KINETIC NONHOLONOMIC DYNAMICS IS NEITHER HAMILTONIAN NOR VARIATIONAL AND HOWEVER......

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Preprint arXiv: 2010.12444
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Geometry, Dynamics and Mechanics Seminar
December 1, 2020

The aim of the talk:

Given a $\boxed{\text{Kinetic nonholonomic system}}$ with constraint distribution D and a point q of the configuration space Q

$$\Downarrow$$

- $\exists$ M_q^{nh} a submanifold of Q, $q \in M_q^{nh}$, $\dim M_q^{nh} = \operatorname{rank} D$ and
- $\exists$ a family of Riemannian metrics on M_q^{nh} such that the geodesics of every one of these metrics with starting point q are the nonholonomic trajectories with the same starting point

$$\Downarrow$$

The previous nonholonomic trajectories, for sufficiently small times, minimize length in M_q^{nh}!

The key points:

- The nonholonomic exponential map
- A result on the geodesics of Gauss Riemannian metrics on vector spaces

$\boxed{\text{PLAN OF THE TALK}}$

1. Riemannian geometry without constraints
2. Riemannian geometry with constraints
3. Our result and some comments
4. An example
5. Future work

1. Riemannian geometry without constraints

g a Riemannian metric on Q
∇ the Levi-Civita connection of g

1.1 Geodesics as autoparallel curves

$c : I \longrightarrow Q$ a curve on Q

- c is a geodesic $\Longleftrightarrow$ $\dot{c}$ is autoparallel with respect to ∇ $\Longleftrightarrow$ $\nabla_{\dot{c}(t)} \dot{c}(t) = 0$

- c is a geodesic $\Longrightarrow$ $\|\dot{c}(t)\| = \|\dot{c}(t_0)\|$ $\forall\, t \in I$

1.2 Geodesics as the solutions of a variational problem

$\mathcal{C}(q_0, q_1) = \{ c : [0,1] \to Q$ a smooth curve $\mid c(0) = q_0,\ c(1) = q_1 \}$

$\mathcal{A} : \mathcal{C}(q_0, q_1) \longrightarrow \mathbb{R}$ the action functional

$c \longrightarrow \frac{1}{2} \int_0^1 \| \dot{c}(t) \|^2 \, dt$

$\boxed{\begin{array}{l} c \text{ is a critical point of } \mathcal{A} \\ \iff c \text{ is a geodesic} \end{array}}$

1.3 Geodesics as the trajectories of a kinetic mechanical system

The kinetic energy associated with g

$$K_g : TQ \to \mathbb{R}, \quad v_q \in T_qQ \longrightarrow K_g(v_q) = \frac{1}{2}\|v_q\|^2$$

- The Poincaré-Cartan forms on TQ associated with K_g:

$$\alpha_{K_g} = S^*(d K_g), \quad \omega_{K_g} = -d\alpha_{K_g}$$

S the vertical endomorphism on TQ

$$\Downarrow$$

- ω_{K_g} is a symplectic structure on TQ
- The Hamiltonian vector field Γ_g

$$i_{\Gamma_g}\omega_{K_g} = dK_g$$

is a second order differential equation on TQ (its integral curves are tangent lifts of curves on Q)

- The trajectories of Γ_g are just the geodesics of g
- $\Gamma_g \equiv$ the geodesic flow of g

$$\left(q \in Q,\ v_q \in T_qQ \overset{(1)}{\Longrightarrow} \exists!\ \text{geodesic}\right.$$

$$c_{v_q} : \underset{\substack{\Downarrow \\ 0}}{I_{v_q}} \longrightarrow Q \ / \ c_{v_q}(0) = q$$

and $\left. \dot{c}_{v_q}(0) = v_q \right)$

$c_{v_q} : I_{v_q} \to Q$ the geodesic with initial velocity v_q

$$v_q = 0_q \Longrightarrow c_{v_q}(t) = q,\ \forall t \in \mathbb{R}$$

The exponential map at the point q

1) $\exp^g_q : \mathcal{U} \subseteq T_qQ \longrightarrow U \subseteq Q$

$$v_q \in \mathcal{U} \longrightarrow c_{v_q}(1) \in Q$$
$$= \tau_Q(\Phi_1^{\Gamma_g}(v_q))$$

is a diffeomorphism, with $\mathcal{U}$ an starshaped open subset of T_qQ, $0_q \in \mathcal{U}$ and U an open subset of Q, $q \in U$.

$\tau_Q : TQ \longrightarrow Q$ the canonical projection

$\{\Phi_t^{\Gamma_g}\}$ the (local) flow of Γ_g

ii) $\exp^g_q(0_q) = q$

iii) $\exp^g_q(tv_q) = c_{v_q}(t)$, $v_q \in \mathcal{U}$, $t \in [0,1]$

iv) $T_{0_q}\exp^g_q : T_{0_q}\mathcal{U} \simeq T_qQ \longrightarrow T_qQ$

is the identity map.

$$\boxed{v_q \in LL \Rightarrow T_{v_q}LL = \{ (u_q)^{\vee}_{v_q} \,/\, u_q \in T_qQ \} \cong T_qQ \qquad (u_q)^{\vee}_{v_q} = \frac{d}{dt}\Big|_{t=0} (v_q + t u_q)}$$

Radial directions at $v_q \in LL$:

$$R_q(v_q) = \langle (v_q)^{\vee}_{v_q} \rangle \subseteq T_{v_q}LL$$

Gauss Lemma + iv)

$v_q \in LL,\ u_q \in T_qQ \Rightarrow$

$$\left[(\exp^g_q)^*(g) \right](v_q)\left((v_q)^{\vee}_{v_q}, (u_q)^{\vee}_{v_q} \right)$$

$$g(q)(v_q, u_q) = \left[(\exp^g_q)^*(g) \right](0_q)\left((v_q)^{\vee}_{0_q}, (u_q)^{\vee}_{0_q} \right)$$

$$\Downarrow$$

Minimizing properties of geodesics

$v_q \in \mathcal{U} \Rightarrow c_{v_q} : [0,1] \longrightarrow U \subseteq Q$

minimizes the length

$$\ell(c_{v_q}) = \int_0^1 \|\dot{c}_{v_q}(t)\| \, dt = \|v_q\|$$

in (Q, g) between q and $c_{v_q}(1)$

The definition domain of the exponential map is larger

$$D_q^g = \{ v_q \in T_q Q \mid \text{the geodesic } c_{v_q} \text{ is defined, at least, in } [0,1] \}$$

an open subset of $T_q Q$

$\exp_q^g : D_q^g \longrightarrow Q$ is smooth

- Singularities of the exponential map $\exp_q^g : D_q^g \to Q$ and Jacobi fields along geodesics.
- Global minimizing properties of geodesics and conjugate points
- Global theorems in Riemannian geometry

and much more ......

Standard references: Do Carmo's book, O'Neill's book, Abraham-Marsden's book

2. Riemannian geometry with constraints

g a Riemannian metric on Q

$D \hookrightarrow TQ$ the constraint distribution

(a distribution in Q)

More ingredients:

$q_0, q_1 \in Q$

- $\widetilde{\mathcal{C}}(q_0, q_1) = \{ \tilde{c} : [0,1] \to Q \in \mathcal{C}(q_0, q_1) \,/\, \dot{\tilde{c}}(t) \in D(\tilde{c}(t)), \forall t \}$
- $\tilde{c} \in \widetilde{\mathcal{C}}(q_0, q_1) \implies U_{\tilde{c}} = \{ X \in T_{\tilde{c}}\, \mathcal{C}(q_0, q_1) \,/\, X(t) \in D(\tilde{c}(t)), \forall t \}$

2.1 Sub-riemannian geometry

- A true variational problem

$A : \mathcal{C}(q_0, q_1) \longrightarrow \mathbb{R}$ the action functional

$$\boxed{\begin{array}{c} \tilde{c} \in \tilde{\mathcal{C}}(q_0, q_1) \text{ is a solution of the sub-Riemannian problem } (g, D) \\ \Updownarrow \\ dA(\tilde{c})(\tilde{X}) = 0, \quad \forall \tilde{X} \in T_{\tilde{c}} \tilde{\mathcal{C}}(q_0, q_1) \end{array}}$$

- A Hamiltonian problem

The Hamiltonian function on T^*Q

$H_{(g,D)} : T^*Q \longrightarrow \mathbb{R}$

$$H_{(g,D)}(\alpha_q) = \frac{1}{2}\|v_q\|^2, \quad \text{for } \alpha_q \in T_q^*Q$$

with $v_q \in D_q$ satisfying

$$g(q)(v_q, u_q) = \alpha_q(u_q), \quad \forall u_q \in D_q$$

$$\boxed{\text{The normal solutions of the sub-riemannian problem } (g,D) \text{ are the trajectories of the Hamiltonian vector field of } H_{(g,D)}}$$

2.2 Kinetic nonholonomic systems

- Lagrange–D'Alembert principle

$\boxed{\text{A curve } c: I \to Q \text{ is a nonholonomic trajectory if } \nabla_{\dot c(t)} \dot c(t) \in D^{\perp}(c(t)),\ \dot c(t) \in D_{c(t)}}$

- A non-variational problem

$\tilde c \in \tilde{\mathcal{C}}(q_0, q_1)$ is a solution of the kinetic nonholonomic system (g, D)

$$\Updownarrow$$

$$dA(\tilde c)(\tilde X) = 0, \quad \forall \tilde X \in U_{\tilde c}$$

Note that $U_{\tilde c} \neq T_{\tilde c}\tilde{\mathcal{C}}(q_0, q_1)$!

- A non-Hamiltonian problem

$T^D D \to D$ a vector bundle over D

$v_q \in D_q \Rightarrow T^D_{v_q} D = \{ X \in T_{v_q} D \,/\, (T_{v_q} \tau_D)(X) \in D_q \}$

$\tau_D : D \to Q$ the vector bundle projection

$\Downarrow$

$\boxed{\begin{array}{l} (T^D D, \omega_{Kg} | T^D D \times_D T^D D) \text{ is} \\ \text{a symplectic vector bundle} \\ \text{(it is not a symplectic manifold !!)} \end{array}}$

Moreover

- $\exists !\ \Gamma_{(g,D)} \in \Gamma(T^D D)$ ($\Leftrightarrow \Gamma_{(g,D)} \in \mathfrak{X}(D)$) such that

$$(i_{\Gamma_{(g,D)}} \omega_{K_g})|_{T^D D} = dK_g|_{T^D D}$$

- $\Gamma_{(g,D)}$ is a SODE on D

(its integral curves are tangent lifts of curves on Q)

- The trajectories of $\Gamma_{(g,D)}$

$c^{nh}_{v_q} : I \longrightarrow Q$

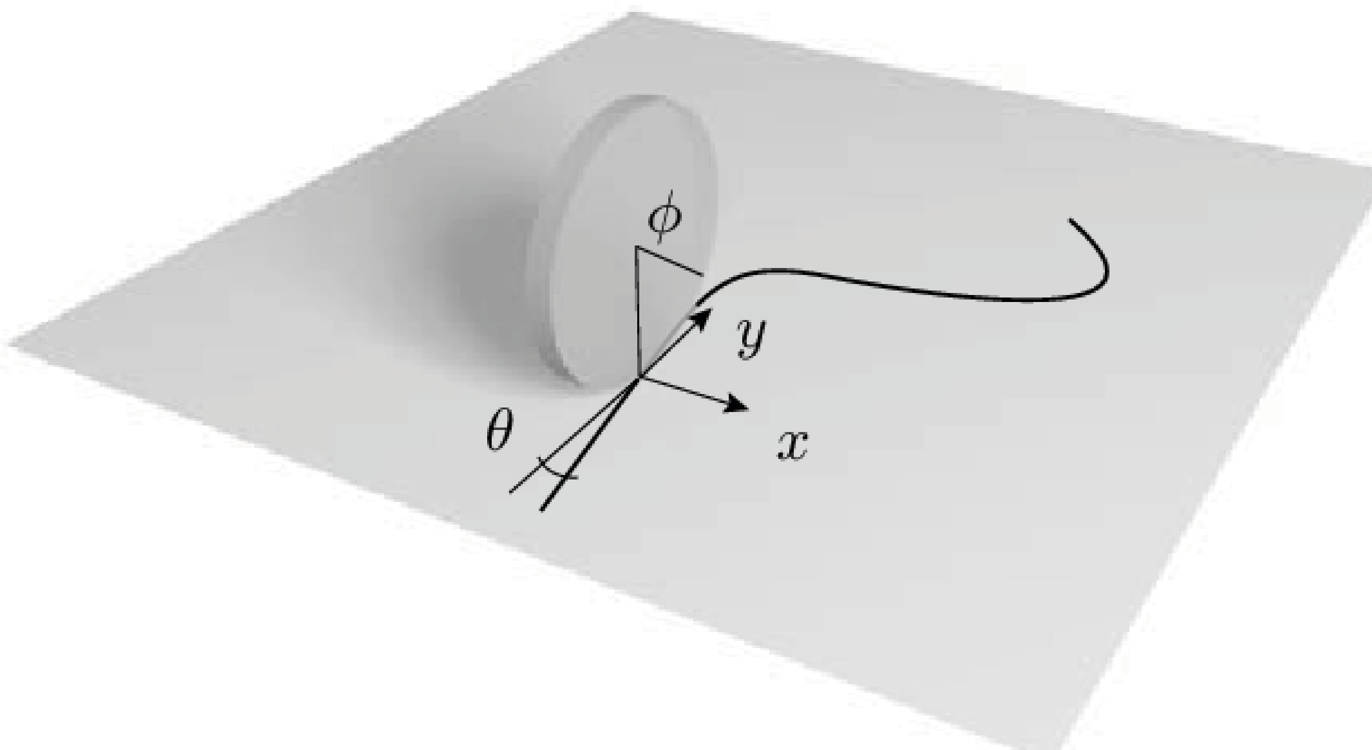
with $v_q \in D_q$ and $\dot{c}^{nh}_{v_q} : I \longrightarrow D$, are just the solutions of the kinetic nonholonomic system.

$$\boxed{\begin{array}{l}\bullet\ \Gamma(g,D) \text{ only preserves the energy} \\ \text{kinetic } K_{g,D}\end{array}}$$

No preservation, in general, of symplectic or Poisson structures or even volume forms

A very simple example:

- The vertical rolling disk



$\boxed{Q = \mathbb{R}^2 \times S^1 \times S^1}$; (x, y, θ, ϕ)

$$\boxed{g = m\,(dx^2 + dy^2) + I\,d\theta^2 + J\,d\phi^2}$$

$m \equiv$ the mass of the disk

$I, J \equiv$ the moments of inertia about an axis perpendicular to the plane of the disk

$$\boxed{D = \{\dot{x} = R(\cos\phi)\dot{\theta},\ \dot{y} = R(\sin\phi)\dot{\theta}\} \hookrightarrow TQ}$$

$R \equiv$ the radius of the disk

D involutive $\Rightarrow$

$$\boxed{\text{the solutions of the sub-riemannian problem and the Kinetic nonholonomic system are just the geodesic of the leaves of the foliation } D}$$

If D is not involutive the solutions of both problems are, in general, different

(Agrachev, Bates-Sniatycki, Bloch, Cantrijn-Sarlet, Cortés, Khesin-Lee, Koiller, Kozlov, León-JCM-Martín de Diego, -Martinet, Lewis-Murray, Marle, Marsden, Montgomery, Ratiu, Zenkov, ...)

3. Our result and some comments

V a real vector space of finite dimension

$\mathcal{U}_0 \subseteq V$ an open subset of V, $0 \in \mathcal{U}_0$

g_0 a Riemannian metric on V

g_0 satisfies Gauss condition on $\mathcal{U}_0$ if

$$g_0(v)\left(v_v^{\mathbf{v}}, u_v^{\mathbf{v}}\right) = g_0(0)\left(v_0^{\mathbf{v}}, u_0^{\mathbf{v}}\right)$$

$\forall v \in \mathcal{U}_0$ and $\forall u \in V$.

Remark: $\bar{g}$ an arbitrary Riemannian metric on V and $\exp_0^{\bar{g}} : \mathcal{U}_0 \subseteq T_0V \xrightarrow{\simeq} U \subseteq V$ the exponential map

$\Downarrow$

$$g_0 = \left(\exp_0^{\bar{g}}\right)^*(\bar{g})$$

is a Riemannian metric on $\mathcal{U}_0$ which satisfies the Gauss condition. In particular, if g is the flat metric induced by a scalar product on V then g satisfies the Gauss condition on $\mathcal{U}_0$

Gauss Lemma + iv)

$v_q \in \mathcal{U},\ u_q \in T_q Q \Rightarrow$

$$[(\exp_q^g)^*(g)](v_q)\big((v_q)^{\mathbf{v}}_{v_q}, (u_q)^{\mathbf{v}}_{v_q}\big)$$

$$= g(q)(v_q, u_q) = [(\exp_q^g)^*(g)](0_q)\big((v_q)^{\mathbf{v}}_{0_q}, (u_q)^{\mathbf{v}}_{0_q}\big)$$

Theorem [A-JCM-M]

(g, D) a kinetic nonholonomic system

$q \in Q$

- First part of the theorem: $\boxed{\exists \text{ a submanifold } M_q^{nh} \text{ of } Q,\ q \in M_q^{nh}, \text{ and a diffeomorphism } \exp_q^{nh} : U_0 \subseteq D_q \longrightarrow M_q^{nh} \subseteq Q}$

with U_0 a starshaped open subset of D_q, $0_q \in U_0$, and $\boxed{\exp_q^{nh}(0_q) = q}$. Moreover,

$$\boxed{\exp_q^{nh}(t v_q) = c_{v_q}^{nh}(t),\ t \in [0,1],\ v_q \in U_0}$$

and

$$\boxed{T_{0_q}\exp_q^{nh} : T_{0_q}U_0 \simeq D_q \longrightarrow T_q Q \text{ is just the canonical inclusion of } D_q \text{ in } T_q Q.}$$

Comments to the first part

- $\exp^{nh}_q$ is just the nonholonomic exponential map defined by

$$\exp^{nh}_q(v_q) = c^{nh}_{v_q}(1) = \tau_D\left(\phi_1^{\Gamma(g,D)}(v_q)\right)$$

$$\Downarrow$$

$$\boxed{\mathcal{M}^{nh}_q = \{ c^{nh}_{v_q}(1) \in Q \mid v_q \in \mathcal{U}_0 \}}$$

- The kinetic nonholonomic dynamics is homogeneous

$$c^{nh}_{tv_q}(s) = c^{nh}_{v_q}(ts) \quad \left(\underset{\boxed{s=1}}{\Rightarrow} c^{nh}_{tv_q}(1) = c^{nh}_{v_q}(t) \right)$$

$\mathcal{M}^{nh}_q$ essentially consists of the radial kinetic nonholonomic trajectories from the point q

$$t \in [0,1] \longrightarrow c^{nh}_{v_q}(t) \in Q$$

$\boxed{\text{Second part of the theorem}}$ All the nonholonomic trajectories departing from the point q are homothetic reparametrizations of the curves

$$t \longrightarrow \exp_q^{nh}(t v_q), \text{ with } v_q \in \mathcal{U}_0$$

Moreover, they are minimizing geodesics for a Riemannian metric g_q^{nh} on M_q^{nh} if and only if the Riemannian metric $g_0 = (\exp_q^{nh})^*(g_q^{nh})$ on $\mathcal{U}_0$ satisfies the Gauss condition.

$\boxed{\text{Comments to the second part}}$

- $\mathcal{U}_0 \subseteq V$ a starshaped open subset of a vector space V, $0 \in \mathcal{U}_0$

  g_0 a Riemannian metric on V

$$\Downarrow$$

g_0 satisfies the Gauss condition on $\mathcal{U}_0$ if and only if, for every $u \in \mathcal{U}_0$, the line

$$t \in [0,1] \longrightarrow tu \in \mathcal{U}_0$$

is a minimizing geodesic on (V, g).

• The third part of the theorem

Riemannian metrics on $\mathcal{M}_q^{nh}$ as in the second part always exist and if g_q^{nh} is one of them then

$$\exp_q^{g_q^{nh}} = \exp_q^{nh} \; !!$$

Comments to the third part

$\mathfrak{g}_0$ a Riemannian metric on D_q which satisfies Gauss condition

$$\Downarrow$$

$$\boxed{g_q^{nh} = \left[\left(\exp_q^{nh}\right)^{-1}\right]^*(\mathfrak{g}_0)}$$

$$\left(\exp_q^{nh}\right)^{-1} : \mathcal{M}_q^{nh} \longrightarrow \mathcal{U}_0 \subseteq D_q$$

Remark: $i_q : \mathcal{M}_q^{nh} \longrightarrow Q$ the canonical inclusion $\Rightarrow$ $\left(\exp_q^{nh}\right)^*(i_q^* g)$ doesn't satisfy, in general, Gauss condition

Suppose that $\boxed{\mathcal{D} \text{ is involutive}}$

$\Downarrow$ The third part of the theorem

$\boxed{\begin{array}{l} M^{nh}_q \text{ is an open subset of} \\ \text{the leaf } L_q \text{ of the foliation } \mathcal{D} \\ \text{by the point } q \end{array}}$

When $\mathcal{D}$ is not involutive, M^{nh}_q plays the same role, in some sense, that L_q

4. An example

The vertical rolling disk (revisited)

$q = 0 \in Q = \mathbb{R}^2 \times S^1 \times S^1$; $D_0 = \mathbb{R}^2$ (u, v)

The nonholonomic exponential map at 0

- $\exp_0^{nh}(u,v) = \left(u \dfrac{\sin v}{v}, u \dfrac{(1-\cos v)}{v}, u, v\right)$
if $(u,v) \in \mathbb{R}^2$, $v \neq 0$
- $\exp_0^{nh}(u,0) = (u, 0, u, 0)$

A Gauss-Riemannian metric on $D_0 \cong \mathbb{R}^2$

$$g_0 = E\,du^2 + F\,du\,dv + G\,dv^2$$

$$E = \frac{2(v^2 - v\sin v - \cos v + 1)}{v^2}$$

$$F = \frac{u(-(v^2-2)\cos v + 2v\sin v - 2)}{v^3}$$

$$G = \frac{u^2v^2 - 2u^2 v\sin v - 2u^2\cos v + 2u^2 + v^4}{v^4}$$

5 Future work

Program:

"To discuss geometric properties of nonholonomic geodesics"

i) $\boxed{\text{To extend the result in this talk for nonholonomic Lagrangian systems of mechanical type}}$

g a Riemannian metric on Q

$V: Q \to \mathbb{R} \in C^\infty(Q)$ the potential energy

$L = K_g + V \circ \tau_Q$ the Lagrangian function

Our ideas:

- To use the nonholonomic Maupertuis principle (Koiller, Bakša)

+

- The result in this talk

A Anahory, JCM, D Martín, work in preparation

ii) Under what conditions does there exist a Riemannian metric g^{nh} on Q such that the nonholonomic trajectories for the kinetic nonholonomic system (g, D) are just the geodesics of g^{nh} with initial velocity in D?

We believe that it is a hard question!

Our motivation:

- There exists a Riemannian metric g^{nh} on $Q = \mathbb{R}^2 \times S^1 \times S^1$ such that the nonholonomic trajectories for the vertical rolling disk are just the geodesics of g^{nh} with initial velocity in the constraint distribution. D

$$\boxed{g^{nh} = dx^2 + dy^2 + 2d\theta^2 + d\phi^2 - \cos\phi\, dx \cdot d\theta - \sin\phi\, dy\, d\theta}$$

In fact

$$g_0 = \left(\exp_0^{nh}\right)^*(g^{nh})$$

• The results in this talk may be considered as the first steps in order to give an answer to the previous question

If g^{nh} exists then a Gauss Riemannian metric on M^{nh}_q is

$$g^{nh}_q = i_q^*(g^{nh})$$

$i_q : M^{nh}_q \longrightarrow Q$ the canonical inclusion

• Relation with the so-called Hamiltonization problem in nonholonomic mechanics

(Balseiro, Borisov, Fedorov, García-Naranjo, Jovanovic, Koiller, Kozlov, Mamaev, JCM, Montaldi, Veselov, Veselova, Yapu, ...)

iii) Given a kinetic nonholonomic system (g, D) and q a point of the configuration space Q,

$\boxed{\text{can we find a bundle metric } g_D \text{ on } D \text{ such that the normal solutions of the subriemannian problem } (g_D, D) \text{ are just the nonholonomic trajectories with starting point } q\text{?}}$

(Comparison between sub-riemannian problems and kinetic nonholonomic systems)

iv) $\boxed{\text{To extend the results in this talk for kinetic nonholonomic systems with affine constraints}}$

To follow the ideas in recent papers by García-Naranjo, Fassò and Sansonetto

Final conclusion:

$\boxed{\text{Integration of } \Gamma_{(g,0)} \Rightarrow \mathcal{M}^{nh}_q, \text{ with } q \in Q}$

$\Downarrow$

Our result is of theoretical nature

Relevance: Potential applications that one could deduce from it

- New geometric integrators for kinetic nonholonomic systems

or as in Riemannian geometry

- Nonholonomic Jacobi fields
- Global minimizing properties of nonholonomic trajectories and conjugate points
- Global results for kinetic nonholonomic systems

THANKS!!