Basic Todaism

Carlos Tomei, PUC-Rio

Geometry, Dynamics and Mechanics Seminar

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$$M' = [M, \Pi_{\text{skew}} f(M)], \quad \text{where } \Pi_{\text{skew}} + \Pi_{\text{upper}} = I,$$
or, *equivalently*, $M(t) = [\exp(t\, f(M))]_R \; M \; [\exp(t\, f(M))]_R^{-1}$ solve
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For $t \to \infty$, $[T^k]_Q \to Q_\infty$, where $T_0 = Q_\infty^* \Lambda Q_\infty$ (the power method).

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A Hamiltonian flow with limit points...

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- (Leite, Saldanha, T., 10) If Λ has no three eigenvalues in arithmetic progression, Wilkinson iteration leads to cubic convergence of $J_{n,n-1}$. Otherwise, there may be a Cantor-like set in which iteration is quadratic.

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And remember: $|\lambda - J_{n,n}| = O(J^2_{n,n-1})$! Deflate !

The isospectral manifold $\mathcal{T}_\Lambda$

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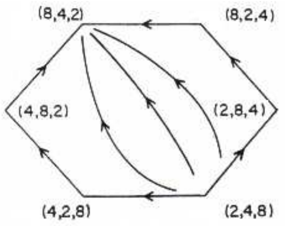

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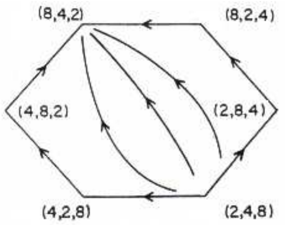

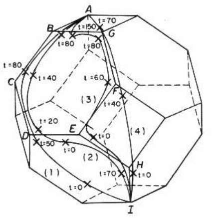

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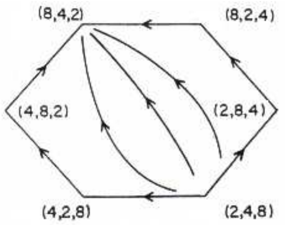

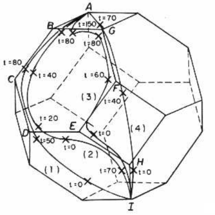

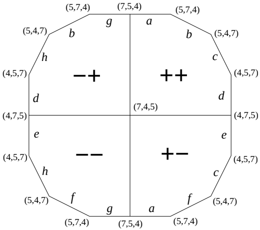

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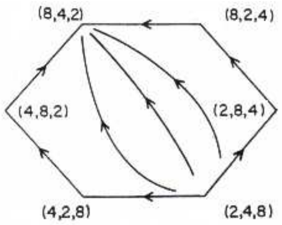

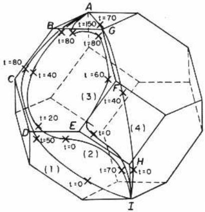

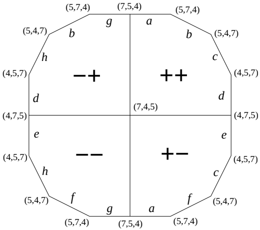

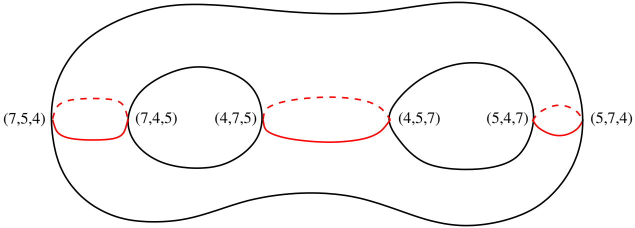

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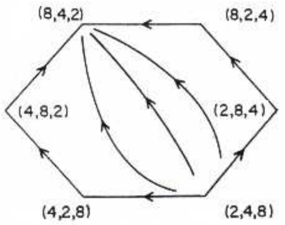

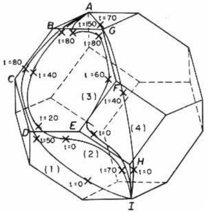

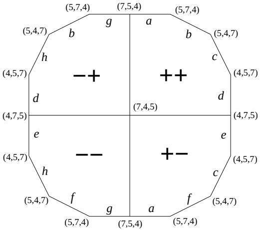

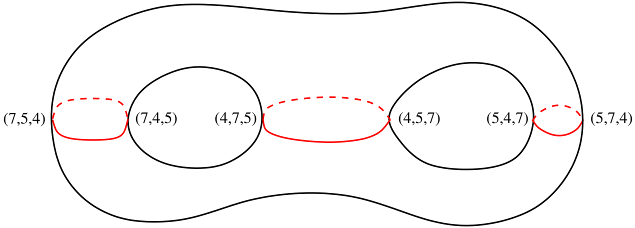

Four hexagons, three (black) deflation components.

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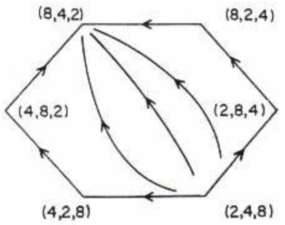

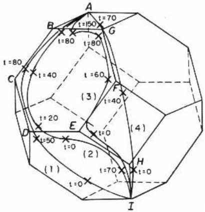

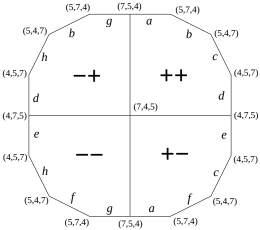

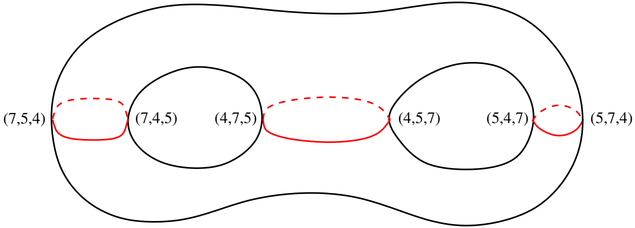

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This is why continuous shift strategies are problematic.

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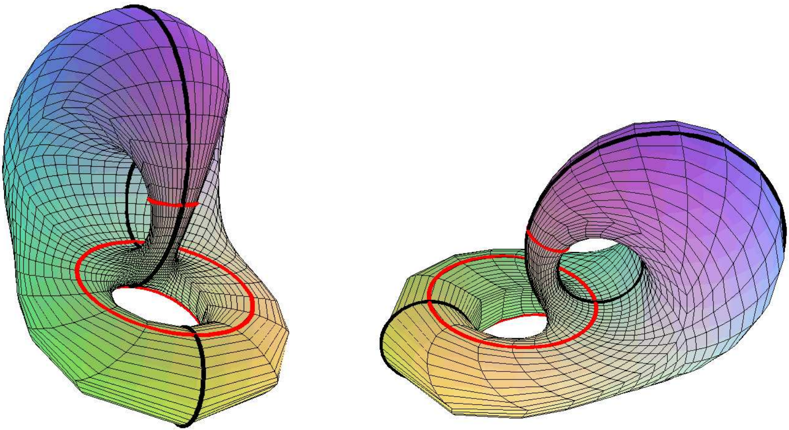

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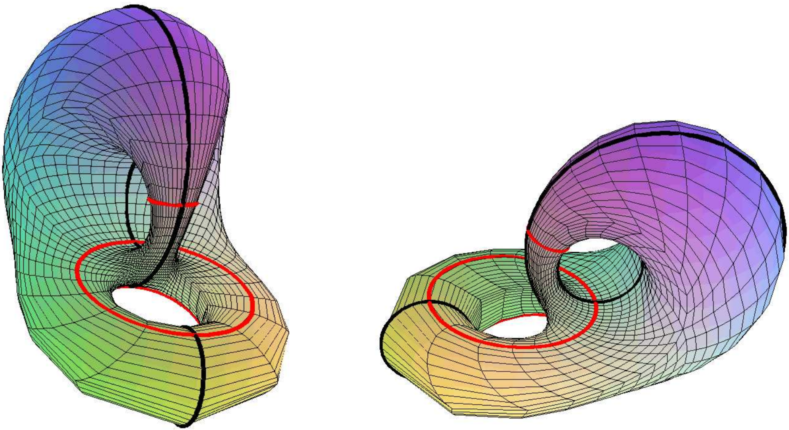

Real, tridiagonal symmetric matrices with eigenvalues 4, 5 e 7.

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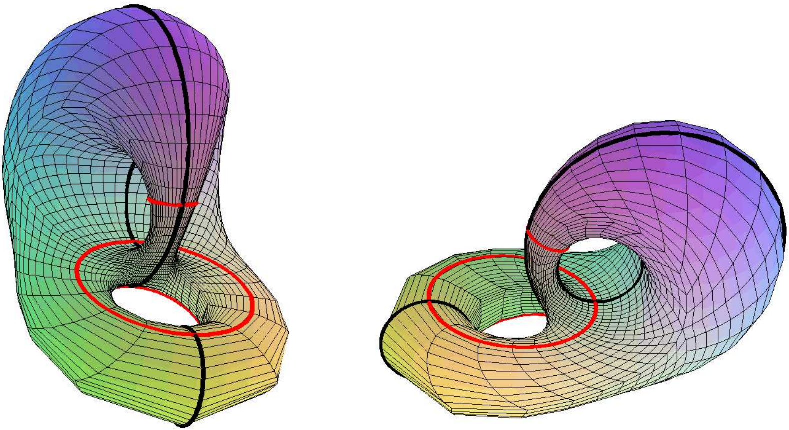

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For points in red, $T_{2,1} = 0$; black stands for $T_{3,2} = 0$.

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What about charts?

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What about charts? Larger isospectral manifolds?

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- Dropping signs (going Jacobi) is as bad as inserting absolute values.

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Simple evolutions: $B' = [B, f(\Lambda)]$, i.e., $\left(new\ \beta_i^\pi \right) = exp(f(\lambda_{i+1}^\pi)/f(\lambda_i^\pi)) \left(old\ \beta_i^\pi \right)$.

- Dropping signs (going Jacobi) is as bad as inserting absolute values.
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Charts – bidiagonal variables

(Leite, Saldanha, T. 08) For $\pi \in S_n$ and $T \in \mathcal{T}_\Lambda$, write $T = Q_\pi^* \Lambda^\pi Q_\pi$.

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$$Q_\pi = L_\pi U_\pi \text{: } L_\pi \text{ uni-lower and } U_\pi \text{ pos-upper,}$$
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Set $B_\pi = L_\pi^{-1} \Lambda^\pi L_\pi$ (lower) $= U_\pi T U_\pi^{-1}$ (upper Hessenberg):

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  This begs for a new inverse algorithm: Gragg and Harrod, '84.

Larger matrices

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*	*	*	*	*	*
*	*	*	*	*	*
0	*	*	*	*	*
0	0	*	*	*	*
0	0	0	*	*	*
0	0	0	0	*	*

*	*	*	*	*	*
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The steps commute. An action of $\mathbb{R}^n$.

A few papers

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