Pentagram Maps and Integrable Hierarchies

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Geometry of the Boussinesq equation

Let $G : \mathbb{R} \to \mathbb{RP}^2$ be a nondegenerate curve, i.e. G' and G'' are not collinear $\forall x \in \mathbb{R}$.



Reminder on envelopes:

Define evolution of G(x) in time. Given $\epsilon > 0$ take the envelope L_{ϵ} of chords $[G(x - \epsilon), G(x + \epsilon)]$. Expand the envelope L_{ϵ} in ϵ :

$$L_{\epsilon}(x) = G(x) + \epsilon^2 B_G(x) + O(\epsilon^4)$$



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In the expansion

$$L_{\epsilon}(x) = G(x) + \epsilon^2 B_G(x) + O(\epsilon^4)$$

view ϵ^2 as the time variable.

Theorem (Ovsienko-Schwartz-Tabachnikov 2010)

The evolution equation $\partial_t G(x, t) = B_G(x, t)$ is equivalent to the Boussinesq equation $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$.

Remark. The Boussinesq equation is the (2,3)-equation of the Korteweg-de Vries hierarchy.

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Joseph Boussinesq and shallow water



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- the Boussinesq shallow water approximation (1872) $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$
- he introduced the KdV equation (as a footnote, 1877) $u_t + (u^2)_x + u_{xxx} = 0$
- it was rediscovered by D.Korteweg and G.de Vries (1895)

Reminder on the KdV hierarchy

Let

$$R = \partial^{m} + u_{m-2}(x)\partial^{m-2} + u_{m-3}(x)\partial^{m-3} + \dots + u_{1}(x)\partial + u_{0}(x),$$

where $\partial^j := d^j/dx^j$. Define its *m*th root $Q = R^{1/m}$ as a formal pseudo-differential operator

$$Q = \partial + a_1(x)\partial^{-1} + a_2(x)\partial^{-2} + \dots,$$

such that $Q^m = R$. (Use the Leibniz rule $\partial f = f \partial + f'$.)

 Define its fractional power R^{k/m} = ∂^k + ... for any k = 1, 2, ... and take its purely differential part Q_k := (R^{k/m})₊.

Example

for
$$k = 1$$
 one has $Q_1 = \partial$,
for $k = 2$ one has $Q_2 = \partial^2 + (2/m)u_{m-2}(x)$.

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The (k, m)-KdV equation is

$$\frac{d}{dt}R=\left[Q_{k},R\right].$$

Given order *m*, these evolution equations on $R = \partial^m + ... + u_0(x)$ commute for different *k* and form integrable hierarchies.

Example

- the Korteweg-de Vries equation $u_t + uu_x + u_{xxx} = 0$ is the "(3,2)-KdV equation". It is the 3rd evolution equation on Hill's operator $R = \partial^2 + u(x)$ of order m = 2.
- the Boussinesq equation $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$ is the "(2,3)-KdV equation". It is the 2nd evolution equation on operator $R = \partial^3 + u(x)\partial + v(x)$ of order m = 3, after exclusion of v.

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In higher dimensions...

Let $G : \mathbb{R} \to \mathbb{RP}^d$ be a nondegenerate curve, i.e. $(G', G'', ..., G^{(d)})$ are linearly independent $\forall x \in \mathbb{R}$.



Given $\epsilon > 0$ and reals $\varkappa_1 < \varkappa_2 < ... < \varkappa_d$ such that $\sum_j \varkappa_j = 0$ define hyperplanes $P_{\epsilon}(x) = [G(x + \varkappa_1 \epsilon), ..., G(x + \varkappa_d \epsilon)].$

Example

In \mathbb{RP}^3 above $\varkappa_i = -1, 0, 1$

Let $L_{\epsilon}(x)$ be the envelope curve for the family of hyperplanes $P_{\epsilon}(x)$ for a fixed ϵ .

The envelope condition means that for each x the point $L_{\epsilon}(x)$ and the derivative vectors $L'_{\epsilon}(x), ..., L^{(d-1)}_{\epsilon}(x)$ belong to the plane $P_{\epsilon}(x)$.

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Expand the envelope in ϵ :

$$L_{\epsilon}(x) = G(x) + \epsilon^2 B_G(x) + O(\epsilon^3)$$

Theorem (K.-Soloviev 2016)

The evolution equation $\partial_t G(x, t) = B_G(x, t)$ is equivalent to the (2, d+1)-KdV equation for any choice of $\varkappa_1, ..., \varkappa_d$. In particular, it is an integrable infinite-dimensional system.

Remark. If $\sum_{j} \varkappa_{j} \neq 0$, then the expansion is

$$L_{\epsilon}(x) = G(x) + \epsilon G'(x) + O(\epsilon^2)$$

and the evolution equation $\partial_t G(x, t) = G'(x, t)$ is equivalent to the (1, d+1)-KdV equation $\frac{d}{dt}R = [\partial, R]$.

Open question. Describe the geometry of higher (k, d + 1)-KdV equations for all $k \ge 3$.

How to associate a curve $G \subset \mathbb{RP}^d$ to a differential operator

$$R = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + \dots + u_0(x) \quad ?$$

• Consider the linear differential equation $R \psi = 0$. Take any fundamental system of solutions

$$\Psi(x) := (\psi_1(x), \psi_2(x), ..., \psi_{d+1}(x)).$$

Regard it as a map $\Psi : \mathbb{R} \to \mathbb{R}^{d+1}$.

• Pass to the corresponding homogeneous coordinates:

 $G(x) := (\psi_1(x) : \psi_2(x) : \ldots : \psi_{d+1}(x)) \in \mathbb{RP}^d, \text{ i.e. } G : \mathbb{R} \to \mathbb{RP}^d.$

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We associated a curve $G \subset \mathbb{RP}^d$ to a linear differential operator R. Moreover,

- Wronskian $\Psi(x) \neq 0 \ \forall x \iff G(x)$ is nondegenerate $\forall x$, i.e. $(G', G'', ..., G^{(d)})$ are linearly independent for all x.
- Solution set Ψ is defined modulo SL_{d+1} transformations.
 The curve G ⊂ ℝP^d is defined modulo PSL_{d+1} (i.e. projective) transformations.
- For differential operator R with periodic coefficients, solutions Ψ of R Ψ = 0 (and hence, the curve G) are quasiperiodic: there is a monodromy M ∈ SL_{d+1} such that Ψ(x + 2π) = MΨ(x) for all x ∈ ℝ.

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The **pentagram map** takes a (convex) *n*-gon $P \subset \mathbb{RP}^2$ into a new polygon T(P) spanned by the "shortest" diagonals of P (modulo projective equivalence):



- T = id for n = 5
- $T^2 = id$ for n = 6
- T is quasiperiodic for $n \ge 7$

Hidden integrability?

Or, rather, a Pentagon map?





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Properties of the pentagram map

- The integrability was proved for closed and twisted (i.e. with fixed monodromy) polygons in 2D (Ovsienko-Schwartz-Tabachnikov 2010, Soloviev 2012).
- There are first integrals, an invariant Poisson structure, and a Lax form.
- Pentagram map is related to cluster algebras, frieze patterns, etc.
- Its continuous limit is the Boussinesq equation.
- Extension to corrugated polygons, polygonal spirals, etc.

How to generalize to higher dimensions?

- No generalizations to polyhedra: many choices of hyperplanes passing through neighbours of any given vertex.
- Diagonal chords of a space polygon may be skew and do not intersect in general.

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Example and integrability of a pentagram map in 3D

For a generic *n*-gon $\{v_k\} \subset \mathbb{RP}^3$, for each *k* consider the two-dimensional "short-diagonal plane" $P_k := [v_{k-2}, v_k, v_{k+2}]$.

Then the **space pentagram map** T_{sh} is the intersection point

$$T_{sh}v_k := P_{k-1} \cap P_k \cap P_{k+1}.$$



Theorem (K.-Soloviev 2012)

The 3D short-diagonal pentagram map is a discrete integrable system (on a Zariski open subset of the complexified space of closed n-gons in 3D modulo projective transformations PSL_4). It has a Lax representation with a spectral parameter. Invariant tori have dimensions $3\lfloor n/2 \rfloor - 6$ for odd n and 3(n/2) - 9 for even n.

Twisted polygons

Remark. There is a version for twisted space *n*-gons, where vertices are related by a fixed monodromy $M \in PSL_4$: $v_{k+n} = Mv_k$ for any $k \in \mathbb{Z}$. The dimension of the space of *closed n*-gons modulo projective equivalence is $3n - \dim PSL_4 = 3n - 15$. This dimension for *twisted n*-gons is 3n - 15 + 15 = 3n.



A differential operator $R = \partial^{d+1} + ... + u_0(x)$ defines a "solution curve" $\Psi : \mathbb{R} \to \mathbb{R}^{d+1}$ such that $\Psi^{(d+1)} + u_{d-1}(x)\Psi^{(d-1)} + ... + u_0(x)\Psi = 0$ $\forall x \in \mathbb{R}$, which defines a nondegenerate curve $G : \mathbb{R} \to \mathbb{RP}^d$ (mod projective equivalence).

A difference operator defines a (twisted) "polygonal curve" $V : \mathbb{Z} \to \mathbb{R}^{d+1}$ such that $V_{i+d+1} + a_{i,d}V_{i+d} + \ldots + a_{i,1}V_{i+1} \pm V_i = 0$ $\forall i \in \mathbb{Z}$, which defines a generic twisted polygon $v : \mathbb{Z} \to \mathbb{RP}^d$ (mod projective equivalence).

As $n \to \infty$ a generic *n*-gon $v_i, i \in \mathbb{Z}$ in \mathbb{RP}^d "becomes" a nondegenerate curve $G(x), x \in \mathbb{R}$ in \mathbb{RP}^d .

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For a generic *n*-gon $\{v_k\} \subset \mathbb{RP}^d$ and any fixed (d-1)-tuple $I = (i_1, ..., i_{d-1})$ of "jumps" $i_{\ell} \in \mathbb{N}$ define an *I*-diagonal hyperplane P_k^I by $P_k^I := [v_k, v_{k+i_1}, v_{k+i_1+i_2}, ..., v_{k+i_1+...+i_{d-1}}]$.



Example

The diagonal hyperplane P_k^I for the jump tuple I = (3, 1, 2) in \mathbb{RP}^4 .

Pentagram map T^{I} in \mathbb{RP}^{d} is

$$T'v_k := P_k' \cap P_{k+1}' \cap ... \cap P_{k+d-1}'$$

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Integrability in higher dimensions

Theorem

The pentagram map T^{I} on (projective equivalence classes of) n-gons in \mathbb{RP}^{d} is an integrable system, i.e. it admits a Lax representation with a spectral parameter, for

- the "short-diagonal case", I = (2, 2, ..., 2);
 (d = 2 O.-S.-T., d = 3 K.-S., d ≥ 4 K.-S.+G.Mari-Beffa)
- the "deep-dented case", I = (1, ..., 1, p, 1, ..., 1) for any $p \in \mathbb{N}$. (any d K.-S.)

Remark. In all cases, the continuous limit $n \to \infty$ is the integrable (2, d + 1)-KdV equation!

A "deep-dented" diagonal hyperplane for I = (1, 1, 3, 1) in \mathbb{RP}^5 :



The corresponding pentagram map T' is defined by intersecting 5 consecutive hyperplanes P_k for each vertex, and it is integrable!

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Theorem (Izosimov 2018)

The pentagram map in $\mathbb{R}P^d$ defined by $Tv_k := P_k^+ \cap P_k^-$ intersecting two planes $P^{\pm} = (v_{k+j}, j \in R^{\pm})$ of complementary dimensions in $\mathbb{R}P^d$, where R^{\pm} are two m-arithmetic sequences, is a discrete integrable system. These pentagram maps can be obtained as refactorizations of difference operators, have natural Lax forms and invariant Poisson structures.



Corollary (Izosimov-K. 2019)

1) The (dual) long-diagonal pentagram map T_m^l in $\mathbb{R}P^d$ defined by the jump tuple I = (m, ..., m, p, m, ..., m) (or, more generally, for the union of two m-arithmetic sequences), and $T_m^l := P_k \cap P_{k+m} \cap ... \cap P_{k+m(d-1)}$, is a completely integrable system, i.e. it admits a Lax representation with a spectral parameter and invariant Poisson structure.

2) The continuous limit is the (2, d + 1)-KdV equation for all those cases.

Remark. Such general pentagram maps T_m^l , along with their duals, include all known integrable cases!

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Duality

- for (d 1)-tuple of jumps $I = (i_1, ..., i_{d-1})$, the *I*-diagonal plane is $P'_k := [v_k, v_{k+i_1}, v_{k+i_1+i_2}, ..., v_{k+i_1+...+i_{d-1}}]$.
- for (d − 1)-tuple of intersections J = (j₁,..., j_{d−1}), the pentagram map T^{I,J} in ℝP^d is

$$T^{I,J}v_k := P^I_k \cap P^I_{k+j_1} \cap \ldots \cap P^I_{k+j_{d-1}}$$

Example

- The 2D pentagram map is $T^{(2)(1)}$ for I = (2) and J = (1).
- The short-diagonal map in 3D has I = (2, 2) and J = (1, 1).
- Pentagram map T' has $I = (i_1, ..., i_{d-1})$ and J = (1, ..., 1).

Integrability of $T^{I,J}$ for general I and J is unknown!

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Duality

Let $I^* = (i_{d-1}, ..., i_1)$ be the (d-1)-tuple I in the opposite order.

Theorem (K.-Soloviev 2015)

There is the following duality for the pentagram maps $T^{I,J}$:

 $(T^{I,J})^{-1} = T^{J^*,I^*}.$

Example For the 2D pentagram map: $(T^{(1)(2)})^{-1} = T^{(2)(1)}.$



The **height of a rational number** $a/b \in \mathbb{Q}$ in the lowest terms is $ht(a/b) = \max(|a|, |b|)$. Use projectively-invariant (cross-ratio) coordinates (x_i, y_i, z_i) on the space of *n*-gons in \mathbb{QP}^3 (i.e., having only rational values of coordinates).

The **height of an** *n*-gon *P* is

$$H(P):=\max_{0\leq i\leq n-1}\max(ht(x_i),ht(y_i),ht(z_i)).$$

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We trace how fast the height of an initial 11-gon grows with the number of iterates of different pentagram maps in 3D.

Fix a twisted 11-gon in \mathbb{QP}^3 by specifying vectors in \mathbb{Q}^4 . Their coordinates are randomly distributed in [1,10].

Observed: a sharp contrast in the height growth for different maps. However, the borderline between numerically integrable and non-integrable cases is difficult to describe. We group those cases separately.

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First study the short-diagonal map ${\it T}_{\rm sh}$ in 3D, which is known to be integrable.

After t = 8 iterations, the height of the twisted 11-gon in \mathbb{QP}^3 becomes of the order of 10^{500} :



"Polynomial growth" of log H for the integrable pentagram map $T_{\rm sh}$ in 3D as a function of t.

The height also grows moderately fast for the integrable dented maps $T^{(2,1)}$ and $T^{(1,2)}$, reaching the value of the order of 10^{800} .



Similar moderate growth is observed for the (integrable) deep-dented map $T^{(1,3)}$ in 3D: the height remains around 10^{1000} .

Map $T^{I,J}$	Name of pent. map	Height at $t = 8$
$T_{\rm sh} := T^{(2,2),(1,1)}$	short-diagonal	10 ⁵⁰⁰
$T^{(2,1)} := T^{(2,1),(1,1)}$	dented	10 ⁸⁰⁰
$T^{(3,1)} := T^{(3,1),(1,1)}$	deep-dented	10^{1000}
$T^{(2,2),(1,2)}$	long-diagonal	10 ¹⁰⁰⁰
$T^{(1,2),(1,2)}$		10 ²⁰⁰⁰
$T^{(1,3),(1,3)}$		10 ³⁰⁰⁰
$T^{(2,3),(2,3)}$		10 ³⁰⁰⁰

Observation - **Conjecture.** The pentagram maps $T^{I,I}$, i.e. those with I = J, are integrable.

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The first case not covered by theorems: for the map $T^{(2,3)}$ after 8 iterations the height is already of order 10^{10^7} .



Linear growth of log log H for the map $T^{(2,3)}$ in 3D indicates super-fast growth of its height and apparent non-integrability.

More numerically non-integrable cases:

Мар <i>Т^{1,J}</i>	Height at $t = 8$
$T^{(1,2),(3,1)}$	$10^{3 \cdot 10^7}$
$T^{(1,2),(1,3)}$	$10^{3 \cdot 10^7}$
$T^{(2,3)} := T^{(2,3),(1,1)}$	10 ^{10⁷}
$T^{(2,4)} := T^{(2,4),(1,1)}$	10 ^{10⁷}
$T^{(3,3)} := T^{(3,3),(1,1)}$	10 ^{10⁷}

Open Problems:

- Prove non-integrability in those cases.
- Describe the border of integrability and non-integrability.

Intricate border of integrable and non-integrable cases:



Several references

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THANK YOU!

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