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Material Barriers to Momentum and Vorticity Transport

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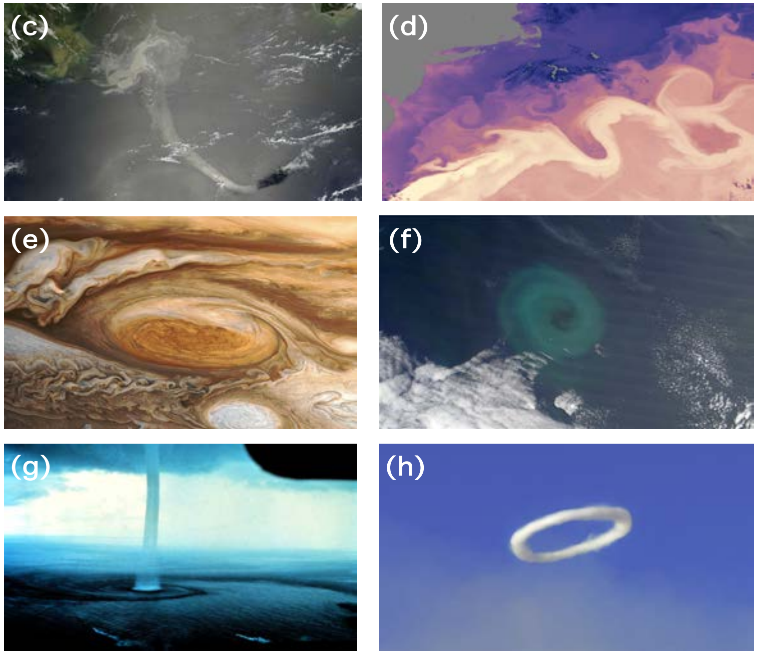
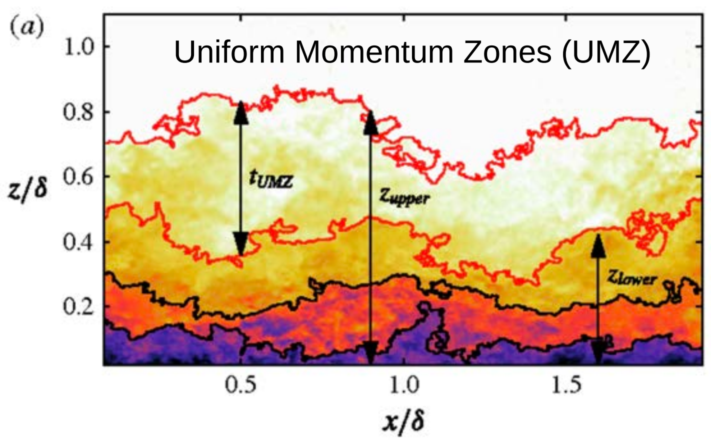
Transport barriers: frequently discussed -- rarely defined



H., *Ann. Rev. Fluid Mech.* [2015]



de Silva, Hutchins & Marusic [2014]

Available results:

(1) Barriers to **advective** transport**:**
Lagrangian coherent structures (LCS)

(2) Barriers to **passive scalar** transport**:**
material barriers to diffusion

H., Karrasch & Kogelbauer, PNAS [2018], SIADS [2020]
Katsanoulis, Farazmand, Serra & H., JFM [2020]

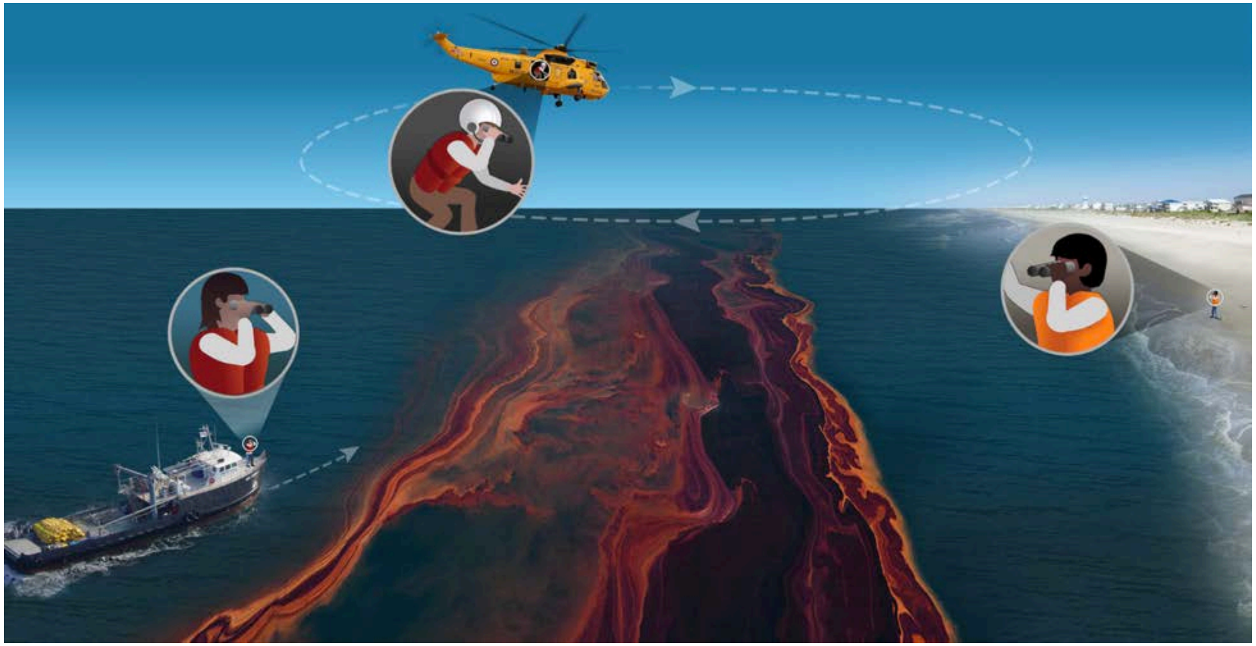
(3) Barriers to **active vectorial** transport**?**
surfaces impeding transport of
momentum, vorticity, …

Requirement: experimentally verifiable
→independent of observer
→theory must be objective (frame-indifferent)

Objectivity: indifference to the observer

"One of the main axioms of continuum mechanics is the requirement that *material response must be independent of the observer*."

M. E. Gurtin, An Introduction to Continuum Mechanics. Academic Press (1981), p. 143



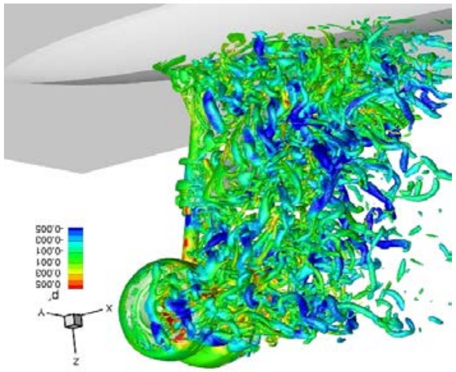
H., Lagrangian Coherent Structures, *Ann. Rev. Fluid Mech.* [2015]

Introduction to Focus Issue: Objective Detection of Coherent Structures

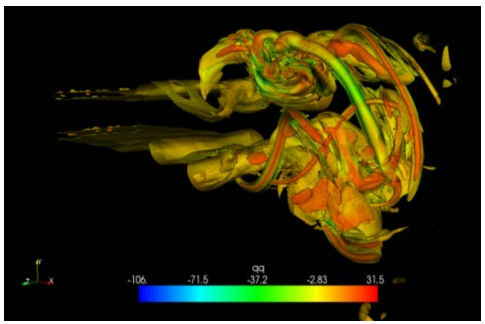
Chaos 25, 087201 (2015); https://doi.org/10.1063/1.4928894

T. Peacock[1, a], G. Froyland[2, b], and G. Haller[3, c]

Classic views on transport barriers (as vortex boundaries) are <u>not</u> objective



NASA



SouVR Co.

$$\mathbf{W} = \tfrac{1}{2}\left(\nabla\mathbf{v} - \left[\nabla\mathbf{v}\right]^T\right), \quad \mathbf{S} = \tfrac{1}{2}\left(\nabla\mathbf{v} + \left[\nabla\mathbf{v}\right]^T\right)$$

Spin Tensor (non-objective) | **Rate of strain tensor (objective)**

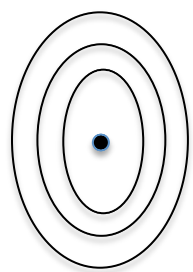
- **Q-criterion:** $Q = \tfrac{1}{2}\left(\|\mathbf{W}\|^2 - \|\mathbf{S}\|^2\right) > 0$
- **Δ-criterion:** $\exists j: \quad \operatorname{Im}\lambda_j(\mathbf{W}+\mathbf{S}) \neq 0$
- **λ_2-criterion:** $\lambda_2\left(\mathbf{W}^2 + \mathbf{S}^2\right) < 0$
- **velocity level sets:** $|\mathbf{v}| = const., \ |v_i| = const.$

Example : Exact linear 2D Navier-Stokes solution

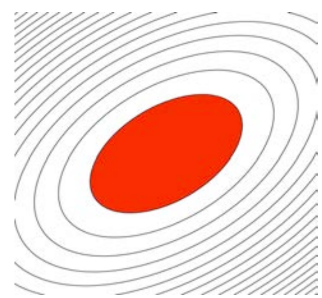
H. [2005], Pedergnana, Oettinger, Langlois & H. [2020]

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x},t) = \begin{pmatrix} \sin 4t & 2+\cos 4t \\ -2+\cos 4t & -\sin 4t \end{pmatrix}\mathbf{x},$$

Coherent vortex by all the above principles



Passive tracers

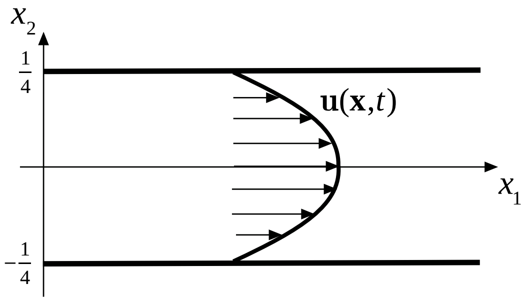


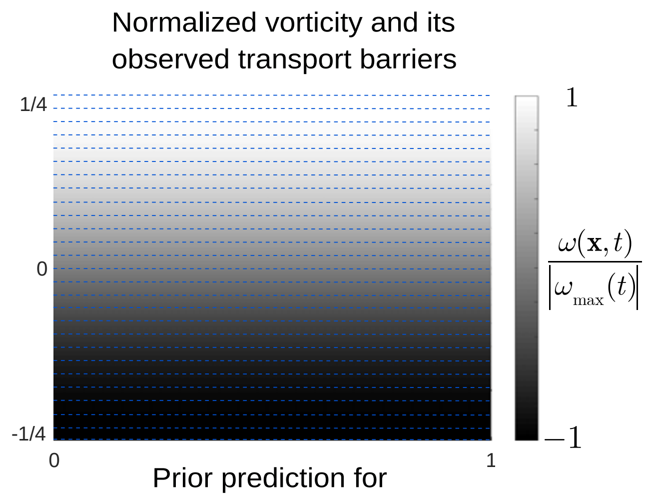
F. J. Beron-Vera

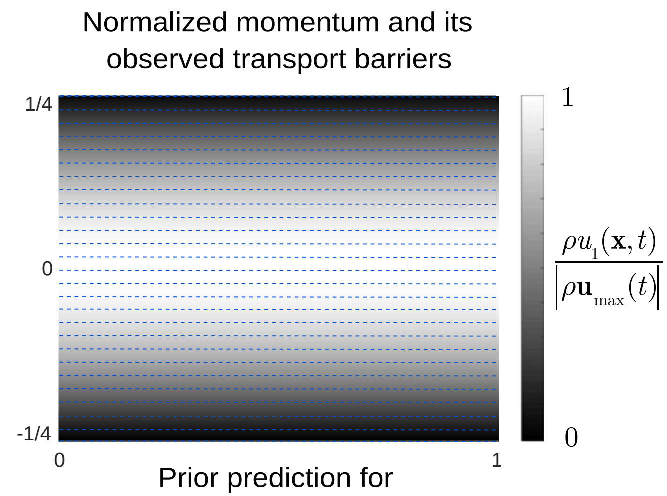
Available results for vorticity and momentum barriers

Example: Decaying 2D channel flow

$$\mathbf{u}(\mathbf{x},t) = e^{-4\pi^2\nu t}\left(a\cos 2\pi x_2, 0\right), \qquad \omega(\mathbf{x},t) = 2\pi a e^{-4\pi^2\nu t}\sin 2\pi x_2$$







Assumptions on the active vector field $\mathbf{f}(\mathbf{x},t)$

- **Consider** general velocity field $\mathbf{v}(\mathbf{x},t)$ solving the momentum equation

$$\rho\dot{\mathbf{v}} = -\nabla p + \nabla\cdot\mathbf{T}_{vis} + \mathbf{q}$$

- compressible, possibly non-Newtonian
- $\mathbf{T}_{vis}(\mathbf{x},t)$: viscous stress tensor
- $\mathbf{q}(\mathbf{x},t)$: external body forces

see, e.g., Gurtin, Fried & Anand [2013]

- **Assume**: - active vector field $\mathbf{f}(\mathbf{x},t)$ satisfies: $\dot{\mathbf{f}} = \mathbf{h}_{vis} + \mathbf{h}_{nonvis}$, $\quad \partial_{\mathbf{T}_{vis}}\mathbf{h}_{nonvis} = \mathbf{0}$

  - $\mathbf{h}_{vis}$ is **objective**: $\quad \mathbf{x} = \mathbf{Q}(t)\mathbf{y} + \mathbf{b}(t) \;\Rightarrow\; \tilde{\mathbf{h}}_{vis} = \mathbf{Q}^T(t)\mathbf{h}_{vis}$.

- **Examples:**

Lin. momentum: $\quad \mathbf{f} := \rho\mathbf{v} \quad \rightarrow \quad \dot{\mathbf{f}} = \nabla\cdot\mathbf{T}_{vis} - \nabla p + \mathbf{q} - \dot{\rho}\mathbf{v}$

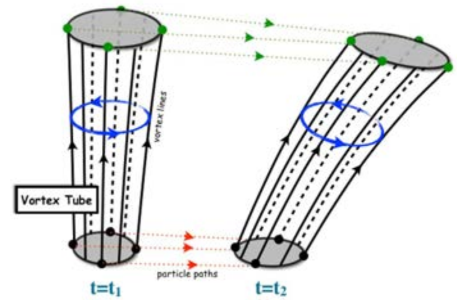
Ang. momentum: $\quad \mathbf{f} := (\mathbf{x}-\hat{\mathbf{x}})\times\rho\mathbf{v} \quad \rightarrow \quad \dot{\mathbf{f}} = (\mathbf{x}-\hat{\mathbf{x}})\times\nabla\cdot\mathbf{T}_{vis} + (\mathbf{x}-\hat{\mathbf{x}})\times\left[\dot{\rho}\mathbf{u} - \nabla p + \mathbf{q}\right]$

Vorticity: $\quad \mathbf{f} := \omega \quad \rightarrow \quad \dot{\mathbf{f}} = \nu\nabla\times\left(\frac{1}{\rho}\nabla\cdot\mathbf{T}_{vis}\right) + \left(\nabla\mathbf{u}\right)\mathbf{f} - \left(\nabla\cdot\mathbf{u}\right)\mathbf{f} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\left(\frac{1}{\rho}\mathbf{q}\right)$

What is the flux of $\mathbf{f}(\mathbf{x},t)$ through a material surface $\mathcal{M}(t)$?

- Vorticity flux: $$\mathrm{Flux}_{\omega}\left(\mathcal{M}(t)\right) = \int_{\mathcal{M}(t)} \omega \cdot \mathbf{n}\, dA$$

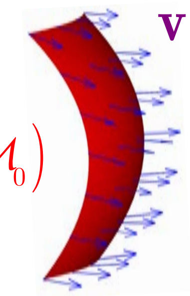
  - ***not*** the physical flux of vorticity (units!)
  - ***not*** objective → vortex tubes are observer-dependent



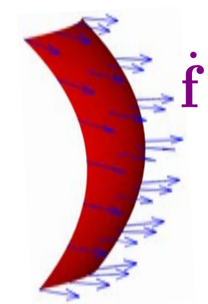
- Momentum flux: $$\mathbf{Flux}_{\rho\mathbf{v}}\left(\mathcal{M}(t)\right) = \int_{\mathcal{M}(t)} \rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}\right) dA$$

  - ***not*** the physical flux of momentum (units!)
  - ***no*** advection through a material surface
  - ***not*** objective

$$\mathcal{M}(t) = \mathbf{F}_{t_0}^{t}\left(\mathcal{M}_0\right)$$



- <u>*Diffusive flux of **f**:*</u> $$\Phi_{\mathbf{f}}\left(\mathcal{M}(t)\right) = \left[\int_{\mathcal{M}(t)} \dot{\mathbf{f}}\cdot\mathbf{n}\, dA\right]_{vis} = \int_{\mathcal{M}(t)} \mathbf{h}_{vis}\cdot\mathbf{n}\, dA$$



  - units **OK**✓
  - **objective**✓

→ *Time-normalized diffusive transport of* **f**: $$\boxed{\psi_{t_0}^{t_1}\left(\mathcal{M}_0\right) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1}\int_{\mathcal{M}(t)} \mathbf{h}_{vis}\cdot\mathbf{n}\, dA\, dt}$$

Active barriers: material surfaces minimizing diffusive transport of ***f***

Theorem 1: $$\psi_{t_0}^{t_1}(\mathcal{M}_0) = \int_{\mathcal{M}_0} \mathbf{b}_{t_0}^{t_1}(\mathbf{x}_0)\cdot \mathbf{n}_0(\mathbf{x}_0)\, dA_0$$

with the objective Lagrangian vector field

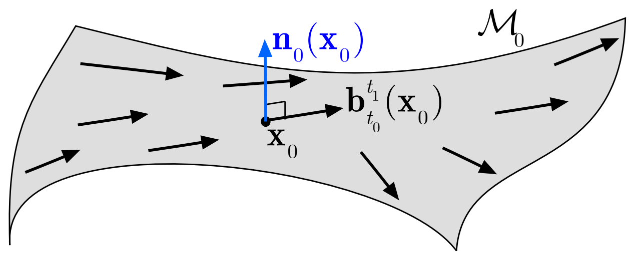
$$\mathbf{b}_{t_0}^{t_1}(\mathbf{x}_0) := \overline{\det \nabla \mathbf{F}_{t_0}^{t} \left(\mathbf{F}_{t_0}^{t}\right)^{*} \mathbf{h}_{vis}}$$

$Notation:$

$$\overline{(\ )} := \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} (\ )\, dt$$

$$\left(\mathbf{F}_{t_0}^{t}\right)^{*} \mathbf{h}_{vis} = \left[\nabla \mathbf{F}_{t_0}^{t}(\mathbf{x}_0)\right]^{-1} \mathbf{h}_{vis}\left(\mathbf{F}_{t_0}^{t}(\mathbf{x}_0), t\right)$$

→ **Perfect active barriers:=**
robust material surfaces with pointwise zero active transport

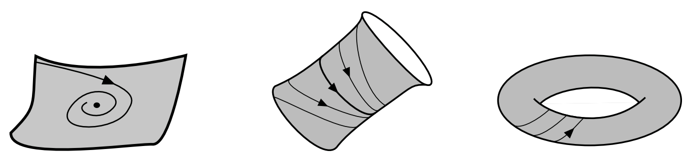


Theorem 2: Active barriers are structurally stable 2D invariant manifolds of

$$\mathbf{x}_0' = \mathbf{b}_{t_0}^{t_1}(\mathbf{x}_0) \qquad \textit{Material (Lagrangian) barrier equation}$$

$$\mathbf{x}' = \mathbf{h}_{vis}(\mathbf{x}; t, \mathbf{v}, \mathbf{f}) \qquad \textit{Instantaneous (Eulerian) barrier equation}$$

- Objective, **steady**, **volume-preserving**
- **Active LCS methods**: passive LCS methods applied to barrier equations



2D stable and unstable manifolds of fixed points

2D stable and unstable manifolds of periodic orbits

2D invariant tori

GH, Katsanoulis, Holzner, Frohnapfel & Gatti, **Objective material barriers to the transport of momentum and vorticity,** JFM, in revision

Example 1: Active barriers in directionally steady 3D Beltrami flows

$$\omega = k(t)\mathbf{v}, \quad \mathbf{v}(\mathbf{x},t) = \alpha(t)\mathbf{v}_0(\mathbf{x}).$$

3D, unsteady, viscous

e.g., unsteady ABC flow:

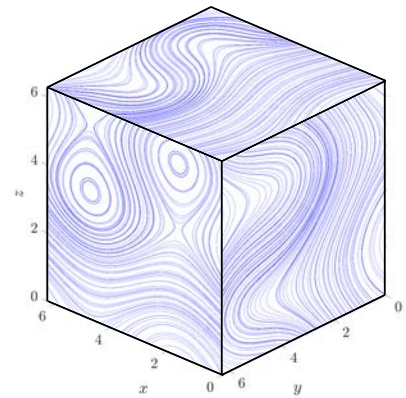
$$\mathbf{v} = e^{-\nu t}\mathbf{v}_0(\mathbf{x}), \qquad \mathbf{v}_0 = (A\sin x_3 + C\cos x_2, B\sin x_1 + A\cos x_3, C\sin x_2 + B\cos x_1)$$

Theorem:
In <u>all</u> directionally steady, 3D Beltrami flows:

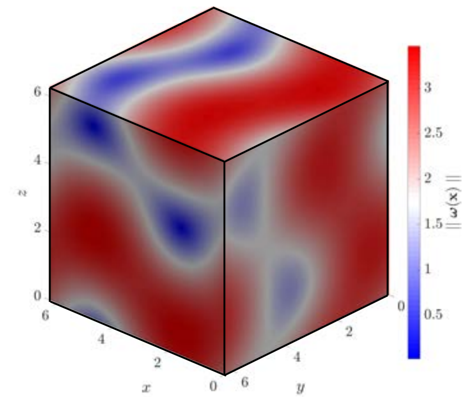
active barriers = classic LCS

$$\mathbf{x}_0' = -\frac{\nu\rho\int_{t_0}^{t_1} k^2\alpha(t)dt}{t_1 - t_0}\mathbf{v}_0(\mathbf{x}_0)$$ *Lagrangian barrier eq.*

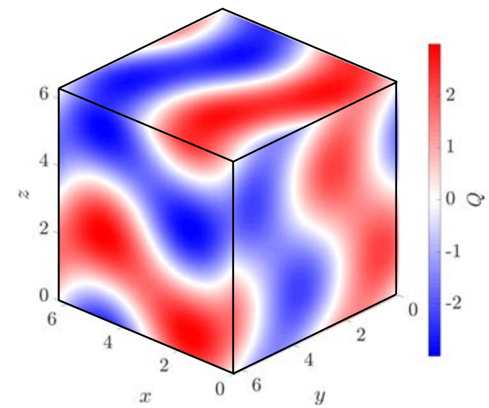
$$\mathbf{x}' = -\nu\rho k^2\alpha(t)\mathbf{v}_0(\mathbf{x})$$ *Eulerian barrier eq.*



sectional streamlines



vorticity norm



values of the Q parameter

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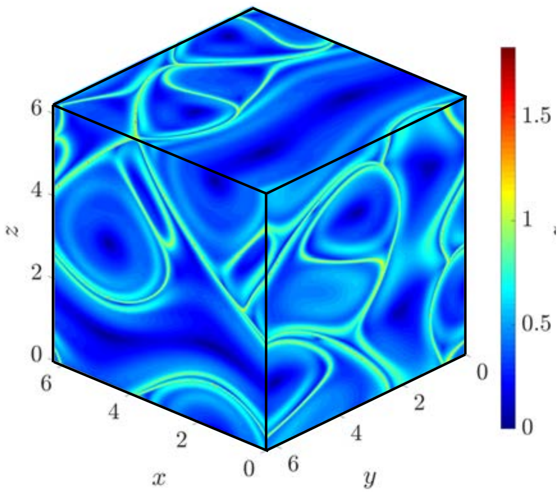
Example 1: Active LCS methods for the ABC flow

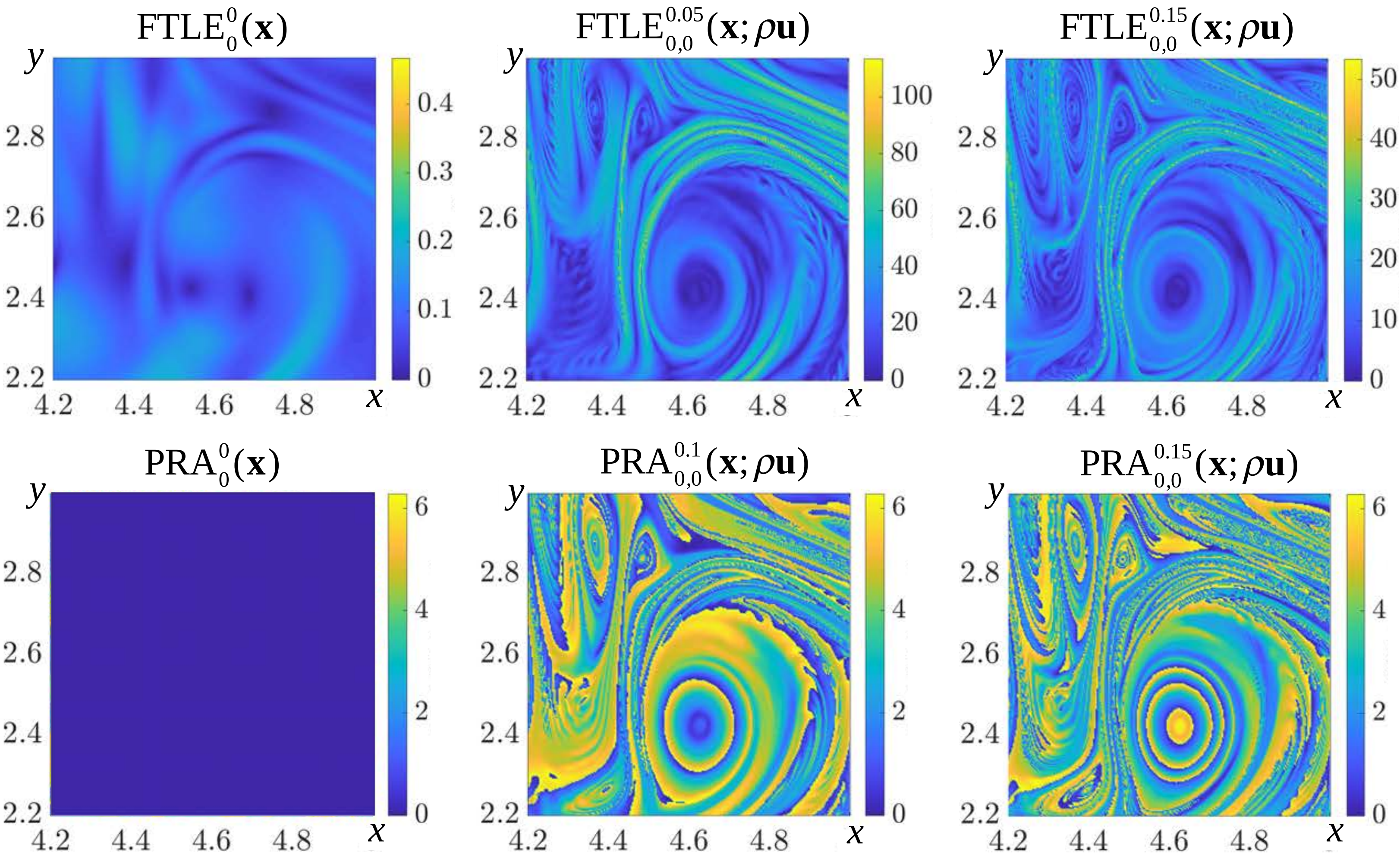
$\mathrm{FTLE}_0^5(\mathbf{x}_0)$



Example 2: Active transport barriers in **2D** incompressible Navier-Stokes flows

data set: Mohammad Farazmand (NCS)

Eulerian momentum barriers at time *t*=0

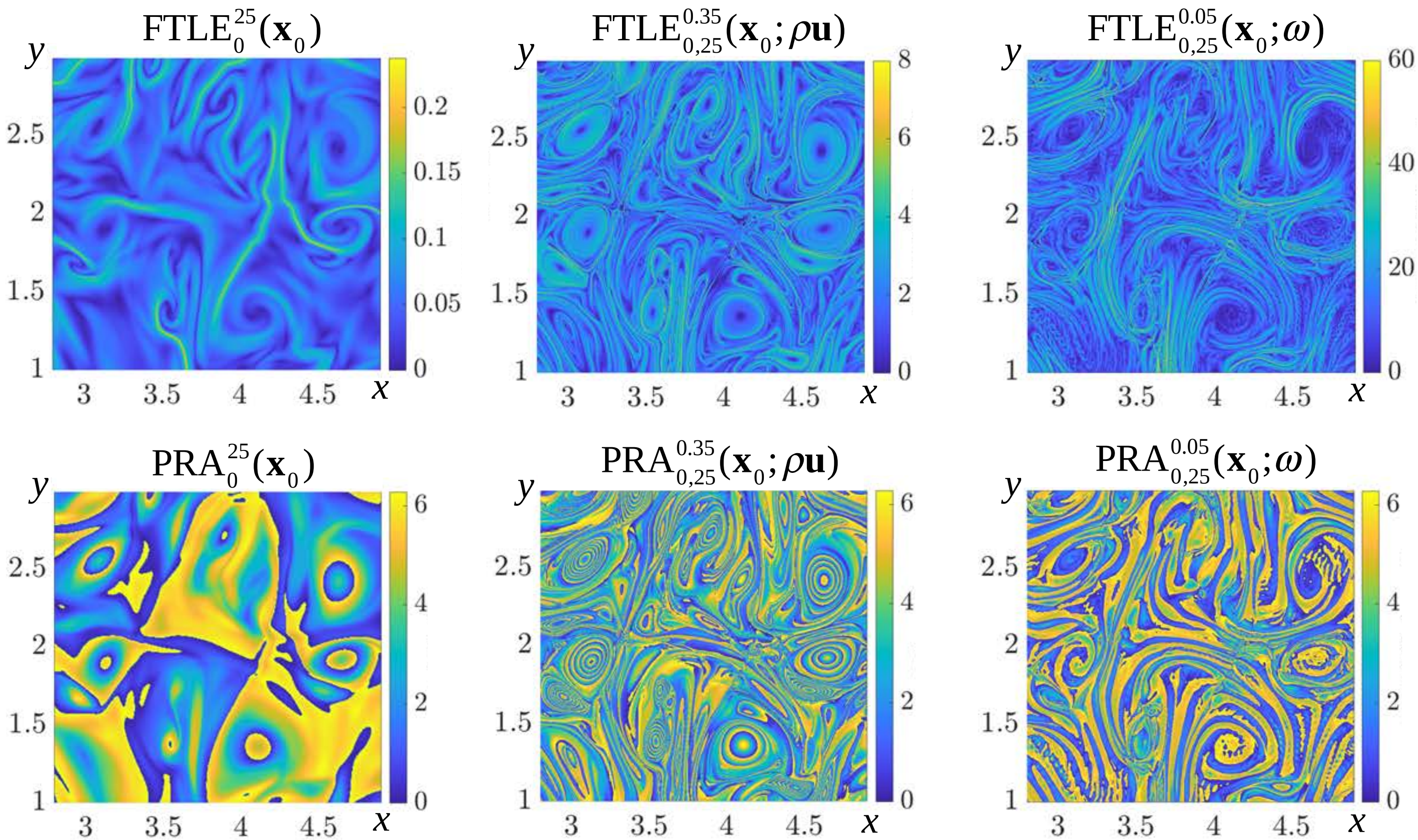


In 2D:
Eulerian momentum barrier eq.
is an
autonomous Hamiltonian system!

$$\mathbf{x}' = \mathbf{J}\nabla H(\mathbf{x}),$$
$$H(\mathbf{x}) = \nu\rho\,\dot{\omega}_z(\mathbf{x};t).$$

GH, Katsanoulis, Holzner, Frohnapfel & Gatti, **Objective material barriers to the transport of momentum and vorticity,** JFM, in revision

Example 2: Lagrangian momentum and vorticity barriers over $[t_0,t_1] = [0,25]$



In 2D:
Lagrangian momentum barrier eq.
is an
autonomous Hamiltonian system!

$$\mathbf{x}' = \mathbf{J}\nabla_0 H(\mathbf{x}_0),$$
$$H(\mathbf{x}_0) = \nu\rho\,\overline{\omega_z(\mathbf{F}_{t_0}^{t}(\mathbf{x}_0),t)}.$$

In 2D:
Lagrangian vorticity barrier eq.
is an
autonomous Hamiltonian system!

$$\mathbf{x}' = \mathbf{J}\nabla_0 H(\mathbf{x}_0),$$
$$H(\mathbf{x}_0) = \nu\rho[\omega_z(\mathbf{F}_{t_0}^{t_1}(\mathbf{x}_0),t_1) - \omega_z(\mathbf{x}_0,t_0)].$$

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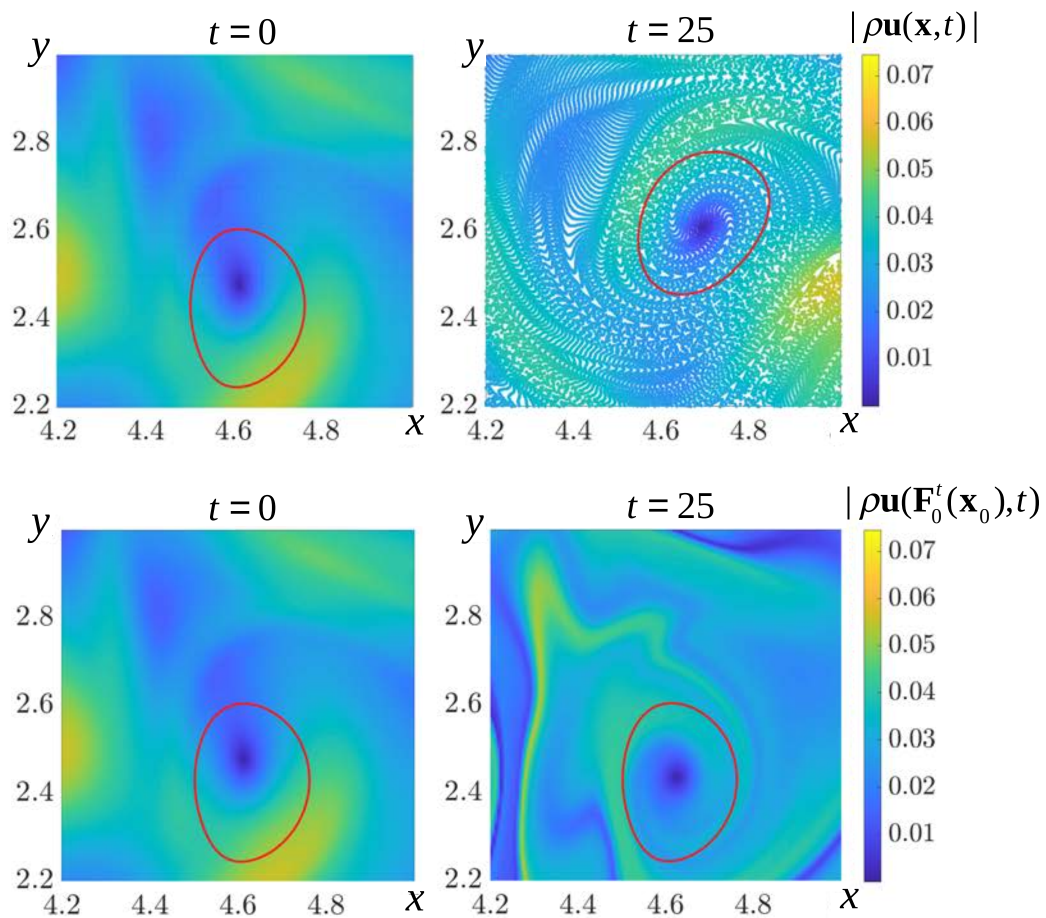
Example 2: Coherence of material barriers to momentum transport

Momentum-barrier evolution and momentum norm

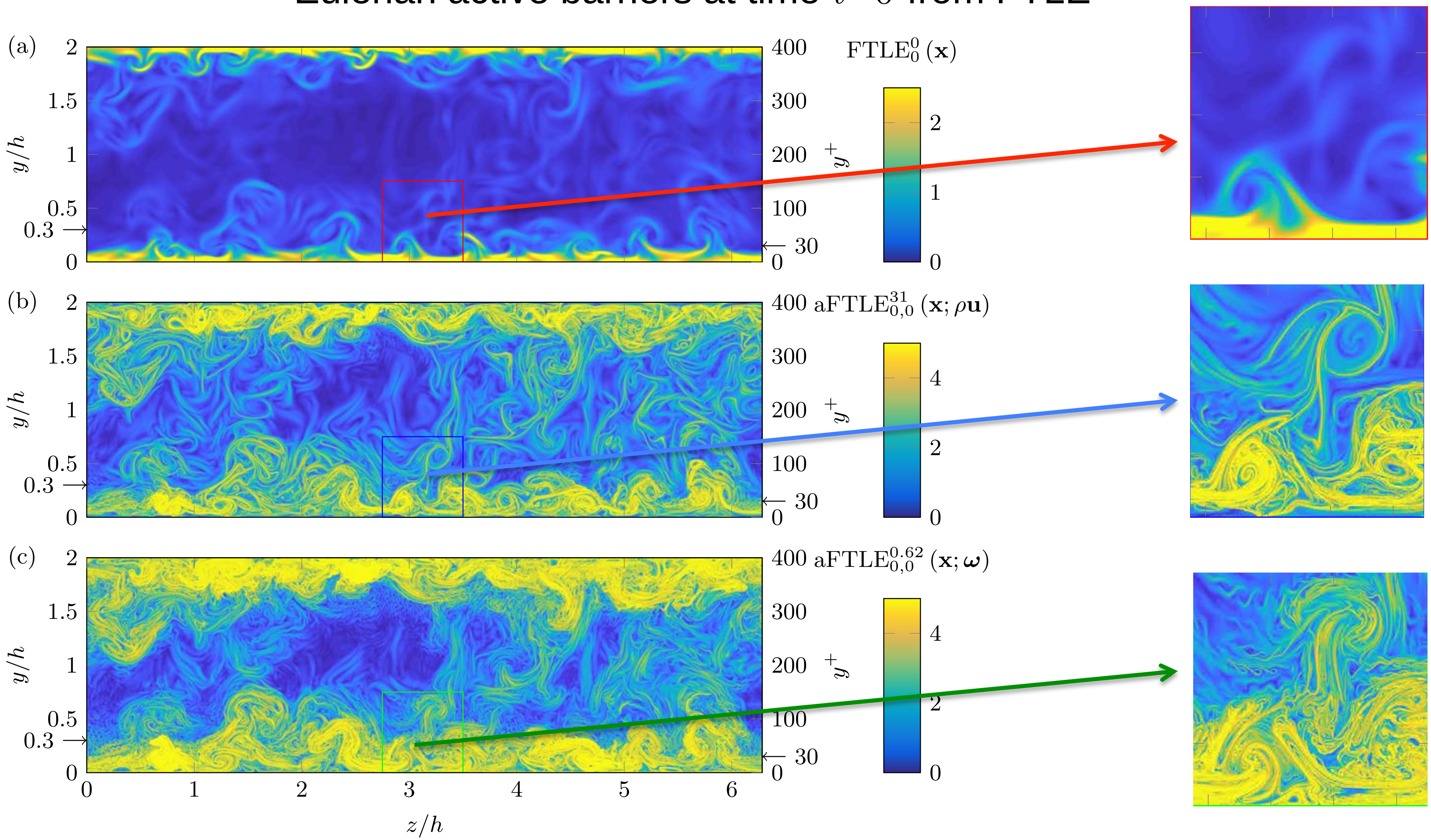
in Eulerian coordinates

in Lagrangian coordinates



Example 3: Active transport barriers in **3D** channel flow (Re=3,000)

Eulerian active barriers at time $t=0$ from FTLE



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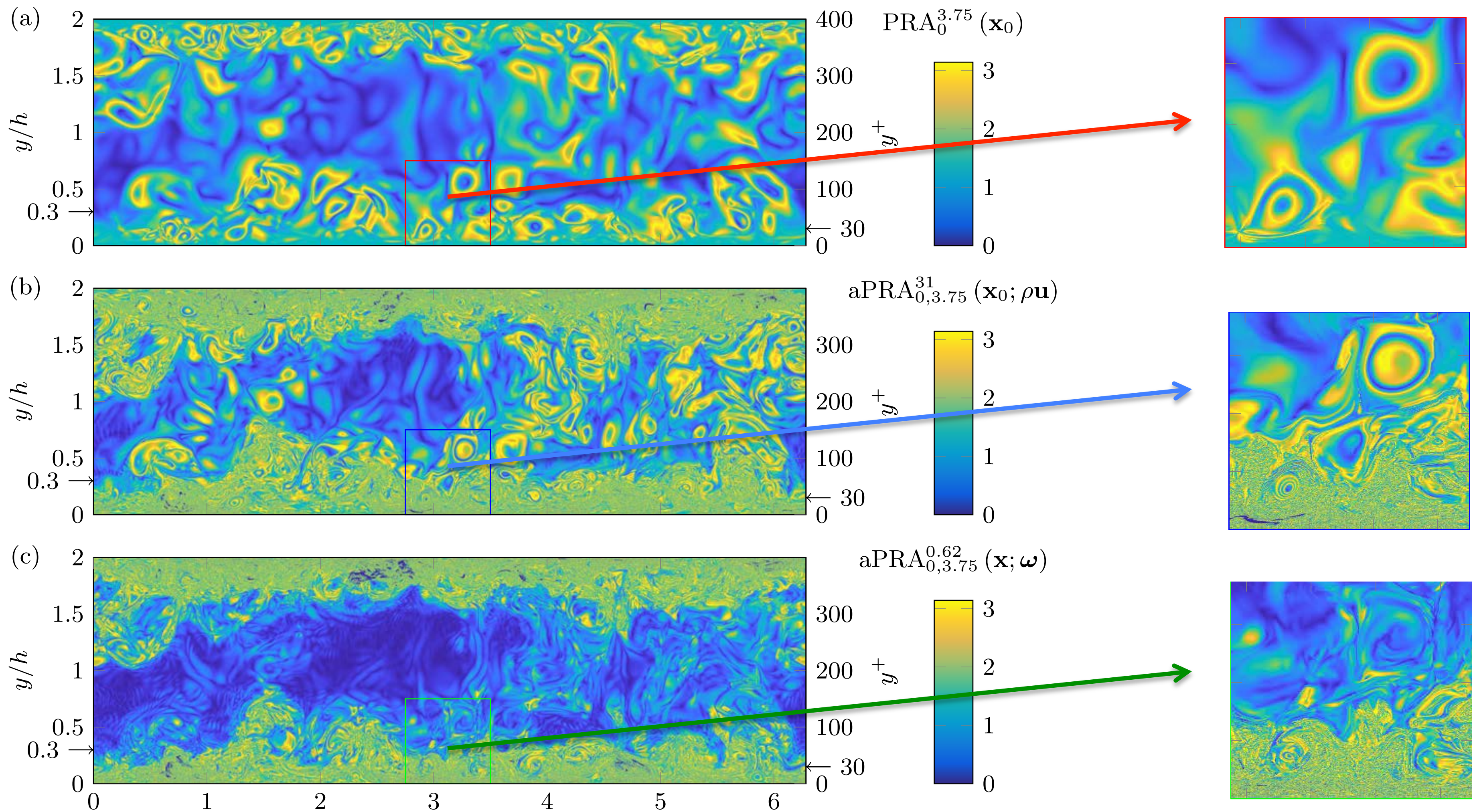
Example 3: Active transport barriers in **3D** channel flow (Re=3,000)

Lagrangian active barriers over $[0, 3.75]$ from PRA



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Conclusions

- Lagrangian and Eulerian active barriers: invariant manifolds of **steady**, **volume-preserving** vector fields (canonical Hamiltonians it 2D).

- Barriers coincide with LCS in directionally steady Beltrami flows

- In more general flows: active barriers differ from LCS

- **Active LCS→** scale-dependent, high-resolution barrier detection

- Need: advanced visualization for invariant manifolds in **3D steady, incompressible flows**