Geometrisation, integrability and knots

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Geometry, Dynamics and Mechanics Seminar, November 3, 2020

Plan

- Geometrisation programmes in dimension 2 and 3
- Liouville-Arnold integrability revisited
- Chaos and integrability in $SL(2, \mathbb{R})$-geometry
- Geodesics on the modular 3-fold and knot theory

References

W.P. Thurston *Hyperbolic geometry and 3-manifolds.* In: LMS Lecture Notes Series **48**, CUP, 1982.
É. Ghys *Knots and Dynamics.* Intern. Congress of Math. Vol. 1. Eur. Math. Soc., Zurich 2007, 247-277.
A. Bolsinov, A.Veselov, Y. Ye *Chaos and integrability in* $SL(2, \mathbb{R})$*-geometry.* arXiv:1906.07958.

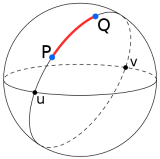
Felix Klein (1849-1925) and Henri Poincaré (1854-1912)

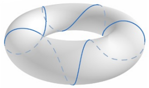
Every conformal class of surface metrics has complete constant curvature representative.

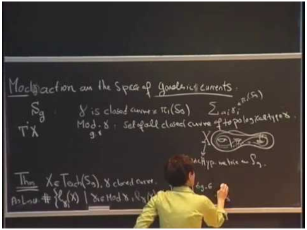
Geodesic flows on surfaces

The behaviour of geodesics on these three types of surfaces are very different.

On the round sphere all geodesics are large circles, on the flat torus they are straight winding lines, while on hyperbolic surfaces their behaviour is known to be very chaotic (**Hedlund 1930s, Anosov 1960s**).







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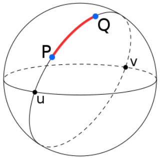

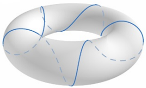

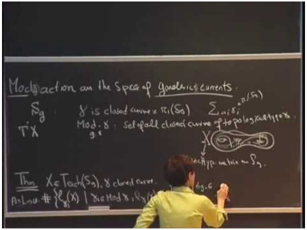

In particular, for the *modular surface* $\mathbb{H}^2/PSL(2,\mathbb{Z})$ the geodesics can be described symbolically using continued fractions (**E. Artin, 1924**).

William Thurston (1946-2012) and Grigori Perelman (1966-)

Every closed 3-manifold can be decomposed into pieces such that each admits one of the following eight types of geometric structures of finite volume

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ Nil,\ Sol,\ \widetilde{SL(2,\mathbb{R})},\ \mathbb{H}^3.$$

$$Nil = \left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right\} \quad Sol = \left\{\begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

and $\widetilde{SL(2,\mathbb{R})}$ is the universal cover of $SL(2,\mathbb{R})$.

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What's about integrability of the corresponding geodesic flows?

Arnold 1963: Hamiltonian system on symplectic manifold M^{2n} is integrable in Liouville sense if it has n independent integrals $F_1, \dots, F_n$ in involution.

When the joint integral level

$$M_c = \{x \in M^{2n} : F_i(x) = c_i,\ i = 1, \dots, n\}$$

is non-critical and compact, then it must be a torus T^n with quasi-periodic dynamics and in its vicinity one can introduce "action-angle" variables I_i, φ_i with $H = H(I)$: $\dot{I} = 0, \dot{\varphi} = \omega(I)$.

Liouville-Arnold integrability revisited

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Tomei 1984, Gaifullin 2006: A natural compactification of integral level in the (extended) Toda system is an aspherical manifold M^{n-1} with

$$\chi(M^{n-1}) = (-1)^{n+1} B_{n+1} \frac{2^{n+1}(2^{n+1}-1)}{n+1},$$

which can be used as universal in Steenrod's cycle realisation problem!

Sol-case: chaotic critical level

In *Sol*-case the principal examples are mapping tori M_A^3 of the hyperbolic maps $A : T^2 \to T^2$, $A \in SL(2, \mathbb{Z})$ (first considered by Poincaré in 1892!):

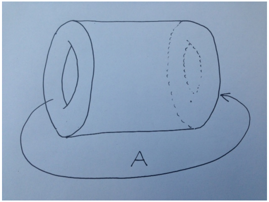

Figure: Torus mapping of A

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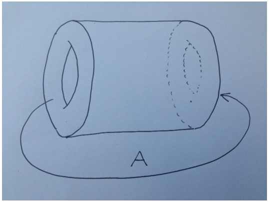

Figure: Torus mapping of A

Bolsinov and Taimanov 2000: On *Sol*-manifolds M^3_A the geodesic flow is Liouville integrable in smooth category, but not in analytic one.

At the degenerate level the system is chaotic (Anosov map), so the system has positive topological entropy!

In $SL(2,\mathbb{R})$-case the principal examples are unit tangent bundles of hyperbolic surfaces

$$\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R}) = S\mathcal{M}^2_\Gamma, \quad \mathcal{M}^2_\Gamma = \Gamma\backslash\mathbb{H}^2,$$

where $\Gamma \subset PSL(2,\mathbb{R})$ is a cofinite Fuchsian group.

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Bolsinov, Veselov and Ye 2019: The corresponding phase space $T^*\mathcal{M}^3_\Gamma$ contains two open regions with integrable and chaotic behaviour.

In the integrable region we have Liouville integrability with analytic integrals, while in the chaotic region the system is not Liouville integrable even in smooth category and has positive topological entropy.

Cf. **Arnold 1961, Taimanov 2004** on magnetic geodesic flow on $\mathcal{M}^2_\Gamma$.

$SL(2,\mathbb{R})$-geometry

Naturally reductive metrics on $SL(n,\mathbb{R})$: left $SL(n,\mathbb{R})$- and right $SO(n)$-invariant

$$\langle X, Y\rangle = \alpha(\mathrm{sym}\, X, \mathrm{sym}\, Y) + \beta(\mathrm{skew}\, X, \mathrm{skew}\, Y), \ \alpha > 0 > \beta,$$

$$(X, Y) := \mathrm{Tr}\, XY, \ \mathrm{skew}\, X := (X - X^\top)/2 \in so(n), \ \mathrm{sym}\, X := (X + X^\top)/2.$$

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For $n = 2$ and $\alpha = 2$, we have the inner product with

$$|\Omega|^2 = 4(u^2 + vw) + k(v - w)^2, \quad k = 1 - \frac{\beta}{\alpha} > 1$$

on the Lie algebra

$$\Omega = \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \in sl(2,\mathbb{R}).$$

$PSL(2,\mathbb{R})$ as unit tangent bundle of hyperbolic plane

$PSL(2,\mathbb{R})$ can be identified with the unit tangent bundle $S\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2 = SL(2,\mathbb{R})/SO(2)$:

$$g = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \longrightarrow (z = \frac{ai+b}{ci+d}, \xi = \frac{i}{(ci+d)^2}) \in S\mathbb{H}^2,$$

where $\mathbb{H}^2$ is realised as the upper half-plane $z = x + iy$, $y > 0$ with the hyperbolic metric $ds^2 = dzd\bar{z}/y^2$.

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In coordinates $x, y, \varphi = \arg \xi$ the metric has the form

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + (k-1)(d\varphi + \frac{dx}{y})^2,$$

which is the generalised **Sasaki metric** on $S\mathbb{H}^2$, considered by **Nagy 1977**. Sasaki metric corresponds to $k = 2$ and can be considered as the "best one".

Euler-Poincare equations and geodesics on $SL(2, \mathbb{R})$

The general Euler-Poincare equations of the corresponding geodesic flow have

$$\dot{M} = [M, \Omega],$$

where $\Omega := g^{-1}\dot{g} \in \mathfrak{g}$ and $M \in \mathfrak{g}^* \cong \mathfrak{g}$ is determined by $(\Omega, M) = \langle \Omega, \Omega \rangle$.

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In our case we have $2M = (\alpha+\beta)\Omega + (\alpha-\beta)\Omega^\top$, so the Euler-Poincare equations have the form

$$\dot{M} = \frac{\beta-\alpha}{2\alpha\beta}[M, M^\top],$$

which can be easily integrated explicitly (e.g. **Mielke 2002**).

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The geodesics on $SL(2,\mathbb{R})$ with $\Omega(0) = \Omega_0$ can be explicitly given by

$$g(t) = g(0)e^{tX_0}e^{tY_0},$$

where

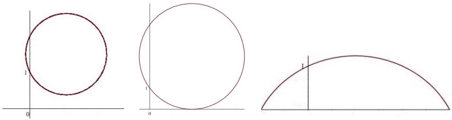
$$X = \frac{1}{\alpha}M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Y = \frac{\alpha-\beta}{2\beta}\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}.$$

Projection of geodesics on hyperbolic plane

Nagy 1977, BVY 2019: The projection of the geodesics on $PSL(2,\mathbb{R}) = S\mathbb{H}^2$ to $\mathbb{H}^2$ are curves of constant geodesic curvature

$$\kappa = \frac{b-c}{\sqrt{4a^2+(b+c)^2}}.$$

They are circles if $\kappa^2 > 1$, or arcs of circles if $\kappa^2 \leq 1$ and can be described as magnetic geodesics on $\mathbb{H}^2$ in constant magnetic field with density $B = b - c$:



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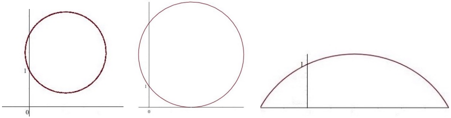

History of magnetic geodesics on $\mathcal{M}^2_\Gamma = \mathbb{H}^2/\Gamma$: **Caratheodory 1932, Hedlund 1936, Arnold 1961, Paternain 1997, Taimanov 2004.**
In particular, Arnold proved that the entropy of the corresponding flow on $S\mathcal{M}^2_\Gamma$ is $h = \sqrt{1-\kappa^2}$ if $\kappa^2 \leq 1$ (and 0 if $\kappa^2 > 1$).

We have two obvious left-invariant Poisson commuting integrals of geodesic flow on $G = SL(2,\mathbb{R})$: Hamiltonian

$$H = \frac{1}{2}(\Omega, M) = \frac{\alpha}{4\beta}(\beta[4a^2 + (b+c)^2] - \alpha(b-c)^2)$$

and

$$\Delta = \det M = a^2 + bc.$$

As the third required for the Liouville integrability integral we can take any non-constant right-invariant function F on T^*G.

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Note that any other such function generates the left shifts and gives an additional integral of the system. Thus the invariant tori of the system have dimension 2 in agreement with the previous picture.

Liouville integrability for Fuchsian quotients M^3_Γ

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a Fuchsian group such that $\Gamma\backslash\mathbb{H}^2 = \mathcal{M}^2_\Gamma$ has finite area and consider the quotient $\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R}) = S\mathcal{M}^2_\Gamma$.

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Matrix elements of right momentum $m = gMg^{-1}$ are not Γ-invariant, so we need to study the invariants of the co-adjoint action of $\Gamma \subset G$ on $m \in \mathfrak{g}^*$.

Liouville integrability for Fuchsian quotients M_Γ^3

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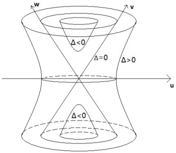
It is known that this action is discrete if $\Delta = \delta < 0$ (which is a model of $\mathbb{H}^2$) and has some dense orbits if $\Delta = \delta > 0$ (**Hedlund, Dal'Bo**).



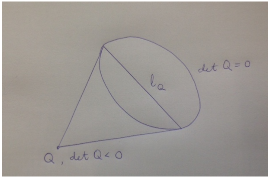


Figure: $sl(2,\mathbb{R})$-symplectic leaves and Klein's correspondence

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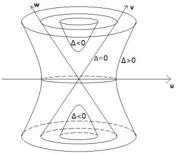


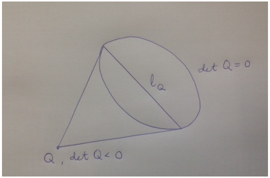
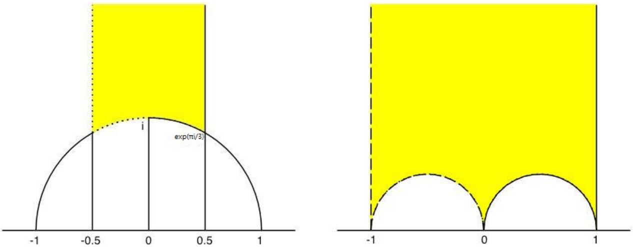


Figure: $sl(2,\mathbb{R})$-symplectic leaves and Klein's correspondence

Corollary: The geodesic flow on $T^*\mathcal{M}^3_\Gamma$ has no smooth right-invariant integrals F independent from Δ in the part of the phase space $T^*\mathcal{M}^3_\Gamma$ with $\Delta \geq 0$.
In the domain $\Delta < 0$ we can use any real analytic automorphic function as the additional third analytic integral F.

Special case: modular groups

Consider now the special case of modular group $\Gamma = PSL(2, \mathbb{Z})$ and its principal congruence subgroup Γ_2.



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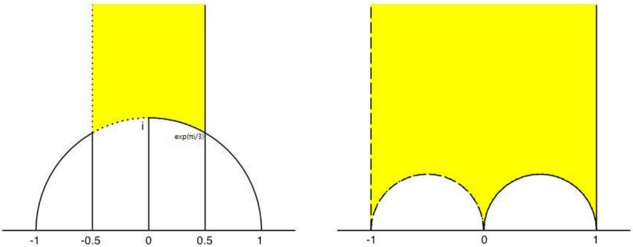

Figure: The fundamental domains of Γ and Γ_2

In the first case the quotient $\mathcal{M}^2 = PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$ is the orbifold with two orbifold points corresponding to the elliptic elements in $PSL(2,\mathbb{Z})$ of order 2 and 3 respectively. In the second case we have the 3-point punctured sphere.

Modular 3-fold and knots

Let $\Gamma = PSL(2,\mathbb{Z})$ be the modular group and consider the *modular 3-fold*

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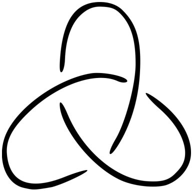
There is a remarkable observation due to **Quillen (1970s):**

$$\mathcal{M}^3 = SL(2, \mathbb{R})/SL(2, \mathbb{Z}) = S^3 \backslash \mathcal{K},$$

where $\mathcal{K}$ is the trefoil knot:



Quillen's proof

Milnor, 1972: Note first that $\mathcal{M}^3 = SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ can be interpreted as the moduli space of the elliptic curves $\mathbb{C}/\mathcal{L}$ up to real scaling. The corresponding $\wp$-function satisfies the Weierstrass equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

which defines an elliptic curve if and only if the discriminant

$$D = g_2^3 - 27g_3^2 \neq 0.$$

The intersection of the unit sphere $S^3 \subset \mathbb{C}^2(g_2, g_3)$ with the set $D = 0$ is $(2, 3)$-torus ($=$ trefoil) knot.

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Alternatively, the projection $\mathcal{M}^3 \to \mathcal{M}^2 = \mathbb{H}^2/PSL(2,\mathbb{Z})$ is the Seifert fibration with two singular fibres corresponding to orbifold points of order 2 and 3 of $\mathcal{M}^2$. The missing Hopf fibre over infinity is thus (2,3)-torus knot.

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Remark. The same arguments show that the complement $S^3 \setminus K_{p,q}$ to any torus knot admit $SL(2,\mathbb{R})$-structure.

Modular knots and Lorenz system

E. Artin, 1924: *Periodic geodesics on modular surface $\mathcal{M}^2$ are labelled by integer indefinite binary quadratic forms Q* (by Klein's correspondence). Their lifts to $\mathcal{M}^3 = S\mathcal{M}^2$ form certain knots called by Ghys *modular*.

Modular knots and Lorenz system

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Birman and Williams (1983), Ghys (2006): Modular knots are exactly those, which appear as periodic orbits in the celebrated Lorenz system

$$\begin{cases} \dot{x} = \sigma(-x + y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}, \quad \sigma = 10,\ b = 8/3,\ r = 28.$$

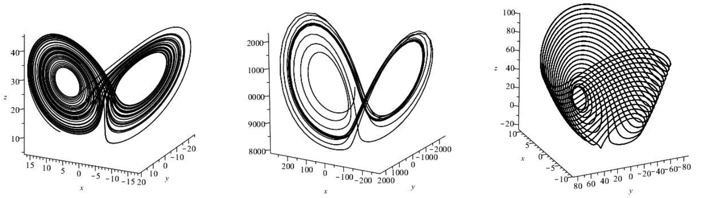
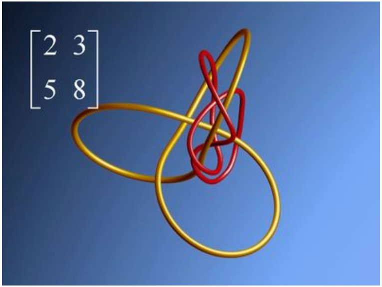


Figure: The Lorenz trajectories for $r = 28$, 10000 and $r = \infty$

Images of modular knots by Jos Leys



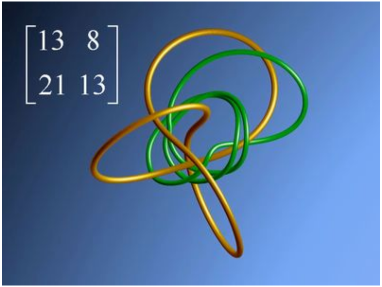


Figure: The images of the lifted modular geodesics in the complement of the trefoil knot from: www.ams.org/featurecolumn/archive/lorenz.html.

Integrable limit and cable knots

Consider the integral

$$\mathcal{C} := \kappa^2 = \frac{(b-c)^2}{4a^2 + (b+c)^2} = \frac{\beta H - \alpha\beta\Delta}{\beta H - \alpha^2\Delta}$$

of the geodesic flow on $\mathcal{M}^3$. We have seen that the system is integrable if $\mathcal{C} > 1$ and non-integrable otherwise.
When $\mathcal{C} = 0$ we have the lifts of the geodesics on the modular surface $\mathcal{M}^2$ considered by Ghys.
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BVY 2019: The periodic geodesics on modular 3-fold $\mathcal{M}^3_\Gamma$ with sufficiently large values of $\mathcal{C}$ represent the trefoil cable knots in $S^3 \setminus \mathcal{K}$.
Any cable knot of trefoil can be realised in such a way.

Thurston (1978): Every knot is either torus, or a satellite, or hyperbolic knot.

Geometric classification of knots

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Torus knot $K_{p,q}$ is winding p times around the axis of rotation of the solid torus in $\mathbb{R}^3$ and q times around the central circle of the torus. Trefoil knot $\mathcal{K} = K_{2,3}$.

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Satellite knot K: take a knot K_1 inside a solid torus in $\mathbb{R}^3$ and knot the torus in the shape of another knot K_2. If K_1 is a torus knot, we have *cable knots of* K_2.

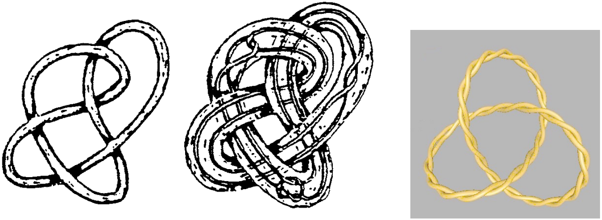

Figure: A knot with its satellite (from Thurston 1982) and trefoil cable knot.

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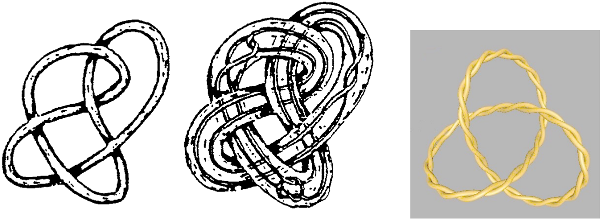

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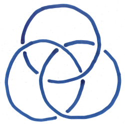
Complements to the torus knots admit $SL(2,\mathbb{R})$-structure, hyperbolic knots - $\mathbb{H}^3$-structure, but the satellite knots do not admit any geometric structure.

Let $\Gamma_2 \subset SL(2, \mathbb{Z})$ consist of matrices congruent to the identity modulo 2:

$$\mathcal{M}_2^3 = \Gamma_2 \backslash SL(2, \mathbb{R}) \cong S^3 \backslash \mathcal{L},$$

where $\mathcal{L}$ is the **Hopf 3-link**



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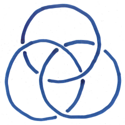

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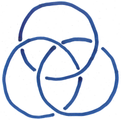

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Llibre, MacKay 1990: iterated torus knots are precisely the knots with zero topological entropy.

Problem. What's about knots at other levels of $\mathcal{C}$?

- Study the types of knots in $\mathcal{M}^3 = S^3 \setminus \mathcal{K}$ outside the integrable limit

  Recall Volume Conjecture **(Kashaev 1997, Murakami et al 2002)**

  $$Vol(S^3 \backslash K) = 2\pi \lim_{N\to\infty} \frac{\ln |J_N(K)|}{N},$$

  where $J_N(K)$ is the *Jones polynomial* of K evaluated at $e^{2\pi i/N}$.

  Volumes for modular knots: **Brandts, Pinsky, Silberman 2017**

Some open questions

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  For the modular surface $\mathcal{M}^2 = \mathbb{H}^2/SL(2,\mathbb{Z})$: **Maas, Selberg, Faddeev, Hejhal (1940-70s)**

Thurston's wisdom