GDM Seminar
November 17, 2020

One can hear semi-toric systems

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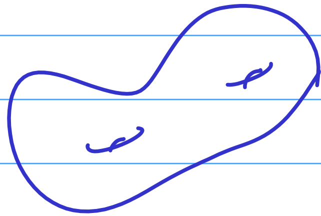
- Inverse spectral problems
- Quantum / Classical integrable systems
- Semitoric systems
- The Proof

Inverse spectral problems

(1) Riemannian manifold



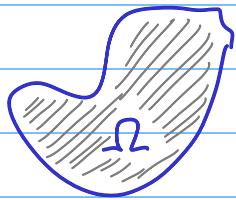
$$\begin{tikzcd}
(X, g) \arrow[r, squiggly, shift left] \arrow[d] & (\lambda_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \arrow[l, shift left, "?"] \arrow[d, phantom, "\parallel"] \\
\Delta_g \arrow[r] & \mathrm{Spectrum}(\Delta_g)
\end{tikzcd}$$

Laplace-Beltrami Δ_g acting on $L^2(X)$

Problem: "Can you hear g?": Can you "recover" g from $(\lambda_j)_{j \in \mathbb{N}}$?"

Inverse spectral problems

(2) (Why "hear") Euclidean domains $\Omega \subset \mathbb{R}^n$



Bounded Ω = "drum"

?

eigen frequencies of the membrane

$\|$

Δ_Ω some Laplacian $\longrightarrow$ $\text{Spectrum}(\Delta_\Omega) = \{\lambda_j, j \in \mathbb{N}\}$

Problem: "hear the drum" = "Recover" Ω from $(\lambda_j)_{j\in\mathbb{N}}$.

$\boxed{\text{Inverse spectral problems}}$ & $\boxed{\text{Quantization}}$

$$
\begin{tikzcd}
\mathcal{C} = \text{classical system} \arrow[d, "\text{Quantization}"] & \\
\mathcal{Q} = \text{Quantum system} \arrow[r] & \text{Spectrum} \arrow[ul, "?"']
\end{tikzcd}
$$

<u>Examples</u>

(X, g)	(Ω, eucl)	$((M, \omega), H)$
$\downarrow$	$\downarrow$	$\downarrow$
$L^2(X), \Delta_g$	$(L^2(\Omega), \Delta_\Omega)$	$(\mathcal{H}_M, \hat{H})$

Inverse spectral problems

③ Hamiltonian version (more general)

Let (M,ω) be a symplectic manifold.

Let $H \in C^\infty(M;\mathbb{R})$ be a Hamiltonian

Let $\mathcal{H}_M$ be a quantization of M ($\mathcal{H}_M$ = Hilbert space)

Let $\hat{H}$ be a quantization of H ($\hat{H}$: acts on $\mathcal{H}_M$)

Problem : From the spectrum of $\hat{H}$ can you recover $((M,\omega), H)$?

But : what does recover mean ?

Problem 1 : (abstract injectivity)

$\boxed{\text{If two classical systems } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ have the same Quantum Spectrum, prove that they are isomorphic}}$

$\Uparrow$ implies

Problem 2 : (Construct a left inverse)

$\boxed{\text{Given the spectrum of some unknown classical system } \mathcal{C}, \text{ show how to construct } \mathcal{C} \text{ (up to isomorphism)}}$

Liouville Integrable Systems

Classical : (M^{2n}, ω) symplectic $f_1, \dots, f_n \in C^\infty(M;\mathbb{R})$

$$\{f_i, f_j\} = 0 \qquad df_1 \wedge \dots \wedge df_n \neq 0 \text{ a.e.}$$

$$F = (f_1, \dots, f_n) : M \to \mathbb{R}^n \qquad \text{(moment map)}$$

Quantum : $\mathcal{H}$ Hilbert space, $P_1, \dots, P_n$ self-adjoint ops (unbounded)

$$[P_i, P_j] = 0 \qquad \text{(spectral measure)}$$

$$\Sigma = \text{joint spectrum}(P_1, \dots, P_n) \subset \mathbb{R}^n \qquad \text{(discrete)}$$

Quantization is not so easy...

1) For many reasons, we need semiclassical analysis. $\hbar \to 0$

Can use
- Pseudodifferential quantization $(M = T^*X)$
- Berezin - Toeplitz quant. (Kähler, ...)

2) OPEN PROBLEM : the Quantization arrow in the category of integrable systems is not known.

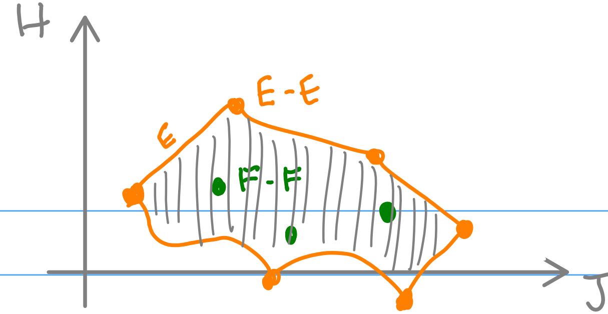
$$F = (f_1, .., f_n) \xrightarrow[?]{??} (\hat{P}_1, .., \hat{P}_n)$$

Hence we shall always assume the existence of an Quantum system.

Note : This is good from physics viewpoint : nature is Quantum.

"From a quantum system, can you recover the classical system ?"

(Spectroscopy, etc ...)



Semi-toric systems

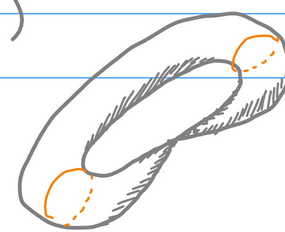
(M^4, ω) $\quad F = (J, H)$ w/ non degenerate singularities and connected fibers (locally)

J is a hamiltonian for an effective S^1-action on M.

(So M is an S^1-Manifold, and an almost toric manifold)

F has 3 types of critical points
1) elliptic - transverse E
2) elliptic - elliptic E-E
3) focus - focus F-F

We assume that the focus-focus fibers are <u>simple</u> (one fixed point)



Semi-toric systems

In a loose way, a semi-toric system is a (Morse) Hamiltonian H on a hamiltonian S^1- manifold (M, J, ω).

Two semi-toric systems $(M_1, \underbrace{J_1, H_1}_{F_1}, \omega_1)$, $(M_2, \underbrace{J_2, H_2}_{F_2}, \omega_2)$ are isomorphic if we have

$$\begin{tikzcd}
(M_1, \omega_1) \arrow[r, "\varphi"] \arrow[d, "F_1"'] & (M_2, \omega_2) \arrow[d, "F_2"] \\
\mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}_2
\end{tikzcd}$$

with $g(x, y) = (x, f(x, y))$ & $\partial_y f > 0$

Main Result: One can hear semitoric systems

Theorem [LE FLOCH – VN, 2020 or 21 ?]

Theorem 1.1 (Theorem 4.16, Theorem 4.36, Theorem 5.7) *From the joint spectrum (modulo $\mathscr{O}(\hbar^2)$) of a quantum semitoric system, one can explicitly recover, in a constructive way, all symplectic invariants of the underlying classical semitoric system. In particular, if two quantum semitoric systems have the same spectrum, then their underlying classical systems are symplectically isomorphic.*

In view of the classification of semi-toric systems, this proves "Problem 2" (and hence "Problem 1")

Some history...

(personal / partial)

Prehistory: (of course S^1-systems are ubiquitous in Physics, Maths...)

1937 MINEUR : action-angle & Quantization

1972-1980 NEKHOROSHEV, DUISTERMAAT : globalize action-angle

1978 VEY : Symplectic Morse lemma for commuting hamiltonians

1982 ATIYAH, GUILLEMIN-STERNBERG : Toric, compact

1982 DUISTERMAAT-HECKMAN : reduction and localization

1984-1990 ELIASSON : C^∞ version of VEY's.

1988 DELZANT : toric classification

1989 DUFOUR-MOLINO : elliptic singularities

1989,... FOMENKO (& school) topological classification

1995 LERMAN : symplectic cuts

1999 KARSHON : S^1-manifolds, classification

Focus-focus singularities come into play

1988 – … BOLSINOV, CUSHMAN, DULLIN, FASSO, … many examples w/o global a.a

1995 ZUNG : focus-focus and monodromy

1999 SADOVSKÍI – ZHILINSKÍI : spin-orbit, eigenvalue redistribution

2001 VNS : Quantum monodromy

2003 ZUNG : topological classification

2003 VNS : Taylor Series invariant

2003 SYMINGTON : almost-toric mfd

2005 VNS : semi-toric systems

2009 – 2011 PELAYO - VNS : symplectic classification

Then many extensions and examples : ALONSO, DULLIN, HOHLOCH, KANE LE FLOCH, DE MEULENAERE, MIRANDA, PALMER, PELAYO, RATIU, SABATINI, SEPE, SYMINGTON, TANG, TOLMAN, WACHEUX, etc …

Quantization

1917 EINSTEIN : Bohr-Sommerfeld quantization rule "Quantensatz"

...

1972 DUISTERMAAT - HÖRMANDER : FIO

1980 COLIN DE VERDIÈRE : B-S for homogeneous pseudo-diff ops (ΨDO)

1988 CHARBONNEL : Semiclassical Bohr-Sommerfeld (ΨDO)

1998 ZELDITCH : Inverse problem for surfaces of revolution

2000 VNS : Quantum focus-focus

2002 IANTCHENKO - SJÖSTRAND - ZWORSKI : Birkhoff NF & Inv. pb.

2003 CHARLES : B-S for Berezin-Toeplitz ops

2010 PELAYO - VNS : "semitoric conjecture" for inverse spectral problem.

... CHARLES, LE FLOCH, PELAYO, POLTEROVICH, ...

2012 DRYDEN - GUILLEMIN - SENA-DIAS : toric Laplacian

2014 LE FLOCH : Singular B-S for Berezin-Toeplitz

2019 DAUGE - HALL - VNS : Asymptotic lattices

The Taylor Series Invariant

(log-degeneration of the integral affine structure)

define $L = \frac{1}{2\pi}\int_\gamma \alpha$

$\Lambda_0 = F^{-1}(c_0)$

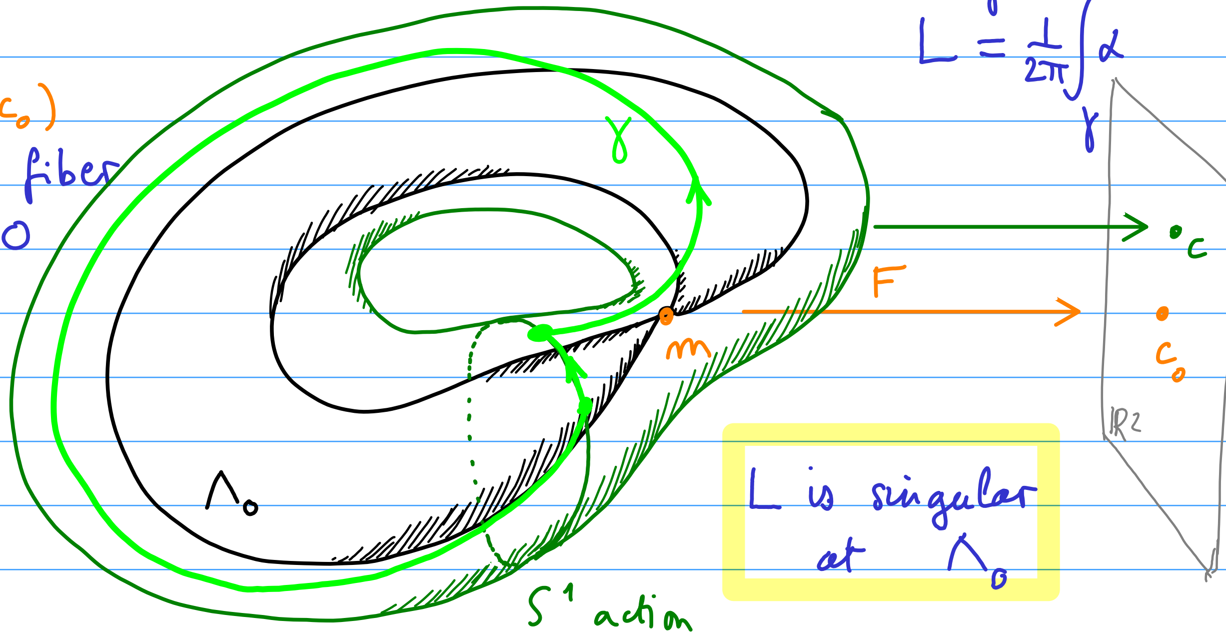
the critical fiber

$dF(m) = 0$

$\Lambda = F^{-1}(c)$

$\omega = d\alpha$

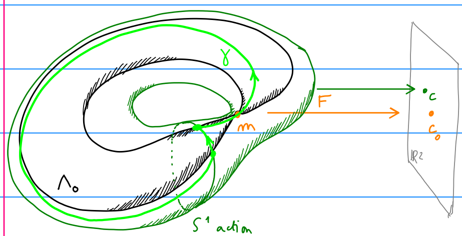
near Λ_0



L is singular at Λ_0

The Taylor Series Invariant

(log - degeneration of the integral affine structure) $F = (J, H)$



<u>Lemma</u>: $\exists!\ H_r = f_r(J, H)$ whose flow is radial (does not wind around m)

In Eliasson coordinates $H_r = x_1 \xi_1 + x_2 \xi_2$

Let $q = (J, H_r)$ (Focus-focus normal form) $q(m) = 0 \in \mathbb{R}^2$

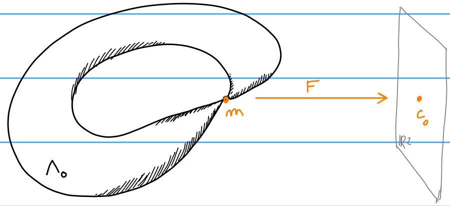
Note: (J, L) are action coordinates

Write $L = \tilde{L} \circ q$

$w = x + iy \simeq (x, y)$

<u>Lemma</u>: The function $S(x,y)$:
$S(w) := \tilde{L}(w) + \operatorname{Im}(w \log w - w)$
is smooth at 0

The Taylor Series Invariant

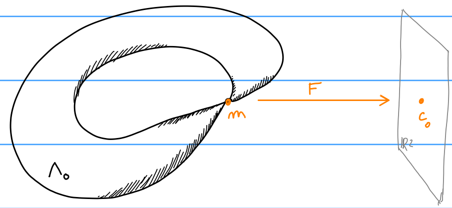


Lemma: The function $S(x,y)$:
$$S(w) := L(w) + \mathrm{Im}(w \log w - w)$$
is smooth at 0

let $S^\infty = \sum S_{\ell,m}\, x^\ell y^m$ be the Taylor Series of S

Theorem [VNS 2003] The class of S^∞ in $\dfrac{\mathbb{R}[[x,y]]}{\mathbb{R} \oplus \mathbb{Z} x}$ is a complete symplectic invariant for the singular foliation near Λ_0

The Taylor Series Invariant



Theorem [VNS 2003] The class of S^∞ in $\dfrac{\mathbb{R}[[x,y]]}{\mathbb{R}\oplus\mathbb{Z}x}$ is a complete symplectic invariant for the singular foliation near Λ_0

$$S^\infty = S_{0,0} + \left[m + \underset{\in\,[0,1[}{S_{1,0}}\right]x + S_{0,1}\,y + O(x,y)^2$$

$S_{0,0} \in \mathbb{R}$ and
$m \in \mathbb{Z}$ are not relevant for the semi-global classification ...

But they can be given a meaning for the global classification

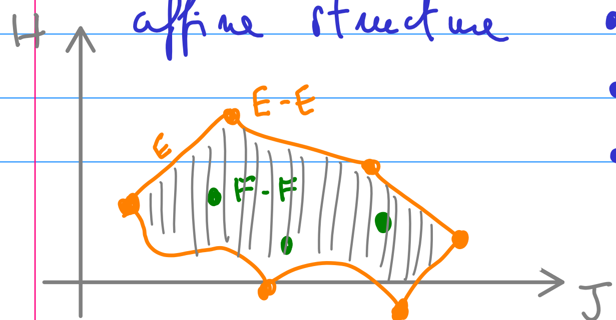
$S_{0,0} \longleftrightarrow$ the height invariant, $m \longleftrightarrow$ the twisting index

Proof : $\boxed{\text{The integral affine structure}}$

From the previous slide (and the classification of PELAYO-VNS) we see that all the information is hidden in the asymptotic behaviour of the action variable L (at the critical fiber) provided we "normalize" it correctly (taking the global picture into account)

In other words, we need to understand the integral affine structure

- its degeneration at singular points
- its boundary values
- how they both relate to each other



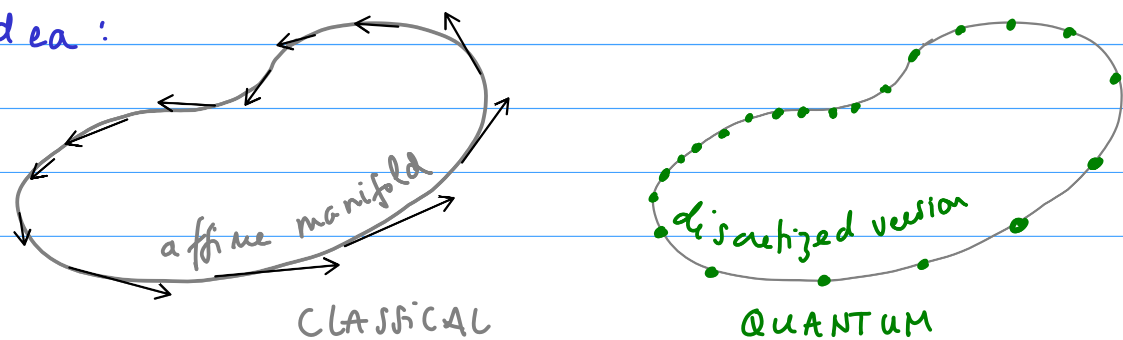
Proof : $\boxed{\text{Asymptotic lattices}}$ [DAUGE-HALL-VNS 2019]

" The joint spectrum is a discretized version of the integral affine structure " (≃ Bohr-Sommerfeld !)

How to push this idea as far as possible ?

$\hookrightarrow$ asymptotic lattices

Basic idea :



Joint Spectra

Integral Affine Manifold B^n	Asymptotic lattice $\Lambda_\hbar \subset \mathbb{R}^n$
$\updownarrow$	$\updownarrow$
affine charts	asymptotic charts
$G : \mathbb{R}^n \longrightarrow B$	$G_\hbar : \hbar\mathbb{Z}^n \longrightarrow \Lambda_\hbar \quad + O(\hbar^\infty)$
$G^{-1} = (a_1, .., a_n)$ action var.	$(\hbar k_1, \dots, \hbar k_n) \longmapsto \lambda_\hbar$

Theorem [... LE FLOCH - VNS '20]

The joint spectrum of a quantum semi-toric system is a global asymptotic lattice (incl. boundaries)

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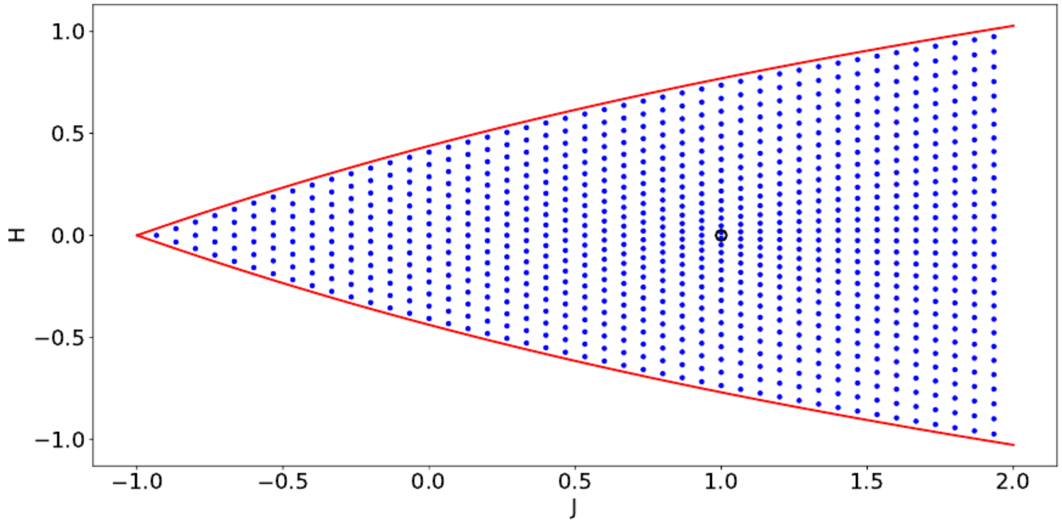


Figure 6: The blue dots are the joint eigenvalues of the spin-oscillator system in the region $-1 \leqslant x \leqslant 2$ for $k = 15$. The red line corresponds to the boundary of the image of the momentum map, and the black circle indicates the focus-focus value.

$$S = \frac{5 \ln 2}{2\pi} Y + \frac{1}{8\pi} XY + \mathscr{O}(3);$$

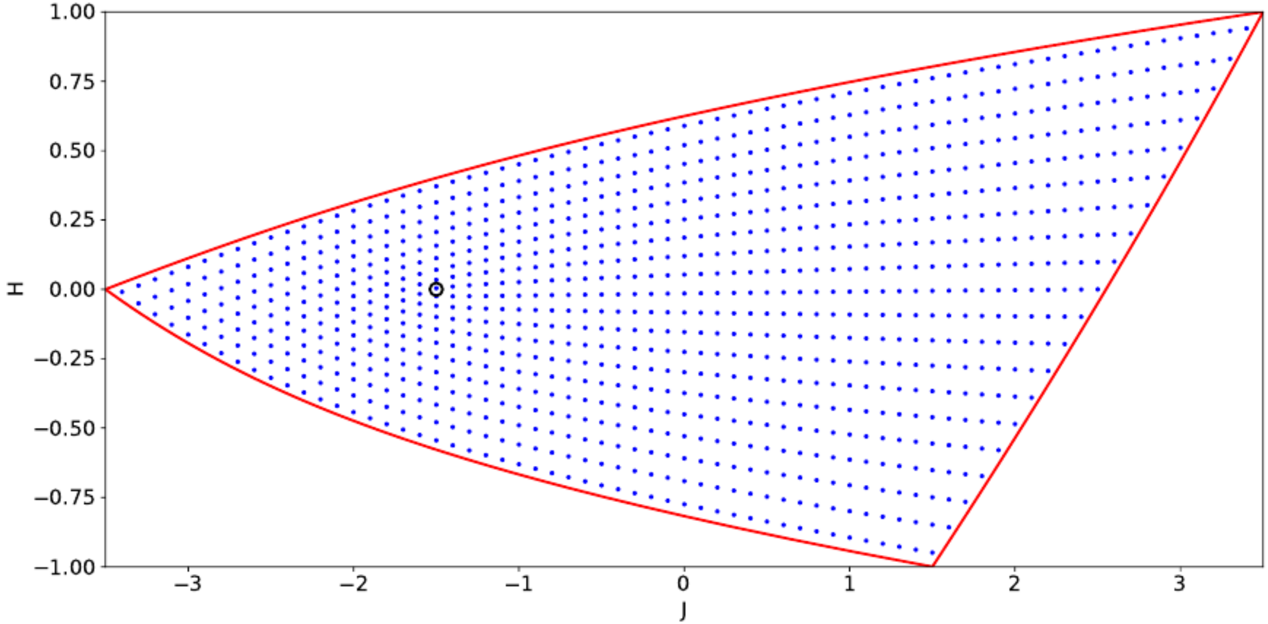


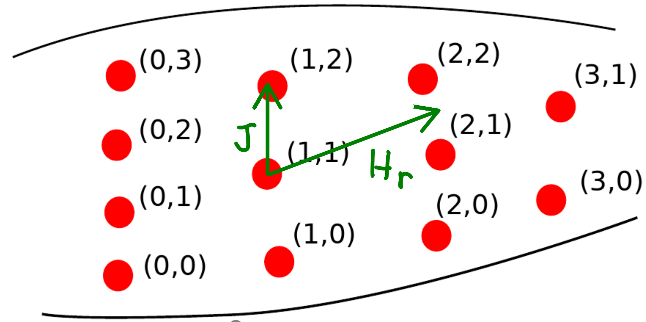
Figure 17: The blue dots are the joint eigenvalues of the quantum coupled angular momenta with $R_1 = 1$, $R_2 = \frac{5}{2}$ and $t = \frac{1}{2}$ for $k = 10$. The red line corresponds to the boundary of the image of the momentum map, and the black circle indicates the focus-focus value.

$$S = \frac{1}{2\pi} \arctan\left(\frac{13}{9}\right) X + \frac{1}{2\pi}\left(\frac{7}{2}\ln 2 + 3\ln 3 - \frac{3}{2}\ln 5\right) Y + \mathscr{O}(2)$$

Quantum Twisting index

- **Step 1:** find a good labelling for the joint spectrum $\Sigma_\hbar$

$\Sigma_\hbar \ni \lambda_\hbar = (J_\hbar, E_\hbar) \qquad E_\hbar = E_{j,\ell}$



- **Step 2:** recover Eliasson's chart

$f_r(x,y) \qquad (H_r = f_r(J,H)) \qquad z_1 = (x, 0) \qquad z_2 = (\mu x, 0)$

$$\partial_x f_r(0) = \lim_{x\to 0^+} \lim_{\hbar \to 0} \frac{2\pi}{\ln \mu} \left(\frac{E_{j_1,\ell_1} - E_{j_1+1,\ell_1}}{E_{j_1,\ell_1+1} - E_{j_1,\ell_1}} - \frac{E_{j_2,\ell_2} - E_{j_2+1,\ell_2}}{E_{j_2,\ell_2+1} - E_{j_2,\ell_2}} \right)$$

$$\partial_y f_r(0) = \lim_{x\to 0^+} \lim_{\hbar \to 0} \frac{2\pi\hbar}{\ln \mu} \left(\frac{1}{E_{j_1,\ell_1+1} - E_{j_1,\ell_1}} - \frac{1}{E_{j_2,\ell_2+1} - E_{j_2,\ell_2}} \right)$$

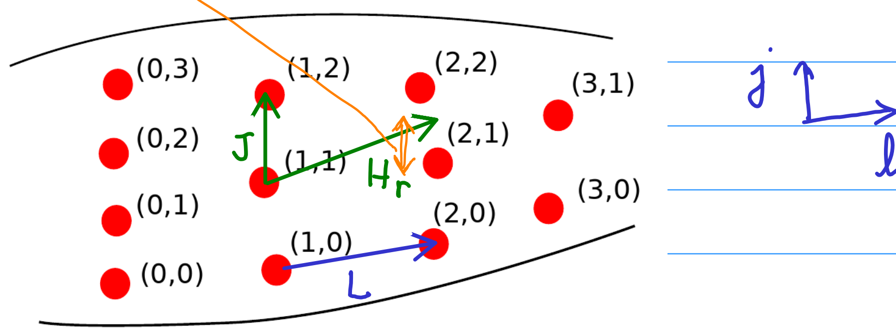
Quantum Twisting index

2

- Step 3 let $s_0 = -\partial_x f_r(0) / \partial_y f_r(0)$ ("radial slope")

Then $$S_{1,0} = \lim_{\hbar \to 0} \lim_{x \to 0} \frac{E_{j,\ell} - E_{j+1,\ell}}{E_{j,\ell+1} - E_{j,\ell}} + \frac{\hbar s_0}{E_{j,\ell+1} - E_{j,\ell}}$$

and $m = \lfloor S_{1,0} \rfloor$
(twisting number)



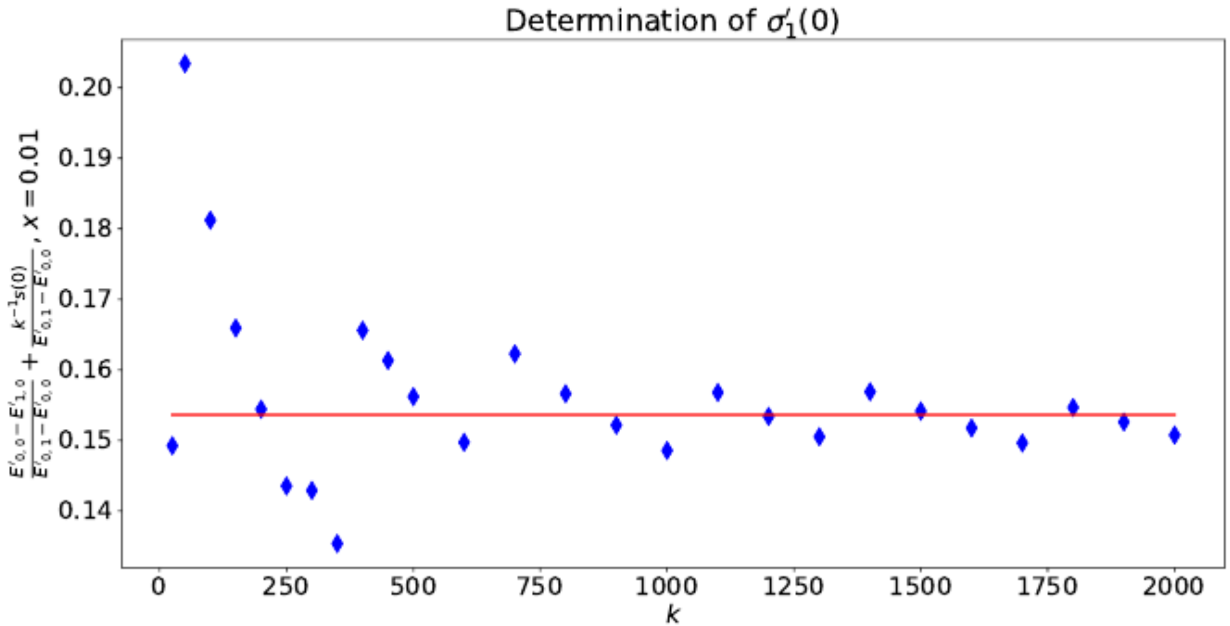


Figure 21: The blue diamonds correspond to Formula (19) evaluated at $(j, k') = (0, 0)$ with $x = 0.01$, for different values of k. The red line corresponds to $\sigma_1'(0) = \frac{1}{2\pi}\arctan(\frac{13}{9})$.

What next ?

- Non-simple semitoric systems (multiple pinches)

  TANG, PALMER, PELAYO

  Would it lead to a counterexample to the inverse problem ?

- Non-proper J (spherical pendulum, cotangent bundles, ...)

  PELAYO - RATIU - VNS, HOHLOCH - SABATINI - SEPE - SYMINGTON, ... ?

  Symplectic classification is still missing

- Higher dimension ? KARSHON - TOLMAN, WACHEUX, ... ?

  " Highly open" ...

  Non degenerate singularities become less generic

That's all. Thank you !