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# Linear symplectic maps and billiard dynamics

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# I. Introduction

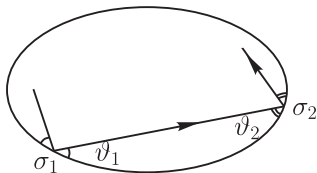
## 1. Billiard

Phase space: cylinder  $Z = \{\sigma, \vartheta\}$ .

Billiard map  $b : Z \rightarrow Z$ ,

$$(\sigma_1, \vartheta_1) \mapsto (\sigma_2, \vartheta_2).$$

Invariant measure:  $\sin \vartheta d\vartheta d\sigma$ .



**2. Motivations.** An important class of discrete Hamiltonian (Lagrangian) systems. Billiard and geodesic flow. Spectrum of the Laplace operator on a 2D domain.

Billiard maps form in the space of symplectic maps a meager set: the number of billiard maps is as big as the number of functions of ONE real variable (curvature as a function of the arclength), while the number of symplectic maps is as big as the number of functions of TWO real variables (generating function)  $\Rightarrow$  rigidity in various problems of billiard dynamics.

## 2. Several classical problems.

(a) Integrability.

**Conjecture.** If a billiard is integrable (as a dynamical system), then any smooth piece of the boundary curve is a line segment or a piece of a conic section.

Partial results: Bolotin 1990, Delshams, Fedorov, Ramirez-Ros 2001, Bialy, Mironov 2016, Avila, De Simoi, Kaloshin 2016. The final proof is announced by Glutsyuk 2017.

Proof in the case of outer billiard: Glutsyuk, Shustin 2016.

(b) Density of the set of periodic points.

(c) **Conjecture** (Ivrii 1980). Suppose the billiard curve is piecewisely smooth. Then the measure of the set of periodic points on the phase cylinder  $Z = \{\sigma, \vartheta\}$  vanishes.

A proof for periodic orbits of period 3: Rychlik 1989; period 4: Glutsyuk and Kudryashov 2012.

## II. The problem.

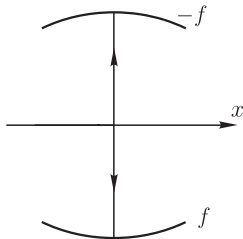
### 1. Symmetric billiard.

$$f(x) = \frac{1}{2}(a_0 + a_2x^2 + a_4x^4 + \dots),$$

$a_0 = -1$ ,  $s_x : Z \rightarrow Z$ , the symmetry in  $x$ .

$b \circ s_x = s_x \circ b \Rightarrow$  “projection” of the map  $b$  to the quotient space  $Z/s_x$  is well-defined. We denote it again  $b$ .

2. The billiard trajectory coinciding with the line segment  $\{x = 0, y \in [-1/2, 1/2]\}$  generates two periodic points of period 2. In the quotient system we obtain a fixed point  $O$ .



In the linear approximation near  $O$   $b$  is a rotation or hyperbolic rotation (except degenerate cases). Rotation  $\Leftrightarrow a_2 \in (0, 2)$ .

**Question.** Is it possible to choose  $f$  so that  $b$  is locally conjugated to its linearization  $\rho$ ?

Several versions:

- (a) The series  $f = \sum a_{2j}x^{2j}$  is formal.
- (b) Radius of convergence is positive.
  - (1)  $\rho$  is the rotation by the angle  $\alpha$ .
  - (2)  $\rho$  is a hyperbolic rotation.

## The same problem in higher dimension

A domain in  $\mathbb{R}^{n+1} = \{x_1, \dots, x_n, y\}$  symmetric w.r.t.  $\{y = 0\}$  (important) and w.r.t.  $\{x_1 = 0\}, \dots, \{x_n = 0\}$  (convenient).

Period-2 “vertical” orbit = fixed point in the quotient (w.r.t. the first symmetry) system.

**Question.** Is it possible to choose  $f$  so that  $b$  is locally conjugated to a linear symplectic map on  $\mathbb{R}^{2n}$ ?

Versions:

- (a) The Taylor series for  $f$  is formal.
- (b) Radius of convergence is positive.

### III. Main equation, 2D

1. Locally near  $0 \in \mathbb{R}^2$  for some  $h : \mathbb{R}^2 \rightarrow Z$  (the conjugacy map)

$$b \circ h = h \circ \rho, \tag{1}$$

where  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the rotation by the angle  $\alpha$ .

2. (Gauge symmetry). Let  $s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  commute with  $\rho$ :  $s \circ \rho = \rho \circ s$ . Then each solution  $(b, h)$  of (1) generates a family of gauge equivalent solutions  $(b, h \circ s)$ .

3. A more explicit form for (1) is needed.

Coordinates  $(\sigma, \vartheta)$  are not convenient ...

We assume that  $b$  is a map of the form

$(\sigma_1, \sigma_2) \mapsto (\sigma_2, \sigma_3)$ . Then

$$\frac{\partial}{\partial \sigma_2} (l(\sigma_1, \sigma_2) + l(\sigma_2, \sigma_3)) = 0.$$

Conjugacy  $h$  has two components:

$$h = (\hat{\chi}, \chi), \quad \hat{\chi}(0, 0) = \chi(0, 0) = 0.$$

4. Easy computation:  $\hat{\chi} = \chi \circ \rho^{-1}$ .

Equation (1) takes the form



$$2f' \circ \chi (\tau_- f \circ \chi \tau_+ f \circ \chi - \tau_- \chi \tau_+ \chi) + (1 - (f' \circ \chi)^2) (\tau_- \chi \tau_+ f \circ \chi + \tau_+ \chi \tau_- f \circ \chi) = 0, \quad (2)$$

where we denote

$$\tau_{\pm} \chi = \chi - \chi \circ \rho^{\pm 1}, \quad \tau_{\pm} f \circ \chi = f \circ \chi + f \circ \chi \circ \rho^{\pm 1}.$$



**5.** (Technical remark). It is convenient to use complex coordinates  $z, \bar{z}$  on  $\mathbb{R}^2 = \{u, v\}$ ,

$$z = u + iv, \quad \bar{z} = u - iv.$$

Then  $\rho(z, \bar{z}) = (\lambda z, \bar{\lambda} \bar{z})$ ,  $\lambda = e^{i\alpha}$ .

**6. Proposition.** For any irrational  $\alpha/(2\pi)$  equation (2) has a formal solution  $f, \chi$ .

Burlakov & Seslavina 2011.

Small divisors of the form  $(\lambda^{k-1} - 1)(\lambda^{-k-1} - 1)$ .

## IV. Numeric results

1. First, we tried to compute  $a_{2j}$  as functions of  $\lambda = e^{i\alpha}$ . Formulas become more and more complicated. About 10 coefficients can be computed by using MAPLE. Nothing is clear ...
2. Take  $\alpha = 2\pi g$ , where  $g$  is the golden mean. We obtain the sequence  $a_2, a_4, \dots$ :

1.7373, 1.2449, 1.7631, 3.1125, 6.1475, 13.002, 28.803,  
65.969, 154.94, 371.18, 903.40, 2227.5, 5552.9, ...

The sequence  $b_j = a_{2j}/a_{2j-2}$  grows monotonically:

0.71658, 1.4161, 1.7653, 1.9750, 2.1151, 2.2151,  
2.2903, 2.3487, 2.3955, 2.4338, 2.4657, 2.4928, ...

We put  $b_\infty = \lim_{j \rightarrow \infty} b_j \leq \infty$ . Then  $b_\infty^{-1/2}$  is the radius of convergence for the series  $f$ .

(a) (Acceleration of convergence I). Suppose that  $b_\infty < \infty$  and

$$b_j \approx b_\infty - \frac{c_1}{j} \quad (3)$$

Having  $b_j$  and  $b_{j+1}$ , we compute from two equations (3)

$$b_\infty(j) = (j+1)b_{j+1} - jb_j, \quad c_1 = j(j+1)(b_{i+1} - b_i).$$

For  $b_\infty(j)$  we obtain the sequence

$$2.81542, 2.81274, 2.81413, 2.81512, 2.81575, 2.81616, \\ 2.81644, 2.81664, 2.81678, 2.81689, 2.81698, \dots$$

(b) (Acceleration of convergence II).

Assuming that  $b_j \approx b_\infty - c_1/j - c_2/j^2$ , we obtain:

$$b_\infty(j) = \frac{(j+2)^2 b_{j+2} - 2(j+1)^2 b_{j+1} + j^2 b_j}{2!},$$

2.8100676, 2.8162207, 2.8170990, 2.8173217, 2.8173956,  
2.8174247, 2.8174376, 2.8174439, 2.8174470, 2.8174487, ...

(c) (Acceleration of convergence III). If we suppose that

$$b_j \approx b_\infty - c_1/j - c_2/j^2 - c_3/j^3,$$

we have:

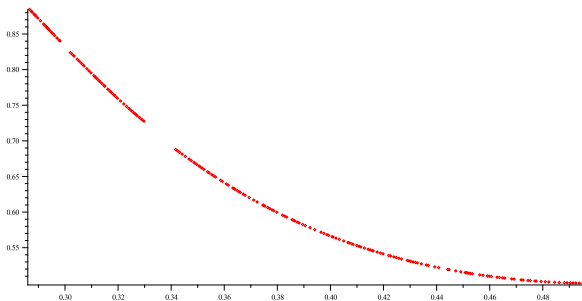
$$b_\infty(j) = \frac{(j+3)^3 b_{j+3} - 3(j+2)^3 b_{j+2} + 3(j+1)^3 b_{j+1} - j^3 b_j}{3!},$$

2.82032285, 2.81797742, 2.81761859, 2.81751871, 2.81748308,  
2.81746786, 2.81746049, 2.81745658, 2.81745436, ...

Continue in the same manner ...

Calculations lead to the following conjecture.

**Conjecture 1.** For good rotation numbers  $\alpha/(2\pi)$  the functions  $f$  and  $\chi$  are real-analytic in a neighborhood of zero.



The quantity  $b_\infty$  can be computed for various values of  $\alpha$ . Let us draw the corresponding points on the plane  $\mathbb{R}^2 = \left\{ \frac{\alpha}{2\pi}, \frac{1}{b_\infty} \right\}$ .

The function is not defined for rational values of the argument. We see gaps near the "strongest" resonances  $\alpha = 3/10$  and  $\alpha = 1/3$ .

**Conjecture 2.** The function  $\alpha \mapsto 1/b_\infty(\alpha)$  is Whitney smooth.

For  $\alpha/(2\pi) < 1/3$  the sequence  $a_{2j}$  is poorly described by the above asymptotics, and for  $\alpha/(2\pi) < 1/4$  becomes sign-alternating.

**3.** For  $\alpha/(2\pi) = 1/2$  equation (2) can be solved w.r.t.  $f$  (not  $\chi!$ ):

$$f = -\frac{1}{2}\sqrt{1 - 4x^2}.$$

A half-circle. Numerics confirm ... Hence, the billiard inside a circle is, in a certain sense, a limit solution when  $\alpha/(2\pi) \rightarrow 1/2$ .

5. Suppose that

$$a_{2j} = c_0 j^\sigma b_\infty^j (1 + O(1/j)). \quad (4)$$

Then

$$b_j = b_\infty \left( 1 + \frac{\sigma}{j} + O\left(\frac{1}{j^2}\right) \right), \quad \sigma = -\frac{c_1}{b_\infty}.$$

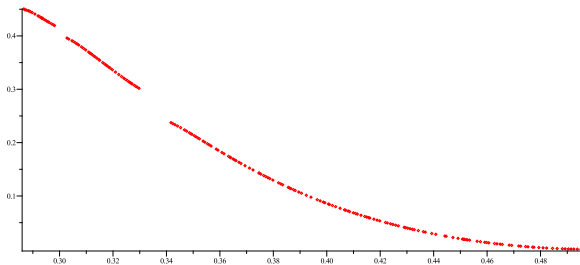
In calculations we always have  $\sigma = -3/2$  with an error  $< 1/1000$ .

**Conjecture 3.** Asymptotic formula (4) is true, where  $\sigma = -3/2$ .

If Conjecture 3 holds, then

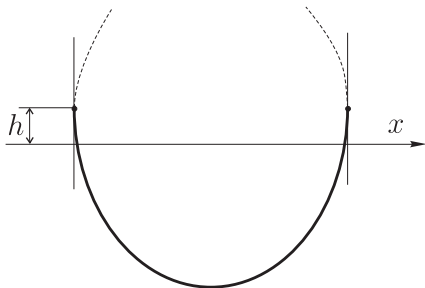
- (a)  $f(x)$  is well-defined and finite at the “border points”  $x_\pm = \pm b_\infty^{-1/2}$ .
- (b)  $f$  has in  $x_\pm$  a singularity of type  $\sqrt{\pm(x_\pm - x)}$ .





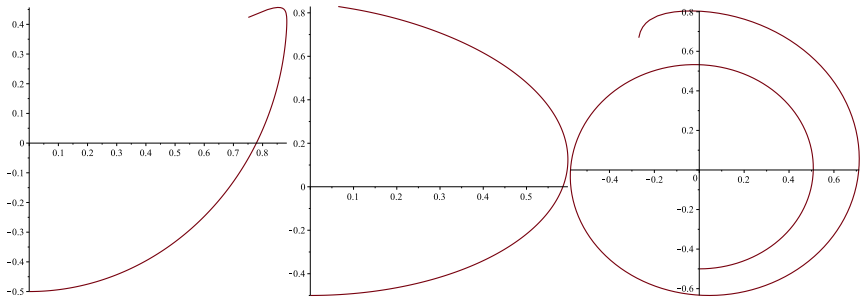
Graph of the function  $\frac{\alpha}{2\pi} \mapsto h(\alpha) := f(x_{\pm})$  is presented in the figure.

**Conjecture 4.** The function  $\alpha \mapsto 1/b_{\infty}(\alpha)$  is Whitney smooth.

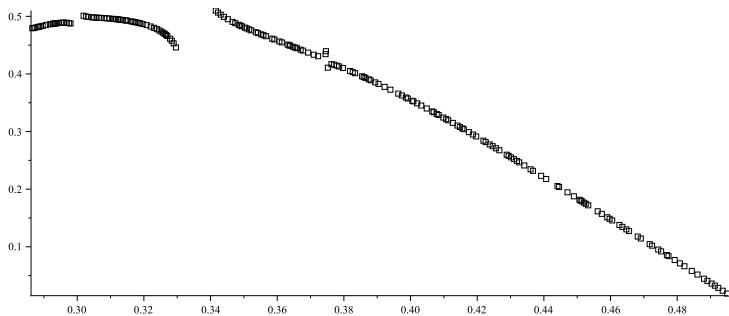


It may happen that the billiard curve  $\gamma_\alpha$  can be continued analytically through the points  $(x_\pm(\alpha), h(\alpha))$ .

By using re-expansions in polar coordinates, we obtain the following pictures of continuation the billiard curve:



Let  $Q$  be the domain in the phase space of the billiard totally occupied by the quasi-periodic dynamics. Here is the graph of the relative measure of  $Q$ . On horizontal axis we have  $\alpha/2\pi$ .



**3D case.** Everything is analogous. Now we have to compute the Taylor coefficients  $a_{j,l}$  ( $j, l$  even).

The coefficients  $a_{0,l}$  and  $a_{j,0}$  can be computed from 2D case because sections of the billiard domain by the vertical planes  $x_1 = 0$  and  $x_2 = 0$  give solutions of the 2D problem.

**Comments on numerics.** Take for example, as the linear map in  $\mathbb{R}^4 = \{q_1, q_2, q_3, q_4\}$  the rotation in  $\alpha_1 = 2\pi \cdot (3, 3, 1, 1, 1, \dots)$  (chain fraction) in the  $(q_1, q_2)$ -plane and the rotation in  $\alpha_2 = 2\pi \cdot (2, 5, 2, 2, 2, \dots)$  in the  $(q_3, q_4)$ -plane.

For any even  $k$  we present the line

$$\frac{a_{0,k}}{\sqrt{C_k^0}}, \frac{a_{2,k-2}}{\sqrt{C_k^2}}, \dots, \frac{a_{k,0}}{\sqrt{C_k^k}}.$$

The multipliers  $1/\sqrt{C_k^l}$  are motivated by the Bombieri metric on the space of homogeneous polynomials.

Here are numeric data beginning from  $k = 4$  (we save only 5 digits):

.50276, 1.0749, 1.8853

.38788, 1.1811, 1.9557, 3.6123

.36853, 1.5808, 2.7866, 4.5700, 8.6479

.39228, 2.3233, 4.4113, 7.3709, 12.080, 23.183

.44643, 3.6066, 7.3683, 12.798, 20.965, 34.380, 66.587

.53202, 5.8039, 12.711, 23.049, 38.630, 62.628, 102.77, 200.34