What can Geometric Mechanics do for Climate Science?

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Context: Oceanic heating due to global warming

Our problem statement

The oceans have absorbed 93% of atmospheric heating due to human greenhouse gas emissions.

What will this absorbed heat do to global ocean circulation?

Besides raising sea level, will atmospheric heating change ocean currents?

What will that change in the ocean climate do to the atmospheric climate?

Wait a moment. What is climate?

Our approach

STUOD (Stochastic Transport in Upper Ocean Dynamics)

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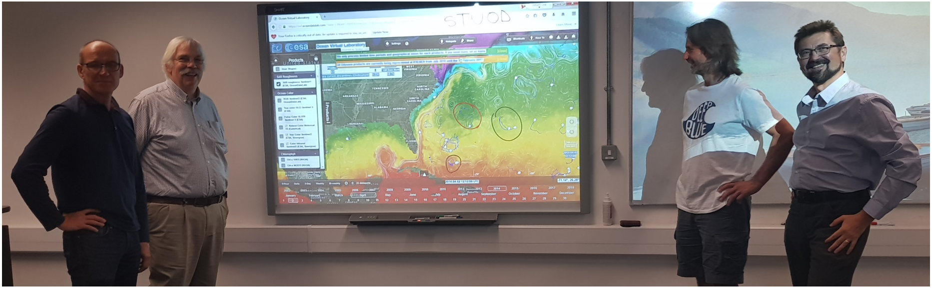
This talk introduces part of the STUOD Synergy Project



Etienne Mémin & Darryl Holm

Bertrand Chapron & Dan Crisan

https://www.imperial.ac.uk/ocean-dynamics-synergy/

We will discuss a single stream of thought

Link ideas $\boxed{\text{Lorenz}} \rightarrow \boxed{\text{Kraichnan}} \rightarrow \boxed{\text{McKean}}$

- *What is climate?* $\boxed{\text{Lorenz}} \rightarrow$ It's what you *expect*, probabilistic.
- *How to make geometric mechanics stochastic?*
  Constrain the variations in reduced Hamilton's principle to follow $\boxed{\text{Kraichnan}} \rightarrow$ stochastic Lagrangian histories.
- *How to derive the dynamics of expectation?*
  Follow $\boxed{\text{McKean}} \rightarrow$ Mean field plus stochastic fluctuations.

Ed Lorenz: Climate is what you expect. (unpublished) (1995)
http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf

DDH: Variational principles for stochastic fluid dynamics.
Proc. R. Soc. A 471(2176), 20140963 (2015)
http://dx.doi.org/10.1098/rspa.2014.0963

Theo Drivas, DDH, James-Michael Leahy: Lagrangian-averaged stochastic advection by Lie transport for fluids. J. Stat. Phys. (2020)
https://doi.org/10.1007/s10955-020-02493-4

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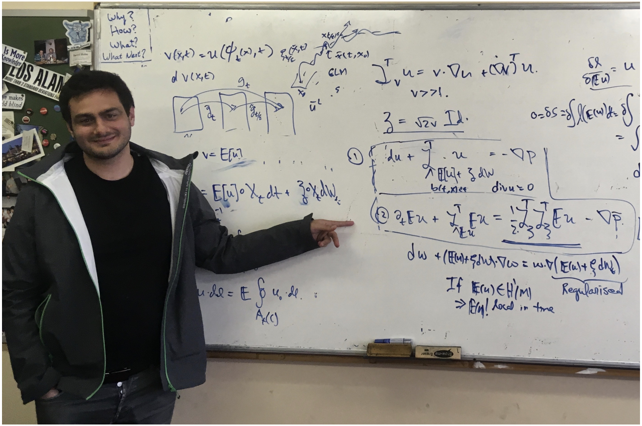
We discuss work with James-Michael Leahy & Theo Drivas

James-Michael Leahy

Theo Drivas



Where are we going in this talk?

1. Ed Lorenz [1995] emphasised that climate is a probabilistic concept.
2. Robert Kraichnan [1959] had postulated stochastic Lagrangian paths!
3. Our problem: Derive fluctuation dynamics around an ensemble-averaged path. Then derive dynamics of the variances.
4. For this, we go "back to basics": What is advection, mathematically?
5. Review role of deterministic advection in Kelvin's Circulation Theorem. Review proof that Kelvin-Noether Theorem $\Leftrightarrow$ Newton's law of motion.
6. Put McKean [1966] mean-field stochastic advection into KN Theorem.
7. We find expectation & fluctuation dynamics separate – variance evolves!
8. Worked examples of LA SALT dynamics: 3D & 2D Euler, Burgers eqn. Ask ourselves, "Does this approach really apply to climate modelling?" For example, "Does it say anything about extreme events?"

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Climate is a probabilistic concept. – Ed Lorenz [1995]

"Climate is what you expect. Weather is what you get." [1]

> *There are many questions regarding climate whose answers remain elusive.*
>
> *For example, there is the question of determinism; was it somehow inevitable at some earlier time that the climate now would be as it actually is?*

Such questions persist as quandaries in the titles of modern papers:

> *On predicting climate under climate change.*
> Daron, J.D. and Stainforth, D.A., 2013.
> Environmental Research Letters, 8(3), p.034021.

[1] Lorenz, E. N., 1995: Climate is what you expect. Unpublished, available at `http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf`
Lorenz, E. N., 1976: Nondeterministic theories of climatic change. Quaternary Research, 6(4), 495-506.

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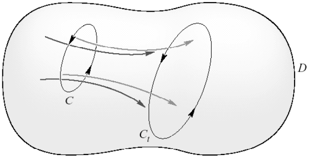
If climate involves expectation, what quantity is stochastic? And how shall we determine its probability distribution?

Here we take a cue from Kraichnan [1959], and propose that *the Lagrangian history of each fluid parcel is Lie-transported by a Stratonovich stochastic vector field*. That is, each history $x_t = \phi_t(x_0)$ is a time dependent diffeomorphic map generated by the stochastic vector field

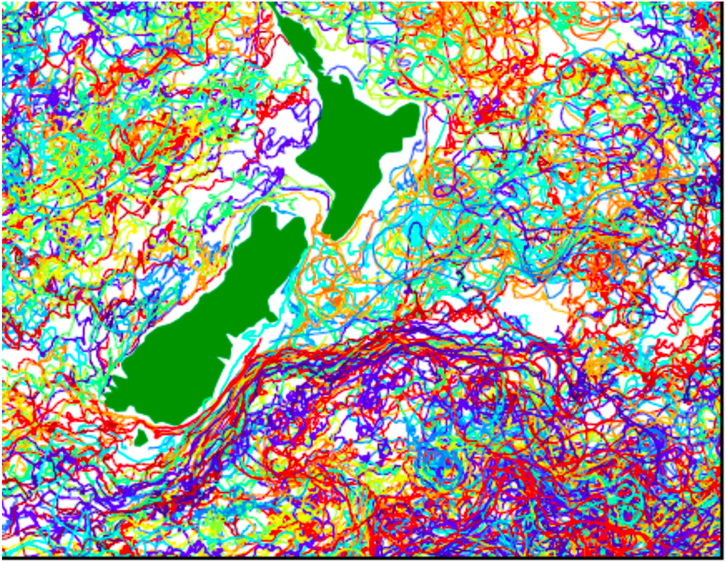
$$\mathbf{d}x_t \overset{\mathfrak{X}}{:=} \underbrace{u_t(x_t)\,\mathrm{d}t}_{\textit{DRIFT VELOCITY}} + \underbrace{\xi(x_t)\circ \mathrm{d}W_t}_{\textit{NOISE}} .$$

Applying this vector field to material loops in the KN thm $\Longrightarrow$ SALT eqns.



The ensemble average will determine the probability distribution, while the determination of the $\xi(x_t)$ must be accomplished from data analysis.

What would a stochastic Lagrangian trajectory look like?

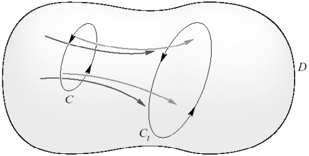


Each Lagrangian path is stochastic. How do we represent the fluctuations away from the ensemble-averaged path?

Suppose histories of fluctuations from the ensemble-averaged path with velocity $\mathbb{E}[u]$ are diffeos $X_t = \Phi_t(X_0)$ generated by stochastic vector field

$$\mathbf{d}X_t \overset{\mathfrak{X}}{:=} \widetilde{u}(X_t, t) := \underbrace{\mathbb{E}[u](X_t, t)\,dt}_{EXPECTED\ \ DRIFT} + \sum_k \underbrace{\xi_k(X_t) \circ dW_k(t)}_{NOISE}, \quad \mathrm{div}\,\widetilde{\mathbf{u}} = 0.$$

Let's substitute this $\widetilde{u}(X_t, t)$ into the material loop in Kelvin's theorem.



The expectation of the drift velocity $\mathbb{E}[u]$ of the stochastic ensemble of pathwise velocities $\{\mathbf{d}x_t\}$ is taken at fixed Lagrangian label on the loop. The loop persists as an ensemble of stochastic paths with a shared expected drift velocity $\mathbb{E}[u]$ since the flow map Φ_t preserves neighbours.

Back to basics: What is fluid advection, mathematically?

According to [Arnold1966], Lagrangian trajectories (histories) are curves on M generated by the action $x_t = \phi_t(x)$ of diffeomorphisms ϕ_t parameterised by time t with $x = \phi_0(x)$ at time $t = 0$.

The *velocity* along the curve is defined as $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) =: u(t, \phi_t(x))$.

Smooth k-form $K(t,x)$ is *advected*, if $\phi_t^* K(t,x) := K(t, \phi_t(x)) = K(0,x)$ where ϕ_t^* is the *pull-back* by ϕ_t. That is, K satisfies an advection equation:

Definition (*Deterministic* Advection by Lie Transport (DALT))

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^* K)(t,x) := \frac{\mathrm{d}}{\mathrm{d}t}K\big(t, \phi_t(x)\big) = \phi_t^*\Big(\partial_t K(t,x) + \mathcal{L}_u K(t,x)\Big) = 0\,.$$

Thus, advection is Lie transport.

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Examples of *Deterministic* Advection by Lie Transport

Definition (Lie derivative is defined via the chain rule)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\phi_t^* K)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} K(\phi_t(x)) =: \mathcal{L}_u K(x)$$

$$\text{with } \frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t}\Big|_{t=0} = u(x).$$

Example (Familiar examples from fluid dynamics:)

(Functions) $(\partial_t + \mathcal{L}_u) b(\mathbf{x}, t) = \partial_t b + \mathbf{u}\cdot\nabla b\,,$

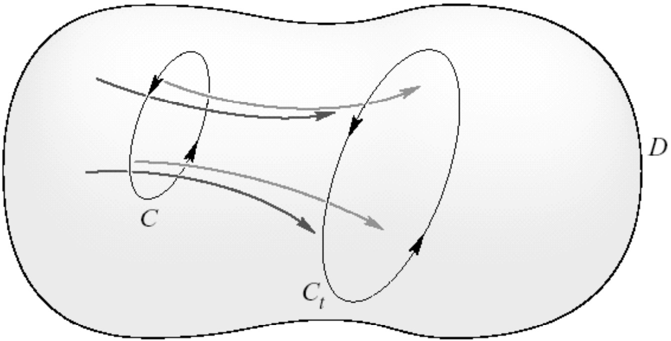
(1-forms) $$(\partial_t + \mathcal{L}_u)(\mathbf{v}(\mathbf{x},t)\cdot d\mathbf{x}) = \big(\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\,\mathbf{v} + v_j \nabla u^j\big)\cdot d\mathbf{x}$$
$$= \big(\partial_t \mathbf{v} - \mathbf{u}\times \mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x} =: \big(\partial_t \mathbf{v} + \mathcal{L}_u^T \mathbf{v}\big)\cdot d\mathbf{x}\,,$$

(2-forms) $(\partial_t + \mathcal{L}_u)(\boldsymbol{\omega}(\mathbf{x},t)\cdot d\mathbf{S}) = \big(\partial_t \boldsymbol{\omega} - \mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}\,,$

(3-forms) $(\partial_t + \mathcal{L}_u)(\rho(\mathbf{x},t)\, d^3x) = (\partial_t \rho + \mathrm{div}\,\rho\,\mathbf{u})\, d^3x\,.$

The *deterministic* Kelvin-Noether theorem coincides with Newton's law for the evolution of (momentum/mass) $\mathbf{v}$ concentrated on an **advecting material loop**, $c_t = \phi_t c_0$ at velocity $\mathbf{u}$,

$$\frac{d}{dt}\oint_{c_t} \mathbf{v}\cdot d\mathbf{x} = \oint_{c_t} \underbrace{(\partial_t + \mathcal{L}_u)(\mathbf{v}\cdot d\mathbf{x})}_{\text{Chain rule}} = \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$



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Proof of the *deterministic* Kelvin-Noether theorem

Proof.

Consider a closed loop moving with the material flow $c_t = \phi_t c_0$.
Its Eulerian velocity is $\frac{d}{dt}\phi_t(x) = \phi_t^* u(t,x) = u(t,\phi_t(x))$.
Compute the time derivative of the loop momentum/mass (impulse)

$$\frac{d}{dt}\oint_{c_t} \mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x} = \oint_{c_0} \frac{d}{dt}\Big(\phi_t^*\big(\mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x}\big)\Big)$$

$$= \oint_{c_0} \underbrace{\phi_t^*\Big(\big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x})\Big)}_{\text{Lie derivative via chain rule}}$$

$$= \oint_{\phi_t c_0 = c_t} \big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x})$$

$$= \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}} = \oint_{c_0} \phi_t^*\Big(\underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Motion eqn}}\Big)$$

- Kelvin-Noether theorem $\Leftrightarrow$ Newton's Law for mass distributed on a material loop.
- KIW theorem: the proof does not change for Stratonovich stochastic vector fields. □

This ends our review of ideal deterministic fluid mechanics

End DALT (Deterministic Advection by Lie Transport).

Begin SALT (Stochastic Advection by Lie Transport).

A Stochastic Kelvin-Noether theorem exists for Lagrangian Averaged (LA) Drift Velocity (McKean [1966])

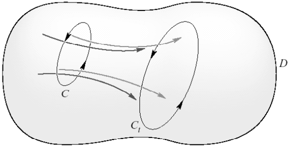
Suppose the divergence-free advection velocity is given by the following stochastic vector field with a Lagrangian Averaged drift velocity:

$$\mathbf{d}X_t \overset{\mathfrak{X}}{:=} \widetilde{u}(X_t) := \underbrace{\mathbb{E}\,[u]\,(x,t)\,dt}_{\text{EXPECTED DRIFT}} + \sum_k \underbrace{\xi_k(x) \circ dW_k(t)}_{\text{NOISE}}, \quad \operatorname{div}\widetilde{\mathbf{u}} = 0.$$

Let $\mathbf{v}$ = momentum/mass. (In Hamilton's principle, $\mathbf{v} = D^{-1}\delta\ell/\delta\mathbf{u}$.)

The **stochastic Kelvin-Noether theorem** represents **Newton's law** for the evolution of momentum concentrated on an advecting loop

$$\mathbf{d}\oint_{c(\widetilde{u})} \mathbf{v}\cdot d\mathbf{x} = \oint_{c(\widetilde{u})} \underbrace{(\mathbf{d} + \mathcal{L}_{\widetilde{u}})(\mathbf{v}\cdot d\mathbf{x})}_{\text{By KIW formula}} = \oint_{c(\widetilde{u})} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$



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We need a final definition before writing the LA SALT eqns

Definition (The diamond operation)

The operation $\diamond : V \times V^* \to \mathfrak{X}^*$ between tensor space elements $a \in V^*$ and $b \in V$ produces an element of $\mathfrak{X}(\mathcal{D})^*$, a one-form density, defined by

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = - \int_{\mathcal{D}} b \cdot \mathcal{L}_u \, a =: \Big\langle b \,,\, -\mathcal{L}_u \, a \Big\rangle_V \,.$$

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Introducing the Lagrangian Averaged (LA) SALT equations

The SALT Euler–Poincaré equations read, with $(\frac{\delta\ell}{\delta u} \in \mathfrak{X}^*, \; \frac{\delta\ell}{\delta a} \in V)$,

$$\mathbf{d}\frac{\delta\ell}{\delta u} + \mathcal{L}_{\mathbf{d}x_t}\frac{\delta\ell}{\delta u} \overset{\mathfrak{X}^*}{=} \frac{\delta\ell}{\delta a} \diamond a \, \mathbf{d}t \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}x_t} a \overset{V^*}{=} 0,$$

where $\mathbf{d}x_t := u_t(x_t)\,\mathrm{d}t + \xi(x_t) \circ \mathrm{d}W_t$ is a stochastic transport vector field. Replace (à la McKean) the Eulerian drift velocity $u_t(x_t)$ in the stochastic transport vector field $\mathbf{d}x_t$ by its expectation, denoted as $\mathbb{E}\,[u_t]\,(X_t)$, so that

$$\mathbf{d}X_t \overset{\mathfrak{X}}{:=} \mathbb{E}\,[u_t]\,(X_t)dt + \sum_k \xi^{(k)}(X_t) \circ dW_t^{(k)}$$

and let's consider the following 'modified' Euler–Poincaré equations

$$\mathbf{d}\frac{\delta\ell}{\delta u} + \mathcal{L}_{\mathbf{d}X_t}\frac{\delta\ell}{\delta u} \overset{\mathfrak{X}^*}{=} \mathbb{E}\left[\frac{\delta\ell}{\delta a}\right] \diamond a \,\mathrm{d}t \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}X_t} a \overset{V^*}{=} 0\,.$$

- The above equations comprise the class of LA SALT theories.

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LA SALT preserves the SALT Lie–Poisson operator

We pass from the Lie–Poisson form of the SALT equations to the corresponding LA SALT form by modifying variational derivatives to $\mathbb{E}\,[\,\cdot\,]$

$$\frac{\delta(\mathbf{d}h)}{\delta\mu} = \mathbf{d}X_t = \mathbb{E}\left[\frac{\delta h}{\delta\mu}\right]\mathrm{d}t + \sum_k \xi^{(k)} \circ \mathrm{d}W_t^{(k)} \quad \text{and} \quad \mathbb{E}\left[\frac{\delta H}{\delta a}\right] = -\,\mathbb{E}\left[\frac{\delta \ell}{\delta a}\right].$$

Taking these expectations transforms the LA SALT equations from their 'modified' Euler–Poincaré form above into their 'Lie–Poisson form',

$$\mathbf{d}\begin{bmatrix}\mu \\ a\end{bmatrix} = -\begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a \\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\mathbb{E}\,[\delta h/\delta\mu]\,dt + \sum_k \xi^{(k)} \circ \mathrm{d}W_t^{(k)} \\ \mathbb{E}\,[\delta h/\delta a]\,dt\end{bmatrix}.$$

The Lie–Poisson operators for DALT, SALT and LA SALT are shared.

- Although they share the same Casimirs and Lagrangian invariants, the LA SALT equations are neither variational nor Hamiltonian. That is, they are not a mean-field theory in the sense of McKean.

The LA SALT expectation dynamics separates & closes

Upon converting LA SALT from Stratonovich into Itô form, we find

$$\mathbf{d}\mu + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mu dt + \mathcal{L}_{\xi^{(k)}}\mu dW_t^{(k)} = \Big(\frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu)dt - \mathbb{E}\left[\frac{\delta H}{\delta a}\right]\diamond a\Big)dt\,,$$

$$\mathbf{d}a + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}a dt + \mathcal{L}_{\xi^{(k)}}a dW_t^{(k)} = \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a)dt\,.$$

Taking the expectation of these equations yields a $\boxed{\textit{PDE sub-system}}$,

$$\partial_t\mathbb{E}\left[\mu\right] + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mathbb{E}\left[\mu\right] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}\left[\mu\right]) = -\,\mathbb{E}\left[\frac{\delta H}{\delta a}\right]\diamond\mathbb{E}\left[a\right]\,,$$

$$\partial_t\mathbb{E}\left[a\right] + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mathbb{E}\left[a\right] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}\left[a\right]) = 0\,.$$

• This sub-system of PDEs for the expectation variables can be **closed** in certain cases of physical interest, some of which we will discuss later.

Fluctuation dynamics can be *linear* and *closed*

The fluctuation variables are defined as

$$\mu' := \mu - \mathbb{E}[\mu] \in \mathfrak{X}^*, \quad a' := a - \mathbb{E}[a] \in V.$$

Taking the difference between the Itô forms and the expectation equations yields the Itô fluctuation equations

$$\mathrm{d}\mu' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}\mu' dt + \mathcal{L}_{\xi^{(k)}}\mu\, dW_t^{(k)} = \Big(\frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu') - \mathbb{E}\left[\frac{\delta h}{\delta a}\right]\diamond a'\Big)dt,$$

$$\mathrm{d}a' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}a' dt + \mathcal{L}_{\xi^{(k)}}a\, dW_t^{(k)} = \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a')dt\,.$$

- When $\delta h/\delta a$ & $\delta h/\delta \mu$ are linear or constant, these equations are **closed**.
- We will pair these two equations with corresponding dual variables to obtain stochastic evolution equations for the resulting quadratic quantities.

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Let's have an interim summary!

The *nonlocality in probability space* (à la McKean) in the LA SALT equations simplifies the dynamics of SALT in three significant ways.

(1) While the Casimirs are still preserved by the full LA SALT dynamics, the equations for the expected physical variables separate into a *dissipative sub-system* embedded into the larger conservative system.

(2) In many cases (including for the LA SALT incompressible Euler fluid) the fluctuation equations are *linear stochastic equations* whose solutions are transported and accelerated by forces involving the expected variables.

(3) In some cases, such as the 2D LA SALT Euler–Boussinesq (EB) equations, this linear stochastic transport property implies *unique global existence*, which is not possessed by the corresponding SALT equations.

(Existence is not discussed here, for lack of time.)

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A special case of the LA SALT Euler eqn is Navier-Stokes.

The LA SALT Euler equation is given as

$$\mathbf{d}u_t + \mathcal{L}^T_{\mathbb{E}[u_t]} u_t dt + \sum_k \mathcal{L}^T_{\xi^{(k)}} u_t \circ dW_t^{(k)} = (-\mathbb{E}[\nabla p_t] + f_t)\mathbf{d}t,$$

with $\operatorname{div} \mathbb{E}\,[u_t] = 0$, $u_t|_{t=0} = u_0(x)$ and $(\mathcal{L}^T_w u_t)_i := w^j \partial_j u_i + (\partial_i w^j) u^j$.

The Itô formulation is, with $\mathrm{div}\, u_t = 0$,

$$\mathbf{d}u_t + \mathcal{L}^T_{\mathbb{E}[u_t]} u_t dt + \sum_k \mathcal{L}^T_{\xi^{(k)}} u_t dW_t^{(k)} = \Big(\frac{1}{2} \sum_k \mathcal{L}^T_{\xi^{(k)}} (\mathcal{L}^T_{\xi^{(k)}} u_t) - \mathbb{E}\,[\nabla p_t] + f_t \Big) dt.$$

Taking the expectation yields a closed equation for $\mathbb{E}\,[u_t]$ given by

$$\partial_t \mathbb{E}\,[u_t] + \mathcal{L}^T_{\mathbb{E}[u_t]} \mathbb{E}\,[u_t] = \frac{1}{2} \sum_k \mathcal{L}^T_{\xi^{(k)}} \big(\mathcal{L}^T_{\xi^{(k)}} \mathbb{E}\,[u_t] \big) - \mathbb{E}\,[\nabla p_t] + f_t\,.$$

This is the *Lie-Laplacian Navier-Stokes equation (LL NS) for $\mathbb{E}\,[u_t]$*.

Theorem (Well-posedness of LL NS)

When LL NS is well-posed, then so is its linear Itô fluctuation equation.

Recall the fluctuation dynamics equations

The fluctuation variables are defined as

$$\mu' := \mu - \mathbb{E}[\mu] \in \mathfrak{X}^*, \quad a' := a - \mathbb{E}[a] \in V.$$

Taking the difference between the Itô forms and the expectation equations yields the Itô fluctuation equations

$$\mathrm{d}\mu' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}\mu' dt + \mathcal{L}_{\xi(k)}\mu\, dW_t^{(k)} = \Big(\frac{1}{2}\sum_k \mathcal{L}_{\xi(k)}(\mathcal{L}_{\xi(k)}\mu') - \mathbb{E}\left[\frac{\delta h}{\delta a}\right]\diamond a'\Big)dt,$$

$$\mathrm{d}a' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}a' dt + \mathcal{L}_{\xi(k)}a\, dW_t^{(k)} = \frac{1}{2}\sum_k \mathcal{L}_{\xi(k)}(\mathcal{L}_{\xi(k)}a')dt\,.$$

- When $\delta h/\delta a$ & $\delta h/\delta\mu$ are linear or constant, these equations are **closed**.
- We now pair these two equations with corresponding dual variables to obtain stochastic evolution equations for the resulting quadratic quantities.

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Fluctuation *variance* dynamics depend on correlations

One takes expectation and integrates in space to find *variance dynamics*

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|\mu'|^2_{\mathfrak{X}}\right] - \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}\mu'\right], \mathbb{E}\left[\frac{\delta H}{\delta\mu}\right]\right\rangle_{\mathfrak{X}} + \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}a'\right], \mathbb{E}\left[\frac{\delta H}{\delta a}\right]\right\rangle_{\mathfrak{X}}$$

$$= -\frac{1}{2}\sum_k \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}(\mathcal{L}_{\xi^{(k)}}\mu') + \mathcal{L}_{\left(\mathcal{L}_{\xi^{(k)}}\mu\right)^\sharp}\mu\right], \xi^{(k)}\right\rangle_{\mathfrak{X}},$$

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|a'|^2_V\right] - \left\langle \mathbb{E}\left[\widehat{a'}\diamond a\right], \mathbb{E}\left[\frac{\delta H}{\delta\mu}\right]\right\rangle_{\mathfrak{X}}$$

$$= -\frac{1}{2}\sum_k \left\langle \mathbb{E}\left[\widehat{a'}\diamond(\mathcal{L}_{\xi^{(k)}}a') + \widehat{\mathcal{L}_{\xi^{(k)}}a}\diamond a\right], \xi^{(k)}\right\rangle_{\mathfrak{X}},$$

where $\mu'^\sharp \in \mathfrak{X}$ is dual to $\mu' \in \mathfrak{X}^*$ and $\widehat{a'} \in V^*$ is dual to $a \in V$.

- The dynamics of the variances of the stochastic system is driven by an intricate variety of correlations among the evolving fluctuation variables.
- The solution behaviour can be seen more easily in examples.

Example: the 2D LA SALT Euler equations

The vorticity in 2D LA SALT, understood as a scalar, is governed by the transport law with $\operatorname{div} \mathbb{E}[u_t] = 0 = \operatorname{div} \xi^{(k)}(x)$,

$$\mathbf{d}\omega_t + \mathbb{E}[u_t] \cdot \nabla\omega_t dt + \sum_k \xi^{(k)}(x) \cdot \nabla\omega_t \circ dW_t^{(k)} = 0.$$

This equation implies the Casimirs

$$\int \varphi(\omega_t) dx = \int \varphi(\omega_0) dx,$$

for any differentiable function φ.

In particular, one may choose $\varphi(x) = x^p$ and find that all of the L^p-norms of the solution are conserved by the dynamics of 2D LA SALT Euler.

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Transform to Itô form of the 2D LA SALT Euler equations

In Itô form, 2D LA SALT Euler is given by

$$\mathbf{d}\omega_t + \mathbb{E}\left[u_t\right] \cdot \nabla\omega_t dt + \sum_k \xi^{(k)}(x) \cdot \nabla\omega_t \, dW_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla\omega_t\right) dt.$$

Its expectation obeys

$$\partial_t \mathbb{E}\left[\omega_t\right] + \mathbb{E}\left[u_t\right] \cdot \nabla\mathbb{E}\left[\omega_t\right] = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla\mathbb{E}\left[\omega_t\right]\right) dt.$$

Its fluctuations satisfy

$$\mathbf{d}\omega_t' + \mathbb{E}\left[u_t\right] \cdot \nabla\omega_t' dt + \sum_k \xi^{(k)}(x) \cdot \nabla\omega_t dW_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla\omega_t'\right) dt.$$

Compute 2D LA SALT Euler vorticity variance dynamics

Let's investigate the dynamics of the variance of the vorticity.
The enstrophy Casimir is constant, so

$$\int \mathbb{E}\left[|\omega_t|^2\right] dxdy = \int \left|\mathbb{E}\left[\omega_t\right]\right|^2 dxdy + \int \mathbb{E}\left[|\omega_t'|^2\right] dxdy \,.$$

The first term on the RHS satisfies

$$\frac{1}{2}\frac{d}{dt}\int \left|\mathbb{E}\left[\omega_t\right]\right|^2 dx = -\sum_k \int |\xi^{(k)}\cdot\nabla\mathbb{E}\left[\omega_t\right]|^2 dx = -\frac{1}{2}\frac{d}{dt}\int \mathbb{E}\left[|\omega_t'|^2\right] dxdy$$

• Without forcing, 2D LA SALT converts enstrophy of $\mathbb{E}\left[\omega_t\right]$ into variance, while preserving the total enstrophy (Casimir).

Summary of stochastic coadjoint motion for 2D LA SALT

Remark

One may regard the expected vorticity equations for 2D LA SALT as a dissipative system embedded into a larger conservative system.

From this viewpoint, the interaction dynamics of the two components of the full LA SALT system dissipates the enstrophy of the expected mean vorticity $\mathbb{E}[\omega_t]$ by converting it into the variance of the vorticity fluctuations, while preserving the mean total enstrophy.

This dynamics occurs because the total (mean plus fluctuation) vorticity field is being linearly transported along the expected mean velocity, while the 2D expected mean vorticity field decays.

The Casimirs are preserved by the full LA SALT dynamics, while the equations for the expected dynamics contain dissipative terms which convert Casimirs of the expected variables into variances of fluctuations.

LA SALT regularises the SALT Burgers equations

Choosing $\ell(u_t) = \frac{1}{2}\int_{S^1} |u_t|^2\, dx$ yields the 1D LA SALT Burgers equation

$$\mathbf{d}u_t + \mathbb{E}\left[u_t\right]\partial_x u\,\mathbf{d}t + \sum_k \xi^{(k)}\partial_x u_t \circ \mathbf{d}W_t^{(k)} = 0\,,$$

$$\mathbf{d}u_t + \mathbb{E}\left[u_t\right]\partial_x u\,\mathbf{d}t + \sum_k \xi^{(k)}\partial_x u_t \mathbf{d}W_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}\partial_x(\xi^{(k)}\partial_x u_t).$$

Theorem (Regularity)

LA SALT Burgers solutions are regular. (SALT Burgers solutions are not.)

The LA SALT expectation $\mathbb{E}\left[u_t\right]$ satisfies a *viscous* Burgers equation,

$$\partial_t\mathbb{E}\left[u_t\right] + \mathbb{E}\left[u_t\right]\partial_x\mathbb{E}\left[u_t\right] = \frac{1}{2}\sum_k \xi^{(k)}\partial_x(\xi^{(k)}\partial_x\mathbb{E}\left[u_t\right])\,.$$

This is *regularization by non-locality in probability space.* The conserved total kinetic energy $\int_{S^1} |u_t|^2\, dx$ again converts the original expected value kinetic energy norm $\int_{S^1} |\mathbb{E}\left[u_0\right]|^2\, dx$ into variance $\int_{S^1} \mathbb{E}\left[|u_t'|\right]^2\, dx$.

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Summary: Lagrangian Averaged (LA) SALT fluid dynamics

The LA SALT equations replace $u_t \to \mathbb{E}[u_t]$ in the SALT Lagrangian path

$$\oint_{C\left(\mathbf{d}x_t = u_t dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \quad \Longrightarrow \quad \oint_{C\left(\mathbf{d}X_t = \mathbb{E}[u_t]dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \,.$$

For example, in the Euler fluid case the modified Kelvin theorem reads,

$$\mathbf{d}\oint_{C(\mathbf{d}X_t)} u_t \cdot dx = \oint_{C(\mathbf{d}X_t)} \left[\mathbf{d}u_t \cdot dx + \mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)\right] = 0\,,$$

where $\mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)$ denotes the Lie derivative of the 1-form $(u_t \cdot dx)$ with respect to the vector field $\mathbf{d}X_t$ given by

$$\mathbf{d}X_t = \mathbb{E}[u_t]\, dt + \sum_k \xi^{(k)}(x) \circ dW_t \,.$$

- *The corresponding 'Euler–Poincaré forms' of the LA SALT eqns are*

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}X_t}\frac{\delta \ell}{\delta u} = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\, dt \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}X_t} a = 0\,.$$

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When the *expected* Euler–Poincaré equations are written out in $\boxed{\text{Itô form}}$, with $\mu := \frac{\delta \ell}{\delta u}$, we find generalised NS and advected-diffusive equations

$$\frac{\partial}{\partial t}\mathbb{E}[\mu] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[\mu] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[\mu]) = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond \mathbb{E}[a] + \mathbb{E}[\mathbb{F}_\mu]\,,$$

$$\frac{\partial}{\partial t}\mathbb{E}[a] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[a] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[a]) = \mathbb{E}[\mathbb{F}_a] \quad \boxed{\text{Climate PDE}}\,.$$

These $\boxed{\text{Climate PDE}}$ predict the expectations $\mathbb{E}[\mu]$ and $\mathbb{E}[a]$ throughout the domain of flow. The Itô $\boxed{\text{Weather}}$ equations for the fluctuations are *linear* drift/stochastic transport relations:

$$\mathbf{d}\mu + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mu + \sum_k \mathcal{L}_{\xi^{(k)}}\mu\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu)\, dt = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\, dt + \mathbb{F}_\mu$$

$$\mathbf{d}a + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}a + \sum_k \mathcal{L}_{\xi^{(k)}}a\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a)\, dt = \mathbb{F}_a \quad \boxed{\text{Weather}}\,.$$

$\boxed{\textbf{Then the variance EVOLVES}}$: $\frac{d}{dt}\mathbb{E}\left[\langle\, |\mu - \mathbb{E}[\mu]\,|^2 \rangle_{L^2}\right] = \mathit{RHS}$

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Thanks for listening!

Let's discuss!

More papers along these lines with up-to-date references at ORCID:

https://orcid.org/0000-0001-6362-9912

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Homework: LA SALT climate of the free rigid body?

(1) 'SALT Rigid Body' equations comprise stochastic coadjoint motion,

$$\mathbf{d}\Pi = \Pi \times \frac{\partial(\mathbf{d}h)}{\partial \Pi} \quad \text{with} \quad \mathbf{d}h(\Pi) = h(\Pi)\, dt + \Pi \cdot \xi \circ dW_t \,,$$

with $h(\Pi) = \frac{1}{2}\Pi \cdot \mathrm{I}^{-1}\Pi$ and $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions.
See arXiv:1601.02249 or `https://doi.org/10.1007/s00332-017-9404-3`.

(2) 'LA SALT Rigid Body' equations may be expressed as

$$\mathbf{d}\Pi = \Pi \times \mathbb{E}\left[\frac{\partial h}{\partial \Pi}\right] dt + \Pi \times \xi \circ dW_t \,,$$

for a constant $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions.
See [?], arXiv:1908.11481.
Hint: $\frac{\mathbf{d}}{\mathbf{d}t}|\Pi|^2 = 0$, $\Pi := \mathbb{E}\,[\Pi] + \Pi'$ with $\mathbb{E}\,[\Pi'] = 0$. Then calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathbb{E}\,[\Pi]\,\right|^2 = -\left|\xi \times \mathbb{E}\,[\Pi]\,\right|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[|\Pi'|^2\right] \,,$$

so the initial expectation magnitude converts into fluctuation variance.

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