Geometric flows in centro-affine space

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Take-home points

- Geometric evolution equations are "integrable" if the evolutions of their geometric invariants are integrable.

  Famous case: vortex filament equation (binormal flow) $\rightarrow$ focusing nonlinear Schrödinger equation:

  $$\gamma_t = \gamma_x \times \gamma_{xx} = \kappa B \longrightarrow \mathrm{i}q_t + q_{ss} + 2|q|^2 q = 0$$

- Invariant moving frames for the evolving geometric object and preservation of the lowest order invariant (e.g. arc length for curves) by the evolution appear to be key to integrability.

- Sometimes, the geometric picture is simpler and clearer than that at the level of the invariants. In this talk, the former will reveal the integrability of the latter.

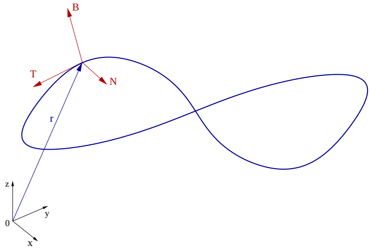
Moving Frames and Integrability

Curves in Euclidean space and moving frames

Recall the Frenet equations of classical curve theory:

$$T_s = \kappa N, \quad N_s = -\kappa T + \tau B, \quad B_s = -\tau N.$$

The unit tangent $T(s)$, normal $N(s)$, and binormal $B(s)$ form an (adapted) orthonormal frame along an arc length parametrized curve $\gamma(s)$. The curvature $\kappa(s) \neq 0$ and torsion $\tau(s)$ describe the shape of of the curve.



Sometimes, *natural Frenet systems* are more convenient:

$$T_s = \kappa_1 U_1 + \kappa_2 U_2, \quad (U_1)_s = -\kappa_1 T + \sigma U_2, \quad (U_2)_s = -\kappa_2 T - \sigma U_1,$$

where σ is a constant. (Frames of least rotation, $\sigma = 0$, are described in Bishop, 1975.) The relation with the classical Frenet system is

$$U_1 + \mathrm{i}U_2 = (N + \mathrm{i}B)\mathrm{e}^{\mathrm{i}\theta}, \quad \kappa_1 + \mathrm{i}\kappa_2 = \kappa \mathrm{e}^{\mathrm{i}\theta}, \quad \theta = \int^s (\tau(u) - \sigma)\, du.$$

The natural frame (T, U_1, U_2) can be represented in terms of skew-hermitian matrices, via the isometry

$$\begin{array}{ccc} (\mathbb{R}^3, \cdot) & \longrightarrow & (\mathfrak{su}(2), <\,,\,>) \\ \mathbf{x} & \longrightarrow & \sum_{k=1}^{3} x_k E_k. \end{array}$$

Here, the triplet

$$E_1 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

forms an orthonormal basis for the Lie algebra $\mathfrak{su}(2)$ with the inner product $< A, B >= -\frac{1}{2}\mathrm{trace}(AB)$, and satisfies $[E_i, E_j] = -2\epsilon_{ijk}E_k$.

Since $SU(2)$ acts transitively on orthonormal frames, there is $\Omega \in SU(2)$ that conjugates (T, U_1, U_2) to (E_1, E_2, E_3):

$$T = \Omega^{-1}E_1\Omega, \quad U_1 = \Omega^{-1}E_2\Omega, \quad U_2 = \Omega^{-1}E_3\Omega.$$

Rewriting the natural frame system in terms of Ω:

$$T_s = \kappa_1 U_1 + \kappa_2 U_2 \quad \Rightarrow \quad [E_1, \Omega_s \Omega^{-1}] = \kappa_1 E_2 + \kappa_2 E_3,$$
$$U_s = -\kappa_1 T + \sigma U_2 \quad \Rightarrow \quad [E_2, \Omega_s \Omega^{-1}] = -\kappa_1 E_1 + \lambda E_3,$$

leads to $\Omega_s \Omega^{-1} = \frac{\sigma}{2} E_1 - \frac{\kappa_2}{2} E_2 + \frac{\kappa_1}{2} E_3$ or, setting $\sigma = 2\lambda$ and

$$q = \tfrac{1}{2}(k_1 + \mathrm{i}k_2) = \tfrac{1}{2}\kappa \mathrm{e}^{\mathrm{i}\theta}.$$

We obtain

$$\Omega_s = \begin{pmatrix} -\mathrm{i}\lambda & \mathrm{i}q \\ \mathrm{i}\bar{q} & \mathrm{i}\lambda \end{pmatrix} \Omega,$$

recognizable as the eigenvalue problem of the AKNS system[1] with λ as spectral parameter. Thus, the AKNS eigenvalue problem arises as the 2λ-natural Frenet system for a space curve of curvature κ and torsion τ.

[1] The "Lax Pair" of several well-known soliton equations including mKdV, NLS, and sG.

Consequences

1. Let Ψ be a fundamental matrix solution of the AKNS spectral problem with $\Psi(0) = \mathrm{I}$. Then, for $\lambda \in \mathbb{R}$, the skew-hermitian matrix

$$\Gamma = \Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \begin{pmatrix} -\mathrm{i}\gamma_1 & \gamma_2 + \mathrm{i}\gamma_3 \\ -\gamma_2 + \mathrm{i}\gamma_3 & \mathrm{i}\gamma_1 \end{pmatrix}$$

gives the position vector of a curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of curvature $\kappa = |q|$ and torsion $\tau = \frac{\mathrm{d}}{\mathrm{d}s}\arg(q) - 2\lambda$. (Sym, Pohlmeyer.)

2. Integrable curve flows[2] arise from adjoining compatible arc length preserving time evolutions to the natural frame system $\Omega_s = \mathcal{U}(q(s,t);\lambda)\Omega$:

$$\Omega_t = \mathcal{V}(q(s,t);\lambda)\Omega, \qquad \mathcal{V} \in \mathfrak{su}(2),$$

so that the resulting *zero curvature condition*

$$\mathcal{U}_t - \mathcal{V}_s + [\mathcal{U}, \mathcal{V}] = 0$$

is independent of the spectral parameter λ.

[2] Those inducing completely integrable PDE for curvature and torsion.

Example: the Vortex Filament Equation

The compatible frame evolution

$$\Omega_t = \begin{pmatrix} \mathrm{i}(|q|^2 - 2\lambda^2) & 2\mathrm{i}\lambda q - q_s \\ 2\mathrm{i}\lambda\bar{q} - \bar{q}_s & \mathrm{i}(2\lambda^2 - |q|^2) \end{pmatrix} \Omega$$

gives rise to the focusing NLS equation

$$\mathrm{i}q_t + q_{ss} + 2|q|^2 q = 0 \qquad (fNLS)$$

for the complex curvature q, and to the *Vortex Filament Equation*

$$\gamma_t = \kappa_2 U_1 - \kappa_1 U_2 = \kappa B \qquad (VFE)$$

for $\gamma = \Gamma|_{\lambda=0}$.

The *Hasimoto map*

$$\mathcal{H}(\gamma) = q := \tfrac{1}{2}\kappa e^{\mathrm{i}\int^s \tau ds} \qquad \text{(Hasimoto, 1972)}$$

maps a solution of the VFE to a solution of the focusing NLS equation.

Integrable hierarchies

VFE

$$\mathcal{J} = T\times$$

$$\mathcal{K} = \mathcal{J}\frac{d}{ds}\mathcal{J}$$

$$\mathcal{R} = T \times \frac{d}{ds}$$

$$\{f,g\}_{\mathrm{MW}} = \oint_{S^1} \nabla f \cdot \mathcal{J}\nabla g$$

$$W_n = \mathcal{R}^n W_0, \quad W_0 = T$$

NLS

$$\tilde{\mathcal{J}}\phi = i\phi$$

$$\tilde{\mathcal{K}}_q\phi = \phi_s + \frac{1}{4}q\left(\int_0^s + \int_{2\pi}^s\right)[\phi\bar{q} - \bar{\phi}q]$$

$$\tilde{\mathcal{R}} = \tilde{\mathcal{K}}_q\tilde{\mathcal{J}}^{-1}$$

$$\{F,G\}_n = \langle\tilde{\mathcal{R}}^n\tilde{\mathcal{J}}F, G\rangle$$

$$X_n = \tilde{\mathcal{R}}^n X_0, \quad X_0 = q_x$$

Given $W = \alpha T + \beta_1 U_1 + \beta_2 U_2$ and arc length-preserving vector field, set $\beta = \beta_1 + \mathrm{i}\beta_2$. Then

Theorem (Langer & Perline)

$$\mathrm{d}\mathcal{H}[W] = \tilde{\mathcal{R}}^4\tilde{\mathcal{J}}\beta \implies \{F,G\}_4(\mathcal{H}(\gamma)) = \{F\circ\mathcal{H}, G\circ\mathcal{H}\}_{MW}(\gamma)$$

What about the KdV equation?

Planar star-shaped curves

A smooth curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ is *star-shaped* if

$$|\gamma(x)\,,\gamma'(x)| \neq 0 \quad \forall x \in I, \qquad (\star)$$

where $|\cdot\,,\cdot|$ denotes the determinant. Condition $(\star)$ is invariant under the linear action of $\mathrm{SL}(2,\mathbb{R})$.

Introduce centroaffine arc length $s = \int |\gamma\,,\gamma'|\,\mathrm{d}x$, with $v = |\gamma\,,\gamma'|$ the *centroaffine speed*.

If γ is arc length parametrized, then $|\gamma\,,\gamma_s| = 1$. Differentiating this relation with respect to s gives $|\gamma,\gamma_{ss}| = 0 \implies$

$$\gamma_{ss} = -p\gamma, \qquad p := |\gamma_s\,,\gamma_{ss}| \text{ (centroaffine curvature)}.$$

• centroaffine speed and curvature generate *differential invariants* of parametrized planar curves under the linear action of $\mathrm{SL}(2,\mathbb{R})$.

Pinkall's flow

The space of closed, unparametrized star-shaped curves in $\mathbb{R}^2$

$$M = \{\gamma : S^1 \to \mathbb{R}^2,\ |\gamma\,, \gamma_s| = 1\}/\{s \to s + c\}$$

has a natural symplectic structure [Pinkall, 1995]

$$\omega(X, Y) = \oint |X\,, Y|\, \mathrm{d}s,$$

with $X, Y \in T_\gamma M$ arc length preserving vector fields of the form

$$W = -\tfrac{1}{2}\beta_s \gamma + \beta \gamma_s, \quad \beta \in C^\infty(S^1).$$

the symplectic form establishes a 1-1 correspondence

$$\mathrm{d}h(X) = \omega(X, X_h), \quad \forall X \in T_\gamma M,$$

between Hamiltonians h and Hamiltonian vector fields X_h.

Taking $h[\gamma] = \oint p(s)\, \mathrm{d}s$ (twice the area swept by γ'), we get

$$\gamma_t = -\tfrac{1}{2} p_s \gamma + p \gamma_s, \qquad \text{Pinkall's Flow}$$

and the associated evolution for the centroaffine curvature p:

$$p_t = -\tfrac{1}{2} p_{sss} + 3 p p_s. \quad \text{(KdV)}$$

Curves in $\mathbb{RP}^1$ and moving frames

$G = PSL(2,\mathbb{R})$ acts on $\mathbb{RP}^1$ via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u = \frac{au+b}{cu+d},$$

(u an affine coordinate on $\mathbb{RP}^1$.)

A curve in $\mathbb{RP}^1$ is a regular map $\phi : I \to \mathbb{RP}^1$. (Take $u(x)$ as the value of ϕ in the affine coordinate, assuming $u'(x) > 0$.)

• A *moving frame* is a smooth equivariant lift $\rho : I \to G$ of the curve into the group, determined by the action of G on the jet space of ϕ.

• A *normalization* process determines the moving frame, as well as a generating set of differential invariants (*curvatures*) of the curve. [Fels & Olver]

Example: normalization conditions for a right-moving frame $g(x)$ (equivariant under the right action of G on itself):

$$(g\cdot u)(x) = 0, \qquad (g\cdot u)'(x) = 1, \qquad (g\cdot u)''(x) = 2\lambda, \quad \forall x \in I.$$

A λ-*normalized left moving frame*[3] is shown below in factored form:

$$\rho(x) = \begin{pmatrix} 1 & u(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u'(x)^{1/2} & 0 \\ 0 & u'(x)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda - \dfrac{u''(x)}{2u'(x)} & 1 \end{pmatrix}.$$

- The entries of the *Maurer-Cartan matrix* $\rho^{-1}\rho'$ contain an independent, generating set of differential invariants for the curve. [Hubert]

- Compute:

$$\rho^{-1}\rho' = \begin{pmatrix} \lambda & 1 \\ \kappa - \lambda^2 & -\lambda \end{pmatrix},$$

where $\kappa = -\frac{1}{2}\mathcal{S}(u)$ is the *projective curvature* of ϕ, and

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative.

[3] equivariant under the left action of G on itself

Invariant evolutions of curves in $\mathbb{RP}^1$

In affine coordinates, the most general $SL(2,\mathbb{R})$-invariant flow takes the form

$$u_t = ru', \tag{1}$$

with r a function of $\kappa = -\frac{1}{2}\mathcal{S}(u)$ and its derivatives. [Sokolov]

- If ϕ evolves by (1), then its λ-normalized projective frame satisfies

$$\rho^{-1}\rho_t = \begin{pmatrix} \lambda r + \frac{1}{2}r' & r \\ -\frac{1}{2}r'' - \lambda r' + (\kappa - \lambda^2)r & -\lambda r - \frac{1}{2}r' \end{pmatrix},$$

and its projective curvature satisfies

$$\kappa_t = -\tfrac{1}{2}r''' + 2\kappa r' + r\kappa',$$

which for $r = \kappa$ reduces to the KdV equation

$$\kappa_t = -\tfrac{1}{2}\kappa''' + 3\kappa\kappa'.$$

Well-known fact: the *Schwarzian KdV* equation $u_t = -\frac{1}{2}\mathcal{S}(u)u'$ induces a KdV evolution for $\frac{1}{2}\mathcal{S}(u)$. [Krichever, Novikov]

From centroaffine to projective

The projectivization map $\pi : \mathbb{R}^2 \to \mathbb{RP}^1$ takes a star-shaped curve $\gamma = (\gamma_1, \gamma_2)$ to a regular map $\phi : I \to \mathbb{RP}^1$ (since $|\gamma, \gamma'| \neq 0$).

- If $\gamma_1 \neq 0$, then $\pi(\gamma)$ in an affine chart is given by $u = \dfrac{\gamma_2}{\gamma_1}$.

- *Projectivization π takes centroaffine invariant evolutions to projective invariant evolutions:*

The most general invariant flow for arclength parametrized curves is

$$\gamma_t = -\frac{1}{2} r_s \gamma + r \gamma_s, \tag{2}$$

r a differential invariant. If γ evolves by (2), then $u = \gamma_2/\gamma_1$ satisfies

$$u_t = r u'.$$

The centroaffine curvature of an arclength parametrized curve satisfies $p(x) = -\frac{1}{2}\mathcal{S}(u)$, thus:

- *Pinkall's flow ($r = p$) is simply the Schwarzian KdV equation in homogeneous coordinates.*

Geometric Hamiltonian structures: framework

Let $\mathfrak{g} \equiv \mathfrak{g}^* = \mathfrak{sl}(2, \mathbb{R})$; the following are compatible Poisson structures on the loop algebra $\mathcal{L}\mathfrak{g}^*$:

$$\begin{aligned} \{\mathcal{H}, \mathcal{G}\}_1(L) &= \int_{S^1} \operatorname{tr}\left(\left(\left(\frac{\delta \mathcal{H}}{\delta L}\right)_x + \left[L, \frac{\delta \mathcal{H}}{\delta L}\right]\right) \frac{\delta \mathcal{G}}{\delta L}\right) dx, \\ \{\mathcal{H}, \mathcal{G}\}_0(L) &= \int_{S^1} \operatorname{tr}\left(\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \frac{\delta \mathcal{H}}{\delta L}\right] \frac{\delta \mathcal{G}}{\delta L}\right) dx, \end{aligned} \tag{3}$$

[Drinfeld, Sokolov]

where $\mathcal{G}, \mathcal{H}$ are functionals on $\mathcal{L}\mathfrak{g}^*$, and $\delta\mathcal{G}/\delta L, \delta\mathcal{H}/\delta L$ ($\in \mathcal{L}\mathfrak{g}$) are their gradients at the point $L \in \mathcal{L}\mathfrak{g}^*$.

Key Observation. (Ivey, Marí-Beffa, C) *These Poisson structures can be reduced on suitable quotient spaces $\mathcal{M}/\mathcal{L}N$ ($\mathcal{M} \subset \mathcal{L}\mathfrak{g}^*$, N subgroup of $SL(2, \mathbb{R})$) to give bi-Hamiltonian structures on: 1. the space of differential invariants of projective curves, and 2. the space of differential invariants of curves parametrized by centroaffine arc length.*

Geometric Hamiltonian structures: projective case

The space of periodic Maurer-Cartan matrices $\rho^{-1}\rho'$ *with* $\lambda = 0$ *can be identified with the quotient*

$$\mathcal{M}/\mathcal{L}N = \left\{ \begin{pmatrix} \alpha & 1 \\ \beta & -\alpha \end{pmatrix} \right\} \Big/ \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\},$$

where $\mathcal{L}N = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$ *acts on* $\mathcal{L}\mathfrak{sl}(2)^*$ *via the gauge action*

$$g \cdot L = g^{-1}g' + g^{-1}Lg, \qquad g \in \mathcal{L}N, L \in \mathcal{L}\mathfrak{sl}(2)^*.$$

The Poisson brackets (3) *can be reduced to* $\mathcal{M}/\mathcal{L}N$ *to produce the second and first Hamiltonian structure for KdV:* [Marí-Beffa]

$$\{h,f\}_1(\kappa) = \int_{S^1} \frac{\delta f}{\delta \kappa} \left(-\frac{1}{2}D^3 - \kappa D - D\kappa \right) \frac{\delta h}{\delta \kappa} dx,$$

$$\{h,f\}_0(\kappa) = 2\int_{S^1} \frac{\delta f}{\delta \kappa} D \frac{\delta h}{\delta \kappa} dx, \qquad D = \mathrm{d}/\mathrm{d}x,$$

with $h(\kappa), f(\kappa)$ functionals on the space of periodic curvatures of projective curves.

Geometric Hamiltonian structures: centroaffine case

Compute the normalized left moving frame for a star-shaped curve γ:

$$\rho = \begin{pmatrix}\gamma & \frac{1}{v}\gamma'\end{pmatrix}, \qquad v = |\gamma, \gamma'|.$$

Results:

- *The space of Maurer-Cartan matrices* $\rho^{-1}\rho' = \begin{pmatrix} 0 & -vp \\ v & 0 \end{pmatrix}$ *can be identified with the quotient*

$$\mathcal{L}\mathfrak{sl}^*(2) \Big/ \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\},$$

where $\mathcal{L}N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ *acts on* $\mathcal{L}\mathfrak{sl}(2)^*$ *via the gauge action*

$$g \cdot L = g^{-1}g' + g^{-1}Lg, \qquad g \in \mathcal{L}N,\ L \in \mathcal{L}\mathfrak{sl}(2)^*.$$

- *The Poisson brackets* (3) *can be reduced to* $\left\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right\} \subset \mathcal{L}\mathfrak{sl}(2)^*$*:*

$$\{h,f\}_1(a,b) = \int_{S^1} \left(\frac{\delta f}{\delta a}, \frac{\delta f}{\delta b}\right) \begin{pmatrix} -\frac{1}{2}D\frac{1}{b}D\frac{1}{b}D + \frac{1}{b}aD + D\frac{1}{b}a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta h}{\delta a} \\ \frac{\delta h}{\delta b} \end{pmatrix} \mathrm{d}x,$$

$$\{h,f\}_0(a,b) = \int_{S^1} \left(\frac{\delta f}{\delta a}, \frac{\delta f}{\delta b}\right) \begin{pmatrix} \frac{2}{b}D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta h}{\delta a} \\ \frac{\delta h}{\delta b} \end{pmatrix} \mathrm{d}x.$$

Further restriction to the subspace with $b = 1$ (arc length parametrized curves) gives the second and first Poisson operators for the KdV equation (where $a = p = -k$, the centroaffine curvature).

Consequence: *The projectivization map* $\pi : \mathbb{R}^2 \longrightarrow \mathbb{RP}^1$ *induces a local bi-Poisson map between the space of periodic Maurer-Cartan matrices for centroaffine curves parametrized by centroaffine arc-length, and the space of Maurer-Cartan matrices for parametrized projective curves.*

Hierarchies

Define the map

$$\mathcal{K}(\gamma) := p = |\gamma_s, \gamma_{ss}|$$

(analogue of the Hasimoto map for Euclidean curves) and let $Y = -\frac{1}{2}\beta_s\gamma + \beta\gamma_s$ be an arc length preserving vector field.

Lemma:

$$\mathrm{d}\mathcal{K}(Y) = \mathcal{P}\beta,$$

where $\mathcal{P}$ is the second Poisson operator of the KdV hierarchy.

Let $f = \int \rho \, \mathrm{d}s$ be an invariant functional on M (ρ must be a function of p and its derivatives). Define

$$X^f = -\tfrac{1}{2}(E\rho)_s\gamma + (E\rho)\gamma_s, \quad \text{where} \quad Ef = \sum_{j\geq 0}(-D)^j \frac{\partial\rho}{\partial p^{(j)}}.$$

If $h^k = \oint \rho_k ds$ is the k-th KdV Hamiltonian, and X_{h_k} is the Hamiltonian vector field defined by

$$\mathrm{d}h^k(Y) = \omega(Y, X_{h_k}),$$

Then,

$$X_{h_k} = X^{\rho_{k+1}}, \qquad E\rho_{k+1} = \mathcal{Q}^{-1}\mathcal{P}E\rho_k.$$

In summary,

Theorem

1. *Each flow of the KdV hierarchy is the curvature evolution induced by a geometric flow for centroaffine curves in* $\mathbb{R}^2$.
2. *Defining* $\{H_j, H_k\}(\gamma) := \omega(X_{H_j}, X_{H_k})$, *we have:*

$$\{H_j, H_k\}(\gamma) = \{H_j, H_{k+1}\}_0(\mathcal{K}(\gamma)) = \{H_j, H_k\}_1(p),$$

Thus $\mathcal{K}$ *is a Poisson map, that maps Pinkall's area form to the second Poisson structure for KdV.*

Discrete setting

Polygonal evolutions

Example

Consider the Volterra model on twisted polygons, a discretization of the Korteweg deVries (KdV) equation

$$(q_n)_t = q_n(q_{n+1} - q_{n-1})$$

and assume that q_n is *the cross ratio* of $u_{n-1}, u_n, u_{n+1}, u_{n+2}$, where $u_n \in \mathbb{RP}^1$ for all n. Let $\{\gamma_n\}$ be a lift of the projective polygon to $\mathbb{R}^2$ with

$$|\gamma_n, \gamma_{n+1}| = 1, \ \forall n.$$

(Possible if N is not even.) Then, $q_n = (|\gamma_{n-1}, \gamma_{n+1}||\gamma_n, \gamma_{n+2}|)^{-1}$ and the *projective tangential flow*

$$(\gamma_n)_t = \frac{1}{2|\gamma_{n-1}, \gamma_{n+1}|}(\gamma_{n+1} - \gamma_{n-1})$$

is a projective realization of the Volterra model .

Note: We assume we have twisted polygons, i.e. $u_{n+N} = g \cdot u_n$, for $g \in \mathrm{PSL}(2, \mathbb{R})$, the monodromy and $N \in \mathbb{N}$, the period.

General polygonal evolutions

Let $\{\gamma_n\}$, $\gamma_n \in \mathbb{R}^{m+1}$, be the lift of an N-twisted projective polygon $\{u_n\}$, $u_n \in \mathbb{RP}^m$, such that

$$\ell_n = |\gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m}| = 1, \ \forall n.$$

(This condition can be satisfied when m and N are coprime.)

The projective invariants are defined by the relation:

$$\gamma_{n+m+1} = a_n^m \gamma_{n+m} + \cdots + a_n^1 \gamma_{n+1} + (-1)^m \gamma_n.$$

Introduce the left moving frame $\rho(\gamma) = \{\rho_n(\gamma)\}$:

$$\rho_n(\gamma) = (\gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m}).$$

The discrete equivalent of the Frenet-Serret equations are:

$$\rho_{n+1} = \rho_n K_n, \qquad K_n = \begin{pmatrix} \mathbf{0}^T & (-1)^{m-1} \\ I_{m-1} & \mathbf{a}_n \end{pmatrix}$$

$\{K_n\}$, the Maurer-Cartan matrix, defines (local) coordinates of the moduli space of polygons.

A projective vector field $X = \{X_n\}$, invariant under the projective group, has the form

$$X_n = \sum_{\ell=0}^{m} r_n^\ell \gamma_{n+\ell} = \rho_n \mathbf{r}, \qquad \mathbf{r}^T = (r_n^0, \ldots, r_n^m),$$

where r_n^ℓ are functions of the curvatures a_j^k. The resulting polygon evolution equation

$$(\gamma_n)_t = X_n$$

induces evolution equations for its left-moving frame $\rho = \{\rho_n\}$:

$$(\rho_n)_t = \rho_n Q_n,$$

The Frenet-Serret equations and the frame evolution equations are compatible provided

$$(K_n)_t = K_n Q_{n+1} - Q_n K_n.$$

Arc length preservation during the evolution can be expressed as:

$$\mathrm{tr}(Q_n) = 0.$$

As in the continuous case, compatibility and arc length preservation are key to integrabiilty.

Discrete generalized KdV flows

The following ODE system

$$(\gamma_n)_t = -\frac{m}{m+1}\gamma_n - \frac{a^2_{n-1}}{a^1_{n-1}}\gamma_{n+1} \ldots - \frac{a^m_{n-1}}{a^1_{n-1}}\gamma_{n+m-1} + \frac{1}{a^1_{n-1}}\gamma_{n+m}. \quad (4)$$

preserves the centroaffine arc length ℓ_n, defining a projective evolution.

Theorem (Marí-Beffa & Wang, 2013)

If $\{\gamma_n(t)\}$ *is a solution of* (4) , *then the geometric invariants* $\{\mathbf{a}_n(t)\}$ *satisfy the following system of ODEs:*

$$\begin{cases} (a^k_n)_t = \dfrac{a^{k+1}_{n+1}}{a^1_{n+1}} - \dfrac{a^{k+1}_n}{a^1_{n-k}} & k = 1, 2, \ldots, m-1 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} a^m_n = \dfrac{1}{a^1_{n+1}} - \dfrac{1}{a^1_{n-m-1}} & \end{cases}$$

a discretization of the m-AGD flow: a generalized KdV equation introduced by Lax ('68) and proven integrable by Adler ('79), Gel'fand and Dikii ('87), who found two compatible Hamiltonian structures (the AGD brackets)

Theorem (Marí-Beffa & Wang, '13)

The evolution equation for $\{\mathbf{a}_n(t)\}$ is Hamiltonian with respect to two Poisson structures $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$. For $m = 1, 2$, the Poisson brackets are compatible and the evolutions are Liouville integrable.

For $m = 2$, the Poisson brackets are given by:

$$\{f,h\}_1(\mathbf{a}) = \sum_n \begin{pmatrix} \frac{\partial f}{\partial a_n^1} & \frac{\partial f}{\partial a_n^2} \end{pmatrix}^T \begin{pmatrix} \mathcal{T}^{-1}a_n^2 - a_n^2\mathcal{T} & \mathcal{T} - \mathcal{T}^{-2} \\ \mathcal{T}^2 - \mathcal{T}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial a_n^1} \\ \frac{\partial h}{\partial a_n^2} \end{pmatrix},$$

$$\{f,h\}_2(\mathbf{a}) = \sum_n \begin{pmatrix} \frac{\partial f}{\partial a_n^1} & \frac{\partial f}{\partial a_n^2} \end{pmatrix}^T \begin{pmatrix} a_n^1\mathcal{R}^{-1}(\mathcal{T} - \mathcal{T}^{-1})a_n^1 & a_n^1\mathcal{R}^{-1}(1 - \mathcal{T}^{-1})a_n^2 \\ a_n^2\mathcal{R}^{-1}(1 - \mathcal{T}^{-1})a_n^1 & \mathcal{T}a_n^1 - a_n^1\mathcal{T}^{-1} + a_n^2\mathcal{R}^{-1}(\mathcal{T} - \mathcal{T}^{-1})a_n^2 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial a_n^1} \\ \frac{\partial h}{\partial a_n^2} \end{pmatrix},$$

where $\mathcal{T}a_n^k := a_{n+1}^k$ and $\mathcal{R} := \mathcal{T}^{-1} + 1 + \mathcal{T}$ (invertible for $N \neq 0 \bmod 3$)For $m \geq 3$, the compatibility question was left open due to unyielding computations.

Lifting the Poisson structures to projective polygons

On the space of projective vector fields on twisted polygons, define the operator $L(X) = \{L_n(X)\}$ with

$$L_n(X) = (-1)^m((-1)^m X_n + a_n^1 X_{n+1} + \cdots + a_n^m X_{n+m} - X_{n+m+1}).$$

Let θ be the 1-form

$$\theta(X) = \sum_{n=1}^{N} |X_n, \gamma_{n+1}, \ldots, \gamma_{n+m}|$$

and define the following two forms ω_1 and ω_2

$$\omega_1(X,Y) = d\theta(X,Y)$$
$$\omega_2(X,Y) = X(\theta(L(Y))) - Y(\theta(L(X)) - \theta\left((XL(Y) - YL(X))\right).$$

Theorem (Marí-Beffa, C)

Both ω_1 and ω_2 are closed 2-forms on the space of projective vector fields, and have non-trivial kernels (pre-symplectic forms).

Results (M-B, C)

Theorem

Given function f, g of the invariants a_n^r, there exist vector fields X^f, X^h (explicitly constructible) such that:

1. $\omega_1(X^f, X^h)(\gamma) = \{f, h\}_1(\mathbf{a})$ *and* $\omega_2(X^f, X^h)(\gamma) = \{f, h\}_2(\mathbf{a})$.
2. X^f *is the* ω_2*-Hamiltonian vector field of* f.
3. *If* γ *evolves via*
$$(\gamma_n)_t = X_n^f,$$
then the evolution of the $\mathbf{a}_n$*'s is Hamiltonian w.r.t.* $\{\ ,\ \}_2$ *with Hamiltonian* f.
4. *The map* $df \to X^f$ *is the de-facto inverse of* ω_2 *on symplectic leaves.*

The discrete AGD-flows are bi-Hamiltonian

Corollary

The Hamiltonian structures $\{\ ,\ \}_1$ *and* $\{\ ,\ \}_2$ *are compatible. It follows that the discretizations of generalized KdV are bi-Hamiltonian and completely integrable.*

Proof.

$$\sum_{\circlearrowleft} \{h, \{f,g\}_1 + \{f,g\}_2\}_1 + \{h, \{f,g\}_1 + \{f,g\}_2\}_2$$

$$= \sum_{\circlearrowleft} \{h, \{f,g\}_2\}_1 + \{h, \{f,g\}_1\}_2$$

$$= \sum_{\circlearrowleft} \omega_1(X^h, [X^f, X^g]) + X^h(\omega_1(X^f, X^g)) = \sum_{\circlearrowleft} d\omega_1(X^h, X^f, X^g) = 0.$$

□

Further properties of ω_1 and ω_2 (Marí-Beffa, C)

1. The kernel of ω_1 is generated by two vector fields X^1 and X^2

$$X_n^1 = \gamma_{n+1} + \alpha_n \gamma_n, \qquad X_n^2 = \gamma_{n+2} + (\alpha_n + \alpha_{n+1})\gamma_{n+1} + \beta_n \gamma_n,$$

   with Hamiltonians with respect to ω_2 given by $h^1 = \frac{1}{m+1}\theta(X^1)$ and $h^2 = \frac{m+1}{2}\theta(X^2)$, respectively.

2. The kernel of ω_2 is generated by the vector fields

$$(\gamma_n)_t = A\gamma_n$$

   where $A \in \mathfrak{sl}(m+1)$ is an element of the isotropy algebra of the monodromy of γ. These vector fields are Hamiltonian with respect to ω_1, with Hamiltonian $h_A = \theta(A\gamma)$.

3. $\omega_2(X^1, X^2) = 0$.

In progress: Construction of integrable hierarchies.

Thank you!