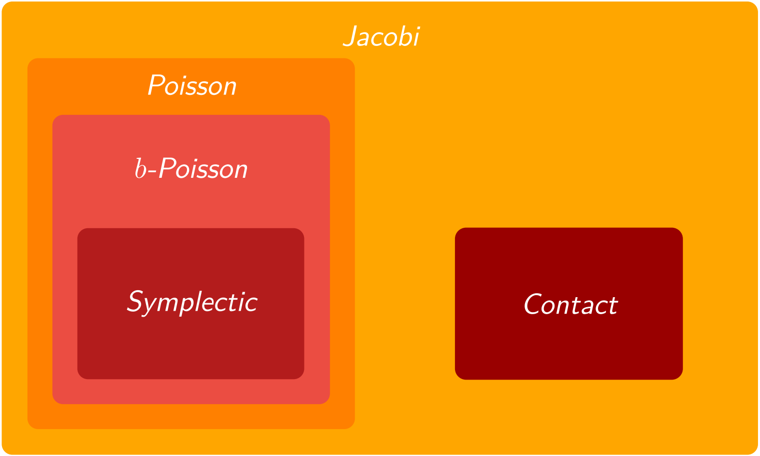
Deblogging everything...



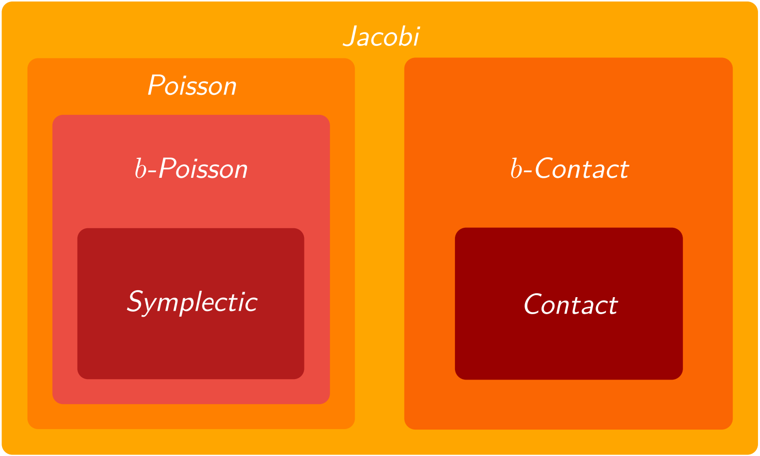
What to deblog?

- **Integrable Systems**
- **Toric actions**
- **Forms..**


Poisson
b-Poisson
Symplectic
Contact


Jacobi
Poisson
b-Poisson
Symplectic
Contact

The big picture



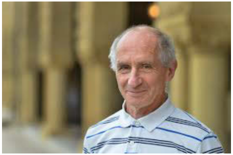
Existence of contact structures

All 3-dimensional manifolds are contact (Martinet-Lutz) in higher dimensions generalized by Casals-Presas-Pancholi in dimension 5 and more generally

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.






Existence of b^{2k}-contact structures

Theorem (Even singularization, M-Oms)

For any pair (M, Z) of contact manifold and convex hypersurface there exists a b^{2k}-contact structure for all k having Z as critical set.

Corollary (of Giroux theorem)

For any 3-dimensional manifold and any generic surface Z, there exists a b^{2k}-contact structure on M realizing Z as the critical set.

What about periodic orbits?

Contact geometry of RPC3BP revisited

In rotating coordinates: $H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

Lemma

The vector field $Y = p\frac{\partial}{\partial p}$ is a Liouville vector field and is transverse to Σ_c for $c > 0$.

- Symplectic polar coordinates: $(r, \alpha, P_r, P_\alpha)$.
  McGehee change of coordinates: $r = \frac{2}{x^2}$.

b^3-symplectic form: $-4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha$.

Singular contact

Is Σ_c b^3-contact after McGehee? Can we apply the results on periodic orbits?

b^3-contact form in the RPC3BP

Theorem

After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y\omega)$ for $c > 0$ are b^3-contact manifolds.

1. *The critical set is a cylinder.* ☹
2. *The Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.* ☺

Proof.

On the critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H) = P_r^2 - P_\alpha = \frac{1}{2}\frac{P_r^2}{2} + c > 0$;

b^3-contact form $\alpha = (P_r\frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha) | x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;

$R_\alpha|_Z = X_{P_r}$ and the cylinder is foliated by periodic orbits.

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$R_\alpha|_Z = X_{P_r}$ and the cylinder is foliated by periodic orbits.

Weinstein conjecture

Weinstein conjecture 

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

- Taubes proved it in dimension 3 for regular contact structures.

  What about singular contact structures?

Key point: Desingularization

Theorem (M-Oms)

Given a b^{2k}-contact manifold with convex critical set Z, there exists a family of contact forms α_ϵ agreeing with a b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The family of bi-vector fields Λ_ϵ and the family of vector fields $\mathbf{R}_\epsilon$ associated to the Jacobi structure of the contact form α_ϵ converges to the bivector field Λ and to the vector field R in the C^{2k-1}-topology as $\epsilon \to 0$.

Theorem (M-Oms)

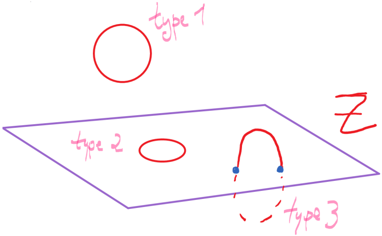
Let (M, α) be a closed b^{2k}-contact manifold of dimension 3, then there exists a family of periodic orbits $\mathcal{O}_\epsilon$ associated to the Reeb vector fields R_ϵ.

What one would like to do: Take the limit $\epsilon \to 0$.

Problem: $\mathcal{O}_\epsilon$ need not be continuous. as it comes from critical point of the action functional

$$\mathcal{A}_{\alpha_\epsilon}(\gamma) = \int_\gamma \alpha_\epsilon$$

for γ in the loop space $C^\infty(S^1, M)$.



Lemma

If there exist $\epsilon > 0$, such that there exists a periodic orbit of R_{α_ϵ} of type (1), then this is a periodic Reeb orbit of (M, α).

Periodic orbits via desingularization

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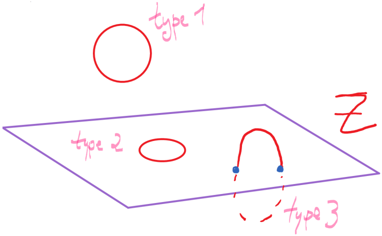

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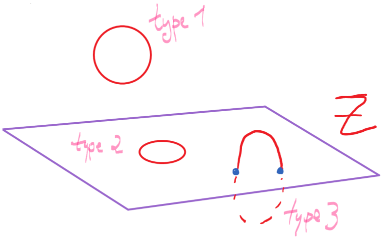

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Attacking b^m-Weinstein's conjecture

Theorem

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Proof.

1. The restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian.
2. $\alpha = u\frac{dz}{z^m} + \beta$
3. u is non-constant on Z.
4. R_α is Hamiltonian on Z for $-u$,
5. $u^{-1}(p)$ where p regular is a circle,
6. R_α periodic on $u^{-1}(p)$.

□

No periodic orbits away from Z

There are compact b^m-contact manifolds (M, Z) of any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits on $M \setminus Z$.

Example

- $S^3 \subset (\mathbf{R}^4, \omega)$
- $X = \frac{1}{2}x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2})$ Liouville v.f.
- $R_\alpha = 2x_1^2\frac{\partial}{\partial x_1} - x_1y_1\frac{\partial}{\partial y_1} + 2x_2\frac{\partial}{\partial y_2} - 2y_2\frac{\partial}{\partial x_2}$
- On $Z = S^2$: rotation, way from Z, no periodic orbits.

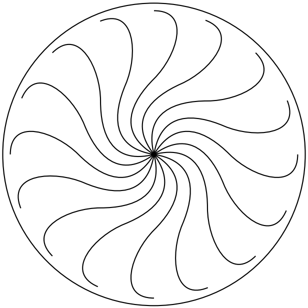
Example

- $(\mathbb{T}^3, \sin\phi\frac{dx}{\sin x} + \cos\phi dy)$. $R_\alpha = \sin\phi\sin x\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}$.
- $Z =$ two disjoint copies of the 2-torus $\mathbb{T}^2$ and the Reeb flow restricted to it is given by $\cos\phi\frac{\partial}{\partial y}$.

Periodic orbits away from Z?

Definition

$(M^3, \xi = \ker \alpha)$ is *overtwisted* if there exists D^2 s.t. $TD \cap \xi$ defines a 1-dimensional foliation given by



A contact manifold that is not overtwisted is called *tight*.

Overtwisted manifolds and periodic orbits

Theorem (Hofer '93)

Let (M^3, α) a closed OT contact manifold. Then there exists a periodic orbit.

Not true for open OT manifolds!

Definition

A b^m-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface Z.

Definition

A b^m-contact form α is $\mathbf{R}^+$-invariant around the critical set if $\alpha = udz + \beta$, where $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$

Singular Hofer

Theorem

Let (M^3, α) be a closed b^m-contact manifold with critical set Z. Assume there exists an overtwisted disk in $M \setminus Z$ and assume that α is $\mathbf{R}^+$-invariant in a tubular neighbourhood around Z. Then there exists

1. *a periodic Reeb orbit in $M \setminus Z$ or*
2. *a family of periodic Reeb orbits approaching the critical set Z.*

Furthermore, the periodic orbits are contractible loops in the symplectization.

The proof is an adaptation of Hofer's techniques.

An image is worth more than a thousand words...

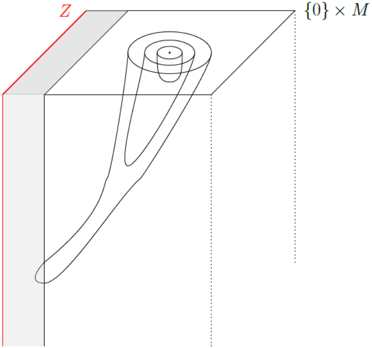
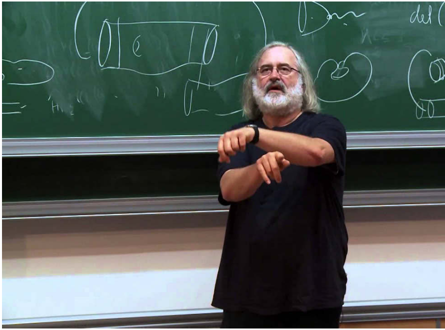


Figure 5.3: Bishop family blowing-up in the $\mathbb{R}^{+}$-invariant part



More generally

Theorem

Let (M^3, α) be a $\mathbf{R}^+$-invariant contact manifold that is OT away from the $\mathbf{R}^+$-invariant part. Then there exists a 1-parametric family of periodic Reeb orbits in the $\mathbf{R}^+$-invariant part of M or a periodic Reeb orbit away from the $\mathbf{R}^+$-invariant part.

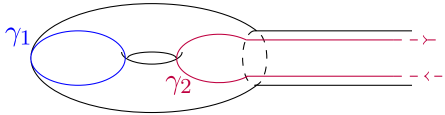
Mantra

Non-compactness $+$ $\mathbf{R}^+$-invariance $=$ compactness

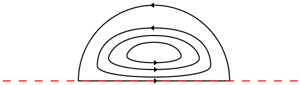
Question: Applications of this theorem?

The singular Weinstein conjecture re-loaded

A true singular Weinstein structures should also admit singular orbits as below:



Or,



Singular Weinstein conjecture

Let (M, α) be a compact b-contact manifold with critical hypersurface Z. Then there exists always a Reeb orbit $\gamma : \mathbb{R} \to M \setminus Z$ such that $\lim_{t\to\pm\infty} \gamma(t) = p_\pm \in Z$ and $R_\alpha(p_\pm) = 0$ (singular periodic orbit).

We need inspiration...

True inspiration comes in a hat...

Escape orbits and Singular orbits

Singular periodic orbits are a particular case of *escape orbits* γ, $\gamma \subset M \setminus Z$ such that $\lim_{t\to\infty} \gamma(t) = p$ where p is an equilibrium point in Z (respectively $\lim_{t\to-\infty} \gamma(t) = p$).

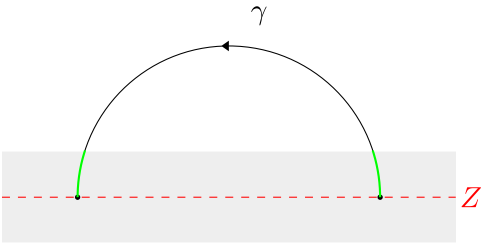


Figure: Singular periodic orbit vs. Escape orbits (in green)

b-Beltrami vector fields to the rescue

b-Beltrami vector field X curl$X = \lambda X$

Theorem (Cardona-M.-Peralta-Salas)

- *Any rotational Beltrami field non-vanishing as a section of* ${}^bTM$ *on* M *is a Reeb vector field (up to rescaling) for some* b*-contact form on* M.
- *Given a* b*-contact form* α *with Reeb vector field* X *then any nonzero rescaling of* X *is a rotational Beltrami field for some* b*-metric and* b*-volume form on* M.

Practical tip

X is a Beltrami vector field on (M, g) $\leftrightsquigarrow$ the Reeb vector field associated to the b-contact form $\alpha = g(X, \cdot)$ is given by $\frac{1}{\|X\|^2}X$.

Escape orbits and Singular orbits

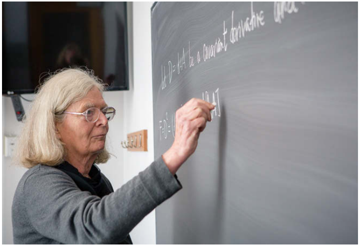
Exact b-metric $\leftrightsquigarrow$ Melrose b-contact forms:

$$g = \frac{dz^2}{z^2} + \pi^* h \tag{1}$$

with h Riemannian metric on Z.

Theorem (M-Oms-Peralta, "lockdown theorem")

There exists at least $2 + b_1(Z)$ escape orbits for Reeb vector fields of generic Melrose b-contact forms on (M, Z).



Proof: The Beltrami equation $\rightsquigarrow$ the Hamiltonian function associated to (R, Z) is an eigenfunction of the induced Laplacian on Z $\rightsquigarrow$ (Uhlenbeck) generically Morse and non-zero critical values.

A garden of singular orbits

A garden of singular orbits

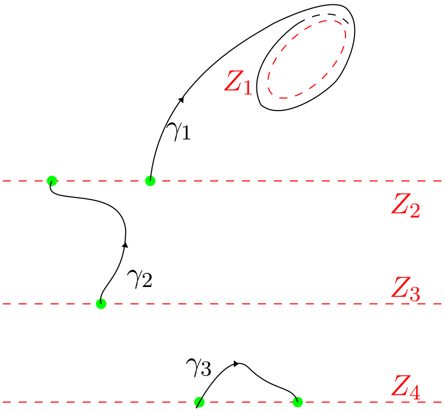


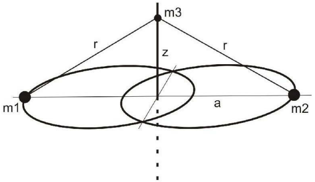
Figure: Different types of escape and singular periodic orbits: γ_1 is a generalized singular periodic orbit, γ_2, γ_3 are singular periodic orbits

Generalized singular periodic orbits

Definition

An orbit $\gamma : \mathbf{R} \to M \setminus Z$ of a b-Beltrami field X is a generalized singular periodic orbit if there exist $t_1 < t_2 < \cdots < t_k \to \infty$ such that $\gamma(t_k) \to p_+ \in Z$ and $t_{-1} > t_{-2} > \cdots > t_{-k} \to -\infty$ such that $\gamma(t_{-k}) \to p_- \in Z$, as $k \to \infty$.

p_+ and p_- may be contained in different components and are not necessarily zeros of X.



This includes oscillatory motions:orbits $(q(t), p(t))$ in the phase space $T^*\mathbb{R}^n$ such that $\limsup_{t\to\pm\infty} \|q(t)\| = \infty$ and $\liminf_{t\to\pm\infty} \|q(t)\| < \infty$.

A more symmetric case

For $g = \frac{dz^2}{z^2} + dx^2 + dy^2$, we can prove more.

Theorem (M-Oms-Peralta Salas)

When g is semi-locally as above and X a generic asymptotically symmetric b-Beltrami vector field, X has a generalized singular periodic orbit. Moreover, it has a singular periodic orbit or at least 4 escape orbits.

In the case of $(\mathbb{T}^3, \alpha = C\cos y dx + B\sin x dy + (C\sin y + B\cos x)\frac{dz}{\sin z})$ for $|B| \neq |C|$, the singular Weinstein conjecture is satisfied.

Final questions

- Can we prove existence of singular Weinstein orbits for generic b-contact forms?
- Extend the apparatus of variational calculus to extend the action functional to this set-up.

$$\mathcal{A}_\alpha(\gamma) = \int_\gamma \alpha$$

- Find higher dimensional applications to celestial mechanics (for instance, escape orbits 5-body problem).