The Noether Theorems a century later

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E. Noether, Invariante Variationsprobleme, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, 1918, p. 235-257.

I propose to sketch the contents of Noether's article,
"Invariante Variationsprobleme"
published in the *Göttingen Nachrichten* of 1918, as it may be seen against the background of the work of her predecessors
and in the context of the question of the conservation of energy that had arisen in the theory of General Relativity.

How original, how modern, how influential were her results?
I shall comment on the curious transmission and the ultimate recognition of the wide applicability of "the Noether theorems".

Situating Noether's theorems in context

- Predecessors
- Circumstances: Erlangen, Göttingen
- Facts: her article of 1918
- Analysis:

– how original was Noether's “Invariante Variationsprobleme”?
– how modern were her use of Lie groups and the introduction of generalized vector fields?
– how influential was her article?

- The reception of her article, 1918-1970.
- Transmission

– translations: Russian, English, Italian, French, others?
– developments: Vinogradov, Olver, ...

- Conclusion: Noether's theorems a century later.

A mathematician in Germany in the early XX$^{\mathrm{th}}$ century

Emmy Noether

Erlangen (Bavaria, Germany), 1882 – Bryn Mawr (Pennsylvania, USA), 1935

A family of mathematicians

• Emmy Noether was born to a Jewish family in Erlangen in 1882. In a manuscript *curriculum vitae*, written circa 1907, she describes herself as "of Bavarian nationality and Israelite confession" (declaring one's religion was compulsory in Germany at the time).

• She was the daughter of the renowned mathematician, Max Noether, professor at the University of Erlangen.

"Extraordinary professor" when he first moved to Erlangen from Heidelberg in 1875, he became an "ordinary professor" in 1888.

• Her brother, Fritz Noether, born in 1884, studied mathematics and physics in Erlangen and Munich, became professor of theoretical mechanics in Karlsruhe in 1902 and submitted his *Habilitation* thesis in 1912.

Later he became professor in Breslau, from where he was forced to leave in 1933. He emigrated to the Soviet Union and was appointed professor at the university of Tomsk. He was accused of being a German spy, jailed and shot in 1941.

The young Emmy Noether

Circumstances: Erlangen

She first studied languages to become a teacher of French and English.

• From 1900 on, she studied mathematics in Erlangen, at first with her father. Then she audited lectures at the university.

• For a semester in 1903-1904, she audited courses in Göttingen.

• In 1904, she was permitted to matriculate in Erlangen.

• In 1907, she completed her doctorate under the direction of Paul Gordan (1837-1912), a colleague of her father.

Warning. Do not confuse the mathematician Paul Gordan, Emmy Noether's „Doktorvater", with the physicist Walter Gordon (1893–1939).

• The "Clebsch-Gordan coefficients" in quantum mechanics bear the name of the mathematician Paul Gordan, together with that of the physicist and mathematician Alfred Clebsch (1833–1872).

• The "Klein-Gordon equation" is named after Walter Gordon and the physicist Oskar Klein (1894–1977) who, in turn, should not be confused with the mathematician Felix Klein (1849-1925) about whom more later.

1907, her thesis at Erlangen University

Über die Bildung des Formensystems der ternären biquadratischen Form

("On the construction of the system of forms of a ternary biquadratic form")

Her thesis dealt with the search for the invariants of those forms (homogeneous polynomials) which are ternary (i.e., in 3 variables) and biquadratic (i.e., of degree 4).

- An extract appeared in the *Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen* in 1907.
- The complete text was published in 1908 in "Crelle's Journal" (*Journal für die reine und angewandte Mathematik*).
- She later distanced herself from her early work as presenting a needlessly computational approach to the problem.
- After 1911, she was influenced by Ernst Fischer (1875–1954) who was appointed professor in Erlangen after Gordan had retired in 1910.
- From 1913 on, she occasionally substituted for her father.

Expertise in invariant theory

Noether's expertise in invariant theory revealed itself in the publications that followed her thesis (1911, 1913, 1915), and it was later confirmed in the four articles on the invariants of finite groups that she published in 1916 in the *Mathematische Annalen*. She studied in particular the determination of bases of invariants that furnish expansions as linear combinations with integral or rational coefficients.

Circumstances: Göttingen

• In 1915, Felix Klein and David Hilbert invited Noether to Göttingen in the hope that her expertise in invariant theory would help them understand some of the implications of Einstein's newly formulated theory of General Relativity.

• In Göttingen, Noether took an active part in Klein's seminar.

• It was in her 1918 article that she solved a problem arising in the theory of General Relativity and proved "the Noether theorems".

She proved and vastly generalized a conjecture made by Hilbert concerning the nature of the law of conservation of energy.

• Shortly afterwards, her work turned to pure algebra, for which she is mainly remembered, as the creator of "abstract algebra".

She is (rightly) considered as one of the greatest mathematicians of the twentieth century.

A timeline

In 1918, Noether published her first article on the problem of the invariants of differential equations in the *Göttinger Nachrichten*, "Invarianten beliebiger Differentialausdrücke":

Invariants of arbitrary differential expressions
presented by F. Klein at the meeting of 25 January 1918
[the meeting is that of the Göttingen Scientific Society]

The article that contains Noether's two theorems is the "Invariante Variationsprobleme":

Invariant variational problems
presented by F. Klein at the meeting of 26 July 1918*

with the footnote:
*The definitive version of the manuscript was prepared only at the end of September.

Hilbert, Einstein, Noether, 1916

Her expertise in the theory of invariants was conceded by both Hilbert and Einstein as early as her first year in Göttingen.

It is clear from a letter from Hilbert to Einstein of 27 May 1916 that Noether had then already written some notes on the subject of the problems arising in the theory of General Relativity:

"My law of [conservation of] energy is probably linked to yours; I have already given Miss Noether this question to study."
Hilbert adds that, to avoid a long explanation, he has appended to his letter "the enclosed note of Miss Noether."

On 30 May 1916, Einstein answered him in a brief letter in which he derived a consequence from the equation that Hilbert had proposed "which deprives the theorem of its sense", and then asks, "How can this be clarified?" and continues, "Of course it would be sufficient if you asked Miss Noether to clarify this for me."

(Einstein, *Collected Papers*, 8A, nos. 222 and 223)

Einstein at the time of the “Invariante Vatiationsprobleme”

Noether's article of 1918. Facts

It was in the Winter and Spring of 1918 that Noether discovered the profound reason for the difficulties that had arisen in the interpretation of the conservation laws in General Relativity. Because she had left Göttingen to visit her father in Erlangen, her correspondence yields an account of her progress in this search.

- In her postcard to Klein of 15 February, she already sketches her second theorem, but only in a particular case.
- It is in her letter to Klein of 12 March that Noether gives a preliminary formulation of an essential consequence of what would be her second theorem (invariance under the action of a group which is a subgroup of an infinite-dimensional group).
- On 23 July, she announced her results before the *Göttingen Mathematische Gesellschaft*.
- On 26 July, Klein presented a communication by Noether on the same subject at the session of the *Gesellschaft des Wissenschaften zu Göttingen*.

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

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1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

What variational problems was Noether considering?

"We consider variational problems which are invariant under a continuous group (in the sense of Lie). [...] What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups."

Noether considers a general n-dimensional variational problem of order κ for an $\mathbb{R}^\mu$-valued function (n, μ and κ are arbitrary integers),

$$I = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots, \frac{\partial^\kappa u}{\partial x^\kappa}\right) dx,$$

where $x = (x_1, \ldots, x_n) = (x_\alpha)$ are the independent variables, and $u = (u_1, \ldots, u_\mu) = (u_i)$ are the dependent variables.

Noether states her conventions: "I omit the indices here, and in the summations as well whenever it is possible, and I write $\frac{\partial^2 u}{\partial x^2}$ for $\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\gamma}$, etc." and "I write dx for $dx_1 \ldots dx_n$ for short."

The theorems of Noether in English translation

"In what follows we shall examine the following two theorems:

I. If the integral I is invariant under a [group] $\mathfrak{G}_\rho$, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences – and conversely, that implies the invariance of I under a [group] $\mathfrak{G}_\rho$. The theorem remains valid in the limiting case of an infinite number of parameters.

II. If the integral I is invariant under a [group] $\mathfrak{G}_{\infty\rho}$ depending upon arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid[1]."

[1]For some trivial exceptions, see §2, note 13.

Noether's proof of Theorem I

Noether assumes that the action integral $I = \int f dx$ is invariant. Actually, she assumes a more restrictive hypothesis, the invariance of the integrand, $f dx$, which is to say that $\delta(f dx) = 0$. This hypothesis is expressed by the relation,

$$\bar{\delta} f + \mathrm{Div}(f \cdot \Delta x) = 0.$$

Here $\bar{\delta} f$ is the variation of f induced by the variation

$$\bar{\delta} u_i = \Delta u_i - \sum \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda.$$

• Noether introduces the components of the evolutionary representative $\bar{\delta}$ of the vector field δ.

Remark. $\bar{\delta}$ is also called the "vertical representative" of δ.

• $\bar{\delta} f$ is the Lie derivative of f in the direction of $\bar{\delta}$.

$\bar{\delta}$ is a generalized vector field

- $\bar{\delta}$ is a generalized vector field, not a vector field on the trivial vector bundle $\mathbb{R}^n \times \mathbb{R}^\mu \to \mathbb{R}^n$.

In fact, if
$\delta = \sum_{\alpha=1}^n X^\alpha(x)\frac{\partial}{\partial x^\alpha} + \sum_{i=1}^\mu Y^i(x,u)\frac{\partial}{\partial u^i}$,
then
$\bar{\delta} = \sum_{i=1}^\mu \left(Y^i(x,u) - X^\alpha(x)u^i_\alpha\right)\frac{\partial}{\partial u^i}$, where $u^i_\alpha = \frac{\partial u^i}{\partial x^\alpha}$.

Locally, generalized vector fields are written,

$$X = \sum_{\alpha=1}^n X^\alpha(x)\frac{\partial}{\partial x^\alpha} + \sum_{i=1}^\mu Y^i\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right)\frac{\partial}{\partial u^i}.$$

End of Noether's proof of Theorem I

By integrating by parts, Noether obtains the identity

$$\sum \psi_i \, \bar{\delta} u_i = \bar{\delta} f + \mathrm{Div}\ A,$$

where the ψ_i's are the "Lagrangian expressions", *i.e.*, the components of the Euler–Lagrange derivative of L, and A is linear in $\bar{\delta} u$ and its derivatives.

In view of the invariance hypothesis which is expressed by $\bar{\delta} f + \mathrm{Div}(f \cdot \Delta x) = 0$, this identity can be written

$$\sum \psi_i \, \bar{\delta} u_i = \mathrm{Div}\ B, \quad \text{with} \quad B = A - f \cdot \Delta X.$$

Therefore B is a conserved current for the Euler–Lagrange equations of L.

QED

The equations $\mathrm{Div}\ B = 0$ are the conservation laws that are satisfied when the Euler–Lagrange equations $\psi_i = 0$ are satisfied.

The converse of Theorem I

Noether proves that the existence of ρ "linearly independent divergence relations" implies the invariance under a (Lie) group of symmetries of dimension ρ, by passing from the infinitesimal symmetries to invariance under their flows, provided that the vector fields Δu and Δx are ordinary vector fields.

In the general case, the existence of ρ linearly independent conservation laws yields infinitesimal invariance under a Lie algebra of infinitesimal symmetries of dimension ρ.

Equivalence relations have to be introduced to make these statements precise.

Theorem II

Theorem II deals with a symmetry group depending on arbitrary functions—such as the group of diffeomorphisms of the space-time manifold and, more generally, the groups of all gauge theories that would be developed, beginning with the paper of Chen Ning Yang and Robert L. Mills on isotopic gauge invariance in 1954. Noether showed that to such symmetries there correspond identities satisfied by the variational derivatives, and conversely.

The assumption is that "the integral I is invariant under a [group] $\mathfrak{G}_{\infty\rho}$ depending upon arbitrary functions and their derivatives up to order σ", *i.e.*, Noether assumes the existence of ρ infinitesimal symmetries of the Lagrangian, each of which depends linearly on an arbitrary function p (depending on $\lambda = 1, 2, \ldots, \rho$) of the variables $x_1, x_2, \ldots, x_n$, and its derivatives up to order σ.

Such a symmetry is defined by a vector-valued linear differential operator, $\mathcal{D}$, of order σ, with components $\mathcal{D}_i$, $i = 1, 2, \ldots, \mu$.

The adjoint operator

Noether then introduces, without giving it a name or a particular notation, the adjoint operator, $(\mathcal{D}_i)^*$, of each of the $\mathcal{D}_i$'s. By construction, $(\mathcal{D}_i)^*$ satisfies

$$\psi_i \, \mathcal{D}_i(p) = (\mathcal{D}_i)^*(\psi_i) p \; + \; \mathrm{Div}\, \Gamma_i,$$

where Γ_i is linear in p and its derivatives. The symmetry assumption and, again, an integration by parts imply

$$\sum_{i=1}^{\mu} \psi_i \, \mathcal{D}_i(p) = \mathrm{Div}\, B.$$

This relation implies

$$\sum_{i=1}^{\mu} (\mathcal{D}_i)^*(\psi_i) \; p = \mathrm{Div}(B - \sum_{i=1}^{\mu} \Gamma_i).$$

Identities

Since p is arbitrary, by Stokes's theorem and the Du Bois-Reymond lemma,

$$\sum_{i=1}^{\mu} (\mathcal{D}_i)^*(\psi_i) = 0.$$

Thus, for each $\lambda = 1, 2, \ldots, \rho$, there is a differential relation among the components ψ_i of the Euler–Lagrange derivative of the Lagrangian f that is identically satisfied.

Noether explains the precautions that must be taken—the introduction of an equivalence relation on the symmetries—for the converse to be valid.

Improper conservation laws

Noether observes that each identity may be written $\sum_{i=1}^{\mu} a_i \psi_i = \mathrm{Div}\,\chi$, where χ is defined by a linear differential operator acting on the ψ_i's.
She then shows that each divergence, $\mathrm{Div}\,B$, introduced above, is equal to the divergence of a quantity C, where C vanishes once the Euler–Lagrange equations, $\psi_i = 0$, are satisfied.
Furthermore, from the equality of the divergences of B and C, it follows that

$$B = C + D$$

for some D whose divergence vanishes identically, which is to say, independently of the satisfaction of the Euler–Lagrange equations.

QED

These are the conservation laws that Noether called improper divergence relations.

Trivial conservation laws

- The quantity C and not only its divergence vanish on $\psi_i = 0$.

C is a *trivial conservation law of the first kind* (in the sense of Olver).

- The divergence of D vanishes identically, *i.e.*, whether or not $\psi_i = 0$.

D is a *trivial conservation law of the second kind* or $\mathrm{Div}\, D$ is a *null divergence* (in the sense of Olver).

Hilbert's conjecture

Hilbert asserted (without proof) in early 1918 that, in the case of general relativity and in that case only, there are no proper conservation laws.

Noether shows that the situation is better understood "in the more general setting of group theory."

She explains the apparent paradox that arises from the consideration of the finite-dimensional subgroups of groups that depend upon arbitrary functions.

"Given I invariant under the group of translations, then the energy relations are improper if and only if I is invariant under an infinite group which contains the group of translations as a subgroup."

Noether concludes with the striking formula:

"The term *relativity* that is used in physics should be replaced by *invariance with respect to a group*."

Noether *circa* 1920

How original were Noether's two theorems?

Noether's article did not appear in a vacuum.

The topic of the important and numerous predecessors of Noether requires a long development. We give a very brief account of some of the most important points.

Lagrange, in his *Méchanique analytique* (1788), claimed that his method for deriving "a general formula for the motion of bodies" yields

> *the general equations that contain the principles, or theorems known under the names of the conservation of kinetic energy, of the conservation of the motion of the center of mass, of the conservation of the momentum of rotational motion, of the principle of areas, and of the principle of least action.*

It is only in the second edition of the *Mécanique Analytique* (1811) that he observed an explicit correlation between these principles of conservation and invariance properties.

After Lagrange

• After Lagrange, the correlation between invariances and conserved quantities was treated by Jacobi in several chapters of his *Vorlesungen über Dynamik*, delivered in 1842-43 (but only published posthumously in 1866).

• The great advances of Sophus Lie – his theory of continuous groups of transformations that was published after 1870 – were the basis of later developments:

in 1904, in the work of Georg Hamel (1877-1954) on the calculus of variations and mechanics, and

in 1911, in a publication of Gustav Herglotz (1881-1853) on the 10-parameter invariance group of Special Relativity.

• Both Hamel and Herglotz were cited by Noether at the beginning of her 1918 article. She also referred to publications, all of them very recent, by "Lorentz and his students (for example, Fokker), [Hermann] Weyl, and Klein" and by [Alfred] Kneser (1862-1930), a specialist in the calculus of variations.

Originality?

Indeed, scattered results in classical and relativistic mechanics tying together properties of invariance and conserved quantities had already appeared in the publications of Noether's predecessors.

However, none of them had discovered the general principle contained in her Theorem I, nor its converse.

Her Theorem II and its converse were, in fact, completely new.
In Thibaut Damour's expert opinion, the second theorem should be considered the most important part of her article.

How modern were Noether's two theorems?

What Noether simply called "infinitesimal transformations" are in fact vast generalizations of the ordinary vector fields. (Above, we called them generalized vector fields.)

Generalized vector fields would eventually be re-discovered by Harold Johnson (1964), Robert Hermann (1965), then in, a joint paper, by Robert L. Anderson from the University of Georgia, Sukeyuki Kumei and Carl Wulfman (1972). In 1979, Anderson and the Russian mathematician Nail Ibragimov called them Lie-Bäcklund transformations (a misnomer) in their monograph, "Lie-Bäcklund Transformations in Applications".

This notion is essential in the theory of integrable systems which became the subject of intense research after 1970. On this topic, Noether's work is modern, half-a-century in advance of the re-discoveries.

A follower

In Göttingen, Noether had only one immediate follower, Erich Bessel-Hagen (1898-1946), who was Klein's student.

In 1921, Bessel-Hagen published an article, *Über die Erhaltungssätze der Elektrodynamik* (On conservation laws in electrodynamics) in the *Mathematische Annalen* in which he determined in particular those which were the result of the conformal invariance of Maxwell's equations.

He explained that Klein had posed the problem of "the application to Maxwell's equations of the theorems stated by Miss Emmy Noether [...] regarding the invariant variational problems."

He formulated the two Noether theorems "slightly more generally" than they had been formulated in her article.

How? By introducing the concept of symmetries up to divergence ("divergence symmetries").

Who invented divergence symmetries?

Bessel-Hagen defined the divergence symmetries.

They correspond, not to the invariance of the Lagrangian fdx, but to the invariance of the action integral $\int fdx$, *i.e.*, they satisfy, instead of $\delta(fdx) = 0$, the weaker condition $\delta(fdx) = \mathrm{Div}\, C$, where C is a vectorial expression.

Noether's fundamental relation remains valid under this weaker assumption provided that $B = A - f \cdot \Delta x$ be replaced by

$$B = A + C - f \cdot \Delta x.$$

Bessel-Hagen wrote: "I owe [these generalized theorems] to an oral communication by Miss Emmy Noether herself", so this type of symmetry was also Noether's invention.

How influential were Noether's two theorems?

The story of the reception of Noether's article in the years 1918-1970 is surprising.

She submitted the "Invariante Variationsprobleme" for her *Habilitation*, that she finally obtained in 1919, but she never referred to her article in any of her subsequent publications.

Hermann Weyl, in *Raum, Zeit, Materie*, first published in 1918, performed computations very similar to hers, but he referred to Noether only in a footnote in the third (1919) and subsequent editions.

Richard Courant must have been aware of her work because a brief summary of a limited form of both theorems appears in all German, and later English editions of "Courant-Hilbert", the treatise on methods of mathematical physics first published in 1924.

The transmission of Noether's results – Soviet Union

The memory of Noether's theorems remained in the Soviet Union among some mathematicians and physicists.

In 1936, the physicist, Moisei A. Markow (1908-1994), a member of the Physics Institute of the U.S.S.R. Academy of Sciences in Moscow, published an article in the *Physikalische Zeitschrift der Sowjetunion* in which he refers to "the well-known theorems of Noether."

The first sentence of the article is:
"Für die Gruppe G_{10} wurden mit Hilfe der Lehrsätze von Noether 10 Erhaltungssätze abgeleitet ($div\, A = T$)."

For the group G_{10} one obtains, with the aid of Noether's theorems, 10 conservation laws ($div\, A = T$).

Markow was a student of V. A. Fock (1898–1974) and of Georg B. Rumer (1901–1985) who had been an assistant to Max Born in Göttingen from 1929 to 1932.

Göttingen to Moscow?

However, it is not obvious that Markow's knowledge of Noether's work was transmitted by Rumer. In fact, Rumer, in an earlier article (1931) that was written in Göttingen and published in the same journal – then simply called the *Physikalische Zeitschrift* –, proved the Lorentz invariance of the Dirac operator but did not allude to any associated conservation laws, while in his articles on the general theory of relativity in the *Göttinger Nachrichten* in 1929 and 1931 he cited Weyl but never Noether.

Was it because she was a woman? Or beause she was Jewish?

Similarly, it seems that Fock never referred to her work in any of his papers to which it is relevant.

Later developments in the Soviet Union

- In 1959, Lev S. Polak published a translation of Noether's 1918 article into Russian.
- Vladimir Vizgin wrote a historical monograph on the *Development of the interconnection between invariance principles and conservation laws in classical physics*, published in Moscow in 1972, in which he analyzes both theorems of Noether.
- The textbook of Israel M. Gelfand and Sergej V. Fomin on the calculus of variations, published in Moscow in 1961 and translated into English in 1963, contains a modern presentation of Noether's first theorem – although not yet using the formalism of jets –, followed by a few lines about her second theorem.
- In the 1970s, Gelfand published several articles with Fomin, Leonid Dikii (Dickey), Irene Dorfman, and Yuri Manin on the "formal calculus of variations".
- "Algebraic theory of nonlinear differential equations" by Manin (1978) and Boris Kuperschmidt's paper of 1980 contain a "formal Noether theorem".

Vinogradov and Krasil'shchik

Alexandre Vinogradov (1938–2019) was a member of Gelfand's seminar and taught in Moscow. He left the Soviet Union for Italy in 1990 and, from 1993 until his retirement from teaching (but not from research!) he held the position of professor at the University of Salerno in Italy.

Beginning in 1977, together with Joseph Krasil'shchik, who also worked in Moscow and later in the Netherlands, he published extensively on symmetries, both at the very general and abstract level – greatly generalizing Noether's formalism and results – and on their applications, notably the Burgers and the Korteweg-de Vries equations.

The theory of local and non-local symmetries and a wealth of applications were later expounded in the book they edited: "Symmetries and Conservation Laws for Differential Equations of Mathematical Physics" (in Russian), Factorial, Moscow (1997); English translation, American Mathematical Society (1999).

In particle physics: Utiyama, 7 years before Yang and Mills

An early, explicit reference to Noether's publication is found in 1947 in the article of Ryoyu Utiyama (1916–1990), then in the department of physics of Osaka Imperial University:
"On the interaction of mesons with the gravitational field. I"
in *Progress of Theoretical Physics*, II (2) (1947), 38–62.

His article begins with "Theory of invariant variation". He cites Noether's 1918 article and Pauli's "Relativitätstheorie" of 1921.

Following Noether closely, he proves the first theorem, introducing "the substantial variation of any field quantity", and also treats the case where the dependent variables "are not completely determined by [the] field equations but contain undetermined functions".

This text dates, in fact, to 1941 as the author reveals in a footnote on the first page: "This paper was published at the meeting[s] of [the] Physico-mathematical Society of Japan in April 1941 and October 1942, but because of the war the printing was delayed".

Other lines of transmission of Noether's results

In the early 1960s Enzo Tonti (later professor at the University of Trieste) translated Noether's article but his translation has remained in manuscript. It was transmitted to Franco Magri in Milan who, in 1978, wrote an article in Italian where he clearly set out the relation between symmetries and conservation laws for non-variational equations, a significant development.

In France, Jean-Marie Souriau (1922–2012), main architect of the applications of symplectic geometry to mechanics, was well aware of Noether's two theorems as early as 1964. In 1970, independently of Bertram Kostant (1928–2017), he introduced the notion of moment map. The conservation of the moment of a Hamiltonian action is the Hamiltonian version of Noether's first theorem. Souriau called this result "le théorème de Noether symplectique", although there is nothing Hamiltonian or symplectic in Noether's article!

What became of Noether's improper conservation laws?

In general relativity the improper conservation laws which are "trivial of the second kind" are called strong laws, while the conservation laws obtained from the first theorem are called weak laws.
The strong laws play an important role in the basic papers of Peter G. Bergmann (1958), Andrzej Trautman (1962) and Joshua N. Goldberg (1980).

In gauge theory, the identities in Noether's second theorem are at the basis of Jim Stasheff's "cohomological physics". They are used as the antighosts in the BV construction for Lagrangians with symmetries, as in Ron Fulp, Tom Lada and Stasheff, "Noether's variational theorem II and the BV formalism" (2003).

Have the Noether theorems been generalized?

Answer: Except for Bessel-Hagen, not until the 1970s.

Until the 1970's, the so-called "generalizations" were all due to physicists and mathematicians who had no direct knowledge of Noether's article but still thought that they were generalizing it, while they were generalizing the truncated and restricted version of her first theorem that had appeared in Edward L. Hill's article, "Hamilton's principle and the conservation theorems of mathematical physics" (1951).

A one-to-one map between equivalence classes of divergence symmetries and of conservation laws

In the late 70's and early 80's, using different languages, both mathematically and linguistically, Olver and Cheri Shakiban, in Minneapolis, and Vinogradov, in Moscow, made great advances in the Noether theory.

Define a symmetry of a differential equation to be trivial if its evolutionary representative vanishes on the solutions of the equation. Then one can formulate the Noether-Olver-Vinogradov theorem (ca. 1985):

For Lagrangians such that the Euler–Lagrange equations are a normal system, Noether's correspondence induces a one-to-one map between equivalence classes of divergence symmetries and equivalence classes of conservation laws.

Non-variational equations

What Magri showed, in his 1978 article, is that, if $\mathcal{D}$ is a differential operator and $V\mathcal{D}$ is its linearization, searching for the restriction of the kernel of the adjoint $(V\mathcal{D})^*$ of $V\mathcal{D}$ to the solutions of $\mathcal{D}(u) = 0$ is an algorithmic method for the determination of the conservation laws for a possibly non-variational equation, $\mathcal{D}(u) = 0$.

For an Euler–Lagrange operator, the linearized operator is self-adjoint. Therefore this result generalizes the first Noether theorem.

This idea is to be found much developed in the work of several mathematicians, most notably Vinogradov, Toru Tsujishita, Ian Anderson, and Olver.

In the language of differential geometry

• Andrzej Trautman in his “Noether equations and conservation laws” (1967, 1972) was the first to present even a part of Noether’s article in the language of manifolds, fiber bundles and, in particular, the jet bundles that had been defined and studied by Charles Ehresmann and his student Jacques Feldbau, and by Norman Steenrod around 1940.

• Stephen Smale published the first part of his article on “Topology and mechanics” in 1970, in which he proposed a geometric framework for mechanics on the tangent bundle of a manifold.

• Hubert Goldschmidt and Shlomo Sternberg wrote a landmark paper in 1973 in which they formulated the Noether theory for first-order Lagrangians in an intrinsic, geometric fashion.

• Jerrold Marsden published extensively on the theory and applications of Noether’s correspondence from 1974 until his death in 2010.

Differential geometric formulation of Noether's results

The ideas that permitted the recasting of Noether's theorems in geometric form and their genuine generalization were first of all that of smooth differentiable manifold (*i.e.*, manifolds of class C^∞), and then the concept of a jet of order k of a mapping ($k \geq 0$), the concept of manifolds of jets of sections of a fiber bundle, and finally of jets of infinite order, not defined directly but as the inverse limit of the manifold of jets of order k, when k tends to infinity.

In the 1970s, many authors contributed to the "geometrization" of Noether's first theorem, notably Jedrzej Śniatycki, Demeter Krupka, and Pedro García.

Cohomological interpretation

It was Vinogradov who showed in 1977 that generalized vector fields are nothing other than ordinary vector fields on the bundle of jets of infinite order of sections of a bundle.
Both Lagrangians and conservation laws then appear as special types of forms.
The divergence operator may be interpreted as a horizontal differential, one that acts only on the independent variables.
Whence one obtains a cohomological interpretation of Noether's first theorem.
The study of the exact sequence of the calculus of variations, and of the variational bicomplex, which constitutes a vast generalization of Noether's theory, was developed in 1975 and later by Włodzimierz Tulczyjew in Warsaw, by Paul Dedecker in Belgium, by Vinogradov in Moscow, by Tsujishita in Japan, and in the United States by Olver and by Ian Anderson.

Discrete versions of the Noether theorems

A pioneer,

John David Logan, "First integrals in the discrete variational calculus", *Aequationes Mathematicae*, 9 (1973), pp. 210–220.

The differentiation operation is replaced by the shift operator.
The independent variables are now integers, and the integral is replaced by a sum, $\mathcal{L}[u] = \sum_n L(n, [u])$, where $[u]$ denotes $u(n)$ and finitely many of its shifts.
The variational derivative is expressed in terms of the inverse shift.

Advances on the discrete analogues of both Noether theorems may be found in a series of papers by Peter Hydon and Elizabeth Mansfield, beginning in 2004 and continuing.

References on the discrete Noether theorem

Elizabeth Mansfield, "Noether's theorem for smooth, difference and finite element systems", in *Foundations of Computational Mathematics* (Santander, 2005), London Math. Society Lecture Note Series 331, 2006.

Peter Hydon, Elizabeth Mansfield, "Extensions of Noether's second theorem: from continuous to discrete systems", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 467 (2011), no. 2135 (contains a modern re-formulation of Noether's second theorem as well as a discrete version)

Mansfield, E. L.; Rojo-Echebura, A.; Hydon, P. E.; Peng, L., "Moving frames and Noether's finite difference conservation laws" I, *Trans. Math. Appl.* 3 (2019), no. 1, 47 pp.

Gonçalves, Tânia M. N.; Mansfield, Elizabeth L., "Moving frames and Noether's conservation laws–the general case", *Forum Math. Sigma* 4 (2016), e29, 55 pp.

And in quantum mechanics

John E. Gough, Tudor S. Ratiu, O.G. Smolyanov,
"Noether's theorem for dissipative quantum dynamical semigroups",
J. Math. Phys. 56 (2015), no. 2, 022108.

From the abstract: The Noether constants are identified with the fixed point of the Heisenberg picture semigroup.

"Noether theorems and quantum anomalies",
Dokl. Akad. Nauk 472 (2017), no. 3, 248252;
Dokl. Math. 95 (2017), no. 1, 26–30.

From the abstract: We show that both infinite-dimensional versions of Noether's theorems, and the explanation of quantum anomalies can be obtained using similar formulas for the derivatives of functions whose values are measures or pseudomeasures,

Were the Noether theorems ever famous?

Both theorems were analyzed by Vizgin in his 1972 monograph on invariance principles and conservation laws in classical physics.

Was the existence of the first and second theorem in one and the same publication expressed in written form in a language other than Russian before the first edition of Olver's book in 1986?

I think not.

Then, "theorems" in the plural appeared in these titles:

• Olver, "Noether's theorems and systems of Cauchy–Kovalevskaya type", in Lect. Appl. Math 23, AMS (1986).

• Hans A. Kastrup's contribution to "Symmetries in Physics", G. Doncel *et al.*, eds., (1987), contains a chapter entitled "The recognition of Noether's theorems within the scientific community".

• yks, "Sur les théorèmes de Noether", in *Géométrie et physique*, Y. Choquet- Bruhat, B. Coll, R. Kerner et A. Lichnerowicz, eds., Hermann, Paris (1987), 147-160.

Fame?

I quote from Gregg Zuckerman's "Action principles and global geometry", in *Mathematical Aspects of String Theory*, S.-T. Yau, ed., 1987:

"E. Noether's famous 1918 paper, "Invariant variational problems" crystallized essential mathematical relationships among symmetries, conservation laws, and identities for the variational or 'action' principles of physics. [...] Thus, Noether's abstract analysis continues to be relevant to contemporary physics, as well as to applied mathematics."

Thus, approximately seventy years after her article had appeared in the *Göttingen Nachrichten*, fame came to Noether for this (very small) part of her mathematical *œuvre*.

At the end of the twentieth century

At the end of the twentieth century, the importance of the concepts Noether had introduced was finally recognized and her name was attached to them by mathematicians and physicists alike.

In 1999, in the twenty-page contribution of Pierre Deligne and Daniel Freed to the monumental treatise, *Quantum Fields and Strings: A Course for Mathematicians*, Noether was credited, not only with "the Noether theorems", but also with "Noether charges" and "Noether currents".

In 1999, the proceedings of a conference held at the then recently named "Emmy Noether Research Institute of Mathematics" at Bar-Ilan University (Israel) were edited by Mina Teicher under the title *The Heritage of Emmy Noether*. The volume contains a comprehensive survey (p. 83–101) by Yuval Ne'eman: "The impact of Emmy Noether's theorems on XXIst century physics".

A century after the "Invariante Variationsprobleme"

Noether's theorems continue to be the basis of discussions by physicists to this day.

Stanley Deser in "Energy in gravitation and Noether's theorems" *Journal of Physics* A, 52 (2019), 381001, writes of "the conflicting roles of Noether's two great theorems in defining conserved quantities, especially Energy in General Relativity and its extension" and of "the physical impact of Noether's theorems".

Deser further published a new article in *Physics Letters* B, also in 2019, on the interpretation of the converse of Noether's first theorem on identically conserved tensors.

A century later, mathematicians and physicists are still citing Noether's 1918 article and developing her fundamental ideas.

Thank you for your attention

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- English translation of Noether's article and history of Noether's theorems

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The centenary of Noether's article

• Olver, "Emmy Noether's enduring legacy in symmetry", *Symmetry, Culture and Science*, 29 (2018), 475-485.
http://www-users.math.umn.edu/ olver/

• yks, "In Noether's words: Invariant variational problems", *IAMP News Bulletin*, October 2018, 5-9.
http://www.iamp.org/bulletins/old-bulletins/Bulletin-October2018-print.pdf

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On arXiv, a search for "Noether theorem"
[excluding "Brill-Noether theorem" and other occurrences in algebraic geometry]
yields 23 preprints for January-June 2020, NEARLY ONE EVERY WEEK...!